Solution: Extra-Exercise 1:

a) Let $\{A_i\}_{i\in\mathbb{N}}$ be a sequence of pairwise disjoint subsets of X.

We have to show that

$$\mu_f\left(\dot{\bigcup_{i=1}^{\infty}} A_i\right) = \sum_{i=1}^{\infty} \mu_f(A_i) \quad , \text{ that is}$$

$$(*) \quad \boxed{\sum_{x\in A} f(x) = \sum_{i=1}^{\infty}\Big(\sum_{x\in A_i} f(x)\Big) \quad \text{where } A \doteq \dot{\bigcup_{i=1}^{\infty}} A_i .}$$

- If there exists $x\in A$ such that $f(x)=\infty$, then we clearly have equality since both sides are then equal to ∞.

- Assume now that $f(x)<\infty$ for all $x\in A$.

  Let B be a <u>finite</u> subset of A and set $B_i = B\cap A_i$, $i\in\mathbb{N}$.

  Set $I=\{i\in\mathbb{N} \mid B_i \neq \emptyset\}$.

  Since $B = \dot{\bigcup_{i\in I}} B_i$, I is finite.

  $$\text{So } \sum_{x\in B} f(x) = \sum_{i\in I}\Big(\sum_{x\in B_i} f(x)\Big) \leq \sum_{i\in I}\Big(\sum_{x\in A_i} f(x)\Big) \quad \left(\begin{array}{l}\text{since } B_i\subset A_i\\ \text{and } B_i \text{ finite}\end{array}\right)$$

  $$\leq \sum_{i=1}^{\infty}\Big(\sum_{x\in A_i} f(x)\Big)$$

  Taking supremum over all such B's, we get

  $$(**) \quad \sum_{x\in A} f(x) \leq \sum_{i=1}^{\infty}\Big(\sum_{x\in A_i} f(x)\Big)$$

  If $\displaystyle\sum_{x\in A} f(x) = \infty$, this gives that $\displaystyle\sum_{i=1}^{\infty}\Big(\sum_{x\in A_i} f(x)\Big)=\infty$

  and we have equality in $(*)$.

So Assume further that $M = \sum_{x\in A} f(x) < \infty$.

In view of $(**)$, it remains to show that

$$(***) \qquad \sum_{i=1}^{\infty}\Big(\sum_{x\in A_i} f(x)\Big) \le M .$$

Note that if C is a finite subset of A_i for some $i \in \mathbb{N}$,

then $\sum_{x\in C} f(x) \in \Big\{ \sum_{x\in B} f(x) \;\Big|\; B \text{ finite subset of } A \Big\}$ (since $C \subset A_i \subset A$),

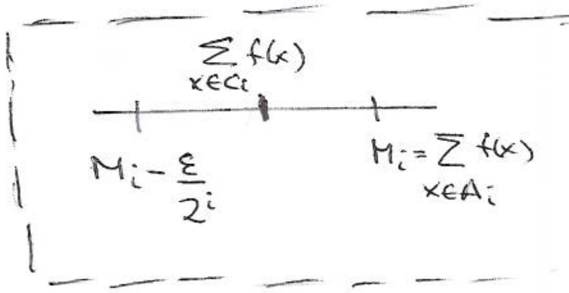
hence $\sum_{x\in C} f(x) \le \sum_{x\in A} f(x) = M < \infty$.

This gives $\sum_{x\in A_i} f(x) \le M < \infty$. Set $M_i = \sum_{x\in A_i} f(x)$.

Let $\varepsilon > 0$.

For each $i \in \mathbb{N}$, pick a finite subset C_i of A_i

Such that $\sum_{x\in C_i} f(x) > M_i - \frac{\varepsilon}{2^i}$.



Then we have

$$\sum_{i=1}^{n} M_i < \sum_{i=1}^{n}\Big(\sum_{x\in C_i} f(x) + \frac{\varepsilon}{2^i}\Big) = \Big(\sum_{x\in \dot{\bigcup}_{i=1}^{n} C_i} f(x)\Big) + \Big(\sum_{i=1}^{n} \varepsilon/2^i\Big)$$

$$\le \Big(\sum_{x\in A} f(x)\Big) + \varepsilon\cdot\sum_{i=1}^{\infty}\frac{1}{2^i} = \Big(\sum_{x\in A} f(x)\Big) + \varepsilon$$

for all $n \in \mathbb{N}$.

Hence $\sum_{i=1}^{\infty} M_i \le \Big(\sum_{x\in A} f(x)\Big) + \varepsilon = M + \varepsilon$.

Since this is true for all $\varepsilon > 0$, we get $\sum_{i=1}^{\infty} M_i < M$, i.e. $(***)$ holds. ∎

b) $X = \mathbb{R}$, $f(x) = \begin{cases} 1/x^2, & x \neq 0 \\ \infty, & x = 0 \end{cases}$, $x \in \mathbb{R}$.

- $A = \{x\}$ gives $\mu_f(A) = f(x)$.
- $A = \mathbb{N}$ gives $\mu_f(A) = \sum_{i \in \mathbb{N}} f(i) = \sum_{i=1}^{\infty} \frac{1}{i^2} = \frac{\pi^2}{6}$.
- $A = \mathbb{Z}$ gives $\mu_f(A) = \infty$ (since $0 \in A$ and $f(0) = \infty$).
- $A = (0,1]$ gives $\mu_f(A) = \infty$:

Indeed, for $n \in \mathbb{N}$, set $A_n = \left\{ \frac{1}{\sqrt{j}} \mid j \in \mathbb{N},\ j \leq n \right\}$

which is a finite subset of A.

Now $\sum_{x \in A_n} f(x) = \sum_{j=1}^{n} \frac{1}{(1/\sqrt{j})^2} = \sum_{j=1}^{n} j = \frac{n(n+1)}{2}$.

So $\sum_{x \in A} f(x) \geq \frac{n(n+1)}{2}$ for all $n \in \mathbb{N}$.

$\Rightarrow \mu_f(A) = \infty$.