# Solution - Extra Exercise 2 - MAT4300

$X$, $f$ and $\mu_f$ are as in Extra Ex. 1. We write $\mu$ for $\mu_f$.

a) We show that

$$\boxed{\mu \text{ is } \sigma\text{-finite}} \iff \boxed{\begin{array}{l} f(x) < \infty \text{ for all } x \in X \\ \text{and } E \stackrel{d}{=} \{x \in X \mid f(x) \neq 0\} \text{ is countable.} \end{array}}$$

($\Rightarrow$) Assume $\mu$ is $\sigma$-finite, i.e. there exists an exhausting sequence $\{A_j\}_{j\in\mathbb{N}} \subset \mathcal{P}(X)$, $A_j \uparrow X$, such that $\mu(A_j) < \infty$ for all $j \in \mathbb{N}$.

Assume (for contradiction) that there exists $x_0 \in X$ such that $f(x_0) = \infty$.

Since $X = \bigcup_{j\in\mathbb{N}} A_j$, there exists $k \in \mathbb{N}$ such that $x_0 \in A_k$.

Now $\mu(A_k) \geq f(x_0) = \infty$, so $\mu(A_k) = \infty$, which is impossible since $\mu(A_k) < \infty$.

Hence, we have $f(x) < \infty$ for all $x \in X$.

Further, set $E_j \stackrel{d}{=} E \cap A_j$, $j \in \mathbb{N}$.

Note that, since $E_j \subset A_j$, we have

$$\mu(E_j) \leq \mu(A_j) < \infty, \quad j \in \mathbb{N}.$$

Note also that $\bigcup_{j\in\mathbb{N}} E_j = E \cap \left(\bigcup_{j\in\mathbb{N}} A_j\right) = E \cap X = E$.

So $E$ will be countable if we show that each $E_j$ is countable.

Fix $j \in \mathbb{N}$ and set $E_{j,n} \stackrel{d.}{=} \{x \in E_j \mid f(x) \geq \frac{1}{n}\}$, $n \in \mathbb{N}$.

Clearly, we have

$$\bigcup_{n \in \mathbb{N}} E_{j,n} = \{x \in E_j \mid f(x) > 0\} = E_j \qquad \left[\begin{array}{l}\text{since } f \geq 0 \text{ and } E_j \subset E \\ \text{we have } f(x) > 0 \text{ for} \\ \text{all } x \in E_j\end{array}\right]$$

So $E_j$ will be countable if we can show that each $E_{j,n}$ is finite.

Assume (for contradiction) that $E_{j,n}$ is infinite for some $n \in \mathbb{N}$.

We can then pick a sequence $(x_k)_{k \in \mathbb{N}}$ in $E_{j,n}$ (without repetitions).

For each $M \in \mathbb{N}$, we then have

$$\mu(E_{j,n}) \geq \sum_{k=1}^{M} f(x_k) \underset{\substack{\uparrow \\ \text{since } f(x) \geq \frac{1}{n} \\ \text{when } x \in E_{j,n}}}{\geq} \sum_{k=1}^{M} \frac{1}{n} = \underbrace{\frac{M}{n}}_{\to \infty \text{ as } M \to \infty}.$$

So $\mu(E_{j,n}) = \infty$. Since $E_{j,n} \subset E_j$ this implies that $\mu(E_j) = \infty$, which is impossible since $\mu(E_j) < \infty$.

Hence, $E_{j,n}$ is finite for all $n \in \mathbb{N}$.

Therefore $E_j = \bigcup_{n \in \mathbb{N}} E_{j,n}$ is countable.

As pointed out before, this implies that E is countable.

($\Leftarrow$) Assume $f(x) < \infty$ and

$E = \{x \in X \mid f(x) \neq 0\}$ is countable.

Let $E = \{x_j\}_{j \in J}$ be an enumeration of $E$ (without repetitions)

where either $J = \{1, 2, \dots, n\}$ for some $n \in \mathbb{N}$, or $J = \mathbb{N}$.

For $k \in J$, set $E_k = \{x_1, x_2, \dots, x_k\}$

and $A_k = \underbrace{\{x \in X \mid f(x) = 0\}}_{\triangleq A} \cup E_k$

In the case $J = \{1, 2, \dots, n\}$ we also set

$E_k = E_n$ and $A_k = A \cup E_k$ for $k \in \mathbb{N}$, $k > n$.

Clearly, $\{A_k\}_{k \in \mathbb{N}}$ is an increasing sequence in $\mathcal{P}(X)$,

with $A_k \uparrow A \cup E = X$.

Further, we have

$$\mu(A) = \sum_{x \in A} \underbrace{f(x)}_{=0} = 0 \quad \text{and}$$

$$\mu(E_k) = \sum_{x \in E_k} f(x) < \infty \quad \left(\text{since } f(x) < \infty \ \forall x \text{ and } E_k \text{ is finite}\right), \text{ so}$$

$$\mu(A_k) = \mu(A \cup E_k) = \underbrace{\mu(A)}_{=0} + \mu(E_k) < \infty \quad \text{for each } k \in \mathbb{N}.$$

This shows that $\mu$ is $\sigma$-finite.

b) Set $X = [0,1]$

$$f(x) = \begin{cases} 1/n, & x = \frac{m}{n} \in \mathbb{Q} \cap [0,1] \quad \left(\text{where } m \in \mathbb{Z}^+, n \in \mathbb{N} \text{ and } \gcd(m,n) = 1\right) \\ 0, & x \in [0,1] \setminus \mathbb{Q} \end{cases}$$

$\mu = \mu_f$.

- Since $f(x) < \infty$ forall $x \in [0,1]$ by definition, and $\{x \in [0,1] \mid f(x) \neq 0\} = \mathbb{Q} \cap [0,1]$ is countable, a) gives that $\mu$ is $\sigma$-finite.

- Since $\mu(X) \geq \underbrace{\mu\left(\left\{\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{j}, \dots\right\}\right)}_{= \sum_{j=1}^{\infty} \frac{1}{j}} = \infty$

  we have $\mu(X) = \infty$, so $\mu$ is not finite.