MAT3300/4300-H09 - Alternative solution of Exercise 7.9 (ii).

Let $F: \mathbb{R} \to \mathbb{R}$ be a Stieljes function, i.e. F is increasing and left-continuous.

Define $\nu_F([a,b)) = F(b) - F(a)$ $\forall a,b \in \mathbb{R}$, $a<b$

and $\nu_F(\emptyset) = 0$.

Then ν_F has a unique extension to a measure on $\mathcal{B} = \mathcal{B}(\mathbb{R})$.

Using Caratheodory's theorem, it suffices to check that ν_F is a premeasure on $\mathcal{J} = \{ [a,b) \mid a,b \in \mathbb{R}, a \le b \}$. The uniqueness will then follow from the fact that $[-k,k) \uparrow \mathbb{R}$ while

$\nu_F([-k,k)) = F(k) - F(-k) < \infty$ $\forall k \in \mathbb{N}$.

We first note that <u>ν_F is finitely additive on $\mathcal{J}$</u>:

Assume $[a,b) = [a,c) \cup [c,b)$.

Then $\nu_F([a,b)) = F(b) - F(a) = \underbrace{F(c) - F(a)}_{= \nu_F([a,c))} + \underbrace{F(c) - F(b)}_{= \nu_F([c,b))}$

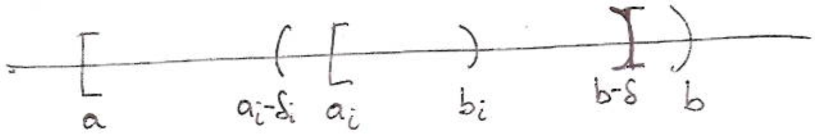
We have to show that <u>ν_F is countably additive on $\mathcal{J}$</u>. This is not easy. The idea of this proof is to use compactness of closed intervals.

Consider $[a,b) = \bigcup_{i \in \mathbb{N}} [a_i, b_i)$.

Let $\varepsilon > 0$. We will show that

$$(*) \qquad \nu_F([a,b)) \le \sum_{i=1}^{\infty} \nu_F([a_i,b_i)) + 2\varepsilon .$$

The following drawing might be helpful:



Using left-continuity of F at b, we can choose $0 < \delta < b-a$

such that $\underbrace{F(b) - F(b-\delta)}_{= \nu_F([b-\delta, b))} < \varepsilon$. Since ν_F is finitely additive,

we get
$$\nu_F([a,b)) = \nu_F([a, b-\delta)) + \nu_F([b-\delta, b))$$
$$< \nu_F([a, b-\delta)) + \varepsilon .$$

Using left-continuity of F at each a_i, we can choose $0 < \delta$

such that $\underbrace{F(a_i) - F(a_i - \delta_i)}_{= \nu_F([a_i - \delta_i, a_i))} < \varepsilon/2^i$

Now we have

$$[a, b-\delta] \subset [a,b] = \bigcup_{i \in \mathbb{N}} [a_i, b_i) \subset \bigcup_{i \in \mathbb{N}} (a_i - \delta_i, b_i) .$$

Using compactness of $[a, b-\delta]$, we get that there exists $M \in \mathbb{N}$

such that $[a, b-\delta] \subset \bigcup_{i=1}^{M} (a_i - \delta_i, b_i)$. It follows that

$$[a, b-\delta) \subset \bigcup_{i=1}^{M} [a_i - \delta_i, b_i) .$$

As ν_F is finitely additive on the semi-ring $\mathcal{J}$, ν_F is also finitely sub-additive (we will explain this at the end).

Hence we get

$$\nu_F([a, b-\delta)) \leq \sum_{i=1}^{M} \nu_F([a_i - \delta_i, b_i)) \leq \sum_{i=1}^{\infty} \nu_F([a_i - \delta_i, b_i))$$
$$= \sum_{i=1}^{\infty} \Big(\underbrace{\nu_F([a_i - \delta_i, a_i))}_{< \varepsilon/2^i} + \nu_F([a_i, b_i)) \Big)$$
$$\leq \underbrace{\sum_{i=1}^{\infty} \varepsilon/2^i}_{= \varepsilon} + \sum_{i=1}^{\infty} \nu_F([a_i, b_i)) .$$

Putting what we have established together, we get

$$\nu_F([a,b)) < \nu_F([a,b-\delta)) + \varepsilon$$
$$< \varepsilon + \sum_{i=1}^{\infty} \nu_F([a_i,b_i)) + \varepsilon = \sum_{i=1}^{\infty} \nu_F([a_i,b_i)) + 2\varepsilon ,$$

which proves that $(*)$ holds.

As this holds for every $\varepsilon > 0$, we get

$$\underline{\nu_F([a,b)) \leq \sum_{i=1}^{\infty} \nu_F([a_i,b_i))} .$$

The reverse inequality is easier to prove:

For each $n \in \mathbb{N}$, we have

$$\sum_{i=1}^{n} \nu_F([a_i,b_i)) = \nu_F\Big(\dot{\bigcup_{i=1}^{n}} [a_i,b_i)\Big) \leq \nu_F\Big(\dot{\bigcup_{i=1}^{\infty}} [a_i,b_i)\Big) = \nu_F([a,b))$$

(where we use that ν_F is monotone on $\mathcal{J}$, as one easily sees).

Letting $n \to \infty$, we get

$$\underline{\sum_{i=1}^{\infty} \nu_F([a_i,b_i)) \leq \nu_F([a,b))}$$

Altogether, this gives $\sum_{i=1}^{\infty} \nu_F([a_i,b_i)) = \nu_F([a,b))$,

and we have shown that ν_F is countably additive on $\mathcal{J}$,

i.e. ν_F is a <u>premeasure on $\mathcal{J}$</u>.

It remains to explain why a premeasure μ on a semi-ring $\mathcal{S}$ is finitely sub-additive, i.e.

$$\{S_j\}_{j\in\{1,\dots,N\}} \subset \mathcal{S} \text{ and } S \in \mathcal{S},\ S \subset \bigcup_{j\in\{1,\dots,N\}} S_j \Rightarrow \mu(S) \leq \sum_{j=1}^{N} \mu(S_j) .$$

This follows from the arguments given in Step 2 of the proof of Caratheodory's theorem (p. 39-41). There it is shown that μ can be uniquely extended to the set $\mathcal{S}_{\uplus} = \{ S_1 \uplus \cdots \uplus S_M \mid M \in \mathbb{N},\ S_j \in \mathcal{S} \}$ by $\bar{\mu}(S_1 \uplus \cdots \uplus S_M) \stackrel{d.}{=} \sum_{j=1}^{M} \mu(S_j)$. Further, $\mathcal{S}_{\uplus}$ is a ring, i.e. it is stable under finite unions, finite intersections and set-differences, and $\bar{\mu}$ is clearly finitely additive on $\mathcal{S}_{\uplus}$.

Now, the proof of Proposition 4.3 gives that $\bar{\mu}$ is finitely sub-additive on $\mathcal{S}_{\uplus}$. As $\bar{\mu}$ restricts to μ on $\mathcal{S}$, it follows that μ is finitely sub-additive on $\mathcal{S}$.