MAT 3300/4300 - Fall 09 - Solutions - Extra Exercises 3, 4, 5.

Extra-Exercise 3 let $f \in \mathcal{M}$ [or $\overline{\mathcal{M}}$].

a) Let $B \in \mathcal{B}$. Then $f^{-1}(B) \in \mathcal{A}$ since $f \in \mathcal{M}$. Hence

$$f_E^{-1}(B) = \{x \in E \mid f_E(x) \in B\} = \{x \in E \mid f(x) \in B\}$$
$$= E \cap f^{-1}(B) \in \mathcal{A}_E .$$

This shows that $f_E \in \mathcal{M}_E$. [The case $f \in \overline{\mathcal{M}}$ is similar].

b) Assume $f \geq 0$. We want to check that

$$\int_E f \, d\mu = \int f_E \, d\mu_E \qquad (*).$$

- We first check that $(*)$ holds when $f = 1_A$, $A \in \mathcal{A}$:

We have $\int_E f \, d\mu = \int_E 1_A \, d\mu = \int 1_E \cdot 1_A \, d\mu = \int 1_{E \cap A} \, d\mu = \mu(E \cap A)$.

On the other hand, if $x \in E$, then $(1_A)_E(x) = \begin{cases} 1, & x \in A \cap E \\ 0, & x \in E \setminus A \end{cases}$.

So $\int f_E \, d\mu_E = \int (1_A)_E \, d\mu_E = \mu_E(A \cap E) = \mu(A \cap E)$, and we see that $(*)$ holds in this case.

- Next we check that $(*)$ holds when $f \in \mathcal{E}(\mathcal{A})^+$:

Let $f = \sum_{i=1}^{M} y_i 1_{A_i}$ be a standard repr. of f. Obviously, we have $f_E = \sum_{i=1}^{M} y_i (1_{A_i})_E$. Therefore, using the first step and linearity of the integral, we get

$$\int f_E \, d\mu_E = \int \Big(\sum_{i=1}^{M} y_i (1_{A_i})_E\Big) d\mu_E = \sum_{i=1}^{M} y_i \int (1_{A_i})_E \, d\mu_E$$
$$= \sum_{i=1}^{M} y_i \int_E 1_{A_i} \, d\mu = \int_E \Big(\sum_{i=1}^{M} y_i 1_{A_i}\Big) d\mu = \int_E f \, d\mu ,$$

as desired.

- Finally we prove $(*)$ assuming only $f \geq 0$:

By Theorem 8.8, we can pick $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{E}(\mathcal{A})^+$ such that $\{f_n\}$ is increasing and $\lim_{n\to\infty} f_n = f$.

Trivially, we have

- $\{1_E f_n\}$ is increasing and $\lim_{n\to\infty} 1_E f_n = 1_E f$,
- $\{(f_n)_E\}$ is increasing and $\lim_{n\to\infty} (f_n)_E = (f)_E$.

Using B. Levi's MCT (twice), we get

$$\int_E f \, d\mu = \int 1_E f \, d\mu = \lim_{n\to\infty} \int 1_E f_n \, d\mu$$

$$= \lim_{n\to\infty} \int_E f_n \, d\mu \underset{\text{Step 2.}}{=} \lim_{n\to\infty} \int (f_n)_E \, d\mu_E = \int f_E \, d\mu_E \, .$$

Let now $B \in \mathcal{A}_E$, so $B = A \cap E$, $A \in \mathcal{A}$. Then

$$\int_B f \, d\mu = \int 1_{A\cap E} f \, d\mu = \int 1_E \cdot 1_A f \, d\mu = \int_E 1_A f \, d\mu$$

$$\underset{(*)}{=} \int (1_A f)_E \, d\mu_E = \int (1_A)_E \, f_E \, d\mu_E = \int_{A\cap E} f_E \, d\mu_E = \int_B f_E \, d\mu_E \, .$$

(Here we have used that $(1_A)_E$ is the indicator function of $A \cap E$ within E, as seen previously).

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c) Assume $f \in \mathcal{L}^1(\mu)$. Then $f \in \mathcal{M}$, so $f_E \in \mathcal{M}_E$ by a).

Further, we know that $\int f^{\pm} \, d\mu < \infty$.

Now, we have $(f_E)^{\pm} = (f^{\pm})_E$:

Indeed, when $x \in E$, we have

$$(f_E)^+(x) = \begin{cases} f_E(x) & \text{if } f_E(x) \geq 0 \\ 0 & \text{otherwise} \end{cases} = \begin{cases} f(x) & \text{if } f(x) \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$= f^+(x) = (f^+)_E(x),$$

and similarly, $(f_E)^-(x) = (f^-)_E(x)$.

Therefore, using this and b), we get

$$\int (f_E)^{\pm} \, d\mu_E = \int (f^{\pm})_E \, d\mu_E = \int_E f^{\pm} \, d\mu \leq \int f^{\pm} \, d\mu < \infty.$$

Hence, $f_E \in \mathcal{L}^1(\mu_E)$, and

$$\int f_E \, d\mu_E = \int (f_E)^+ d\mu_E - \int (f_E)^- d\mu_E = \int (f^+)_E \, d\mu_E - \int (f^-)_E \, d\mu_E$$

$$= \int_E f^+ \, d\mu - \int_E f^- \, d\mu = \int_E f \, d\mu.$$

If $B \in \mathcal{A}_E$, then we can proceed exactly as in b) to deduce that $\int_B f \, d\mu = \int_B f_E \, d\mu_E$.

The case $f \in \overline{\mathcal{L}}^1(\mu)$ is similar.

d) Assume $g \in \mathcal{M}$, $g = f$ μ-a.e.

Let $N = \{x \in X \mid g(x) \neq f(x)\}$. Then $N \in \mathcal{A}$ and $\mu(N) = 0$.

Now $N_E \overset{d}{=} \{x \in E \mid g_E(x) \neq f_E(x)\} = N \cap E \in \mathcal{A}_E$

and $\mu_E(N_E) = \mu(N \cap E) \leq \mu(N) = 0$, so $\mu_E(N_E) = 0$.

Hence $g_E = f_E$ μ_E-a.e. [The case $g \in \overline{\mathcal{M}}$ is similar].

Let now $h \in \mathcal{M}_E$ [The case $h \in \overline{\mathcal{M}}_E$ is similar].

e) Let $\tilde{h}(x) = \begin{cases} h(x), & x \in E \\ 0, & x \in E^c \end{cases}, \; x \in X$.

Then $\underline{\tilde{h} \in \mathcal{M}}$: Let $t \in \mathbb{R}$. We have

$$\tilde{h}^{-1}([t,\infty)) = \{x \in X \mid \tilde{h}(x) \geq t\} = \underbrace{\{x \in E \mid h(x) \geq t\}}_{B} \cup \underbrace{\{x \in E^c \mid \tilde{h}(x) = 0 \geq t\}}_{C}$$

Now $B \in \mathcal{A}_E$ since $h \in \mathcal{M}_E$. Hence $B \in \mathcal{A}$ (since $\mathcal{A}_E \subset \mathcal{A}$).

Further $C = \begin{cases} E^c & \text{if } t \leq 0 \\ \emptyset & \text{if } t > 0 \end{cases}$, so $C \in \mathcal{A}$.

So $\tilde{h}^{-1}([t,\infty)) = B \cup C \in \mathcal{A}$.

f) Assume $h \geq 0$, so $\tilde{h} \geq 0$. Then

$$\underline{\int \tilde{h}\, d\mu} = \int_E \tilde{h}\, d\mu + \underbrace{\int_{E^c} \tilde{h}\, d\mu}_{=0 \text{ since } \tilde{h} = 0 \text{ on } E^c}$$

$$\overset{b)}{=} \int (\tilde{h})_E \, d\mu_E = \underline{\underline{\int h \, d\mu_E}}.$$

Let now $A \in \mathcal{A}$. Then

$$(1_A \tilde{h})(x) = \begin{cases} h(x) & \text{if } x \in A \cap E \\ 0 & \text{if } x \in E \setminus A \\ 0 & \text{if } x \in E^c \end{cases}$$

So $1_A \tilde{h} = \widetilde{(1_A)_E h}$. Therefore we have

$$\int_A \tilde{h}\, d\nu = \int 1_A \tilde{h}\, d\nu = \int \widetilde{(1_A)_E h}\, d\nu$$

$$\underset{\substack{\uparrow \\ \text{from the} \\ \text{first part.}}}{=} \int (1_A)_E h\, d\nu_E = \int_{A\cap E} h\, d\nu_E \, ,$$

g) Let $h \in \mathcal{L}^1(\nu_E)$. Then $h \in \mathcal{M}_E$, so $\tilde{h} \in \mathcal{M}$ by e).

Further, we know that $\int h^{\pm}\, d\nu_E < \infty$.

One checks easily that

$$(\tilde{h})^+(x) = \begin{cases} h(x) & \text{if } x \in E \text{ and } h(x) \geq 0 \\ 0 & \text{otherwise} \end{cases} = \widetilde{(h^+)}(x), \quad x \in X,$$

and similarly that $(\tilde{h})^- = \widetilde{(h^-)}$.

Hence,

$$\int (\tilde{h})^{\pm}\, d\nu = \int \widetilde{(h^{\pm})}\, d\nu = \int h^{\pm}\, d\nu_E < \infty, \quad \text{so}$$

$\tilde{h} \in \mathcal{L}^1(\nu)$ and

$$\int \tilde{h}\, d\nu = \int (\tilde{h})^+\, d\nu - \int (\tilde{h})^-\, d\nu = \int \widetilde{(h^+)}\, d\nu - \int \widetilde{(h^-)}\, d\nu$$

$$= \int h^+\, d\nu_E - \int h^-\, d\nu_E = \int h\, d\nu_E \, .$$

If $A \in \mathcal{A}$, then $\int_A \tilde{h}\, d\nu = \int_{A\cap E} h\, d\nu_E$ follows

precisely as in f).

The case $h \in \overline{\mathcal{L}}^1(\nu_E)$ is similar.

b) Assume $k \in \mathcal{M}_E$, $k = h$ μ_E-a.e.

Set $M = \{x \in E \mid k(x) \neq h(x)\}$, so $M \in \mathcal{A}_E$ and $\mu_E(M) = 0$.

Now $\tilde{M} \stackrel{d.}{=} \{x \in X \mid \tilde{k}(x) \neq \tilde{h}(x)\} = \{x \in E \mid \tilde{k}(x) \neq \tilde{h}(x)\} \cup \underbrace{\{x \in E^c \mid \tilde{k}(x) \neq \tilde{h}(x)\}}_{= \emptyset}$

$= \{x \in E \mid k(x) = h(x)\} = M$.

So $\tilde{M} = M \in \mathcal{A}_E \subset \mathcal{A}$ and $\mu(\tilde{M}) = \mu(M) = \mu_E(M) = 0$.

Hence $\tilde{k} = \tilde{h}$ μ-a.e.

The case $k \in \overline{\mathcal{M}}_E$ is similar.

Extra-Exercise 4: Let $\{f_j\}_{j \in \mathbb{N}} \subset \mathcal{M}$.

Set $E = \{x \in X \mid \lim_{j \to \infty} f_j(x) \text{ exists}\}$.

a) We may proceed as follows:

Let $x \in X$. Then

$x \in E \iff \{f_j(x)\}_{j \in \mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$

$\iff$ for all $n \in \mathbb{N}$, there exists $j_0 \in \mathbb{N}$ such that for all $j, k \geq j_0$, $j, k \in \mathbb{N}$, we have

$$|f_j(x) - f_k(x)| < \frac{1}{n}$$

This gives that we may write E as follows:

$$E = \bigcap_{n=1}^{\infty}\left(\bigcup_{j_0=1}^{\infty}\left(\bigcap_{j,k=j_0}^{\infty} E_{j,k}^{n}\right)\right) \text{ where}$$

$$E_{j,k}^{n} = \{x \in X \mid |f_j(x) - f_k(x)| < \tfrac{1}{n}\}.$$

Now, as all f_j's are measurable, it follows that each $E_{j,k}^{n}$ is in $\mathcal{A}$. As $\mathcal{A}$ is a σ-algebra, we get that $E \in \mathcal{A}$.

b) Set $f(x) = \begin{cases} \lim_{j\to\infty} f_j(x), & x \in E \\ 0, & x \in E^c \end{cases}, \quad x \in X.$

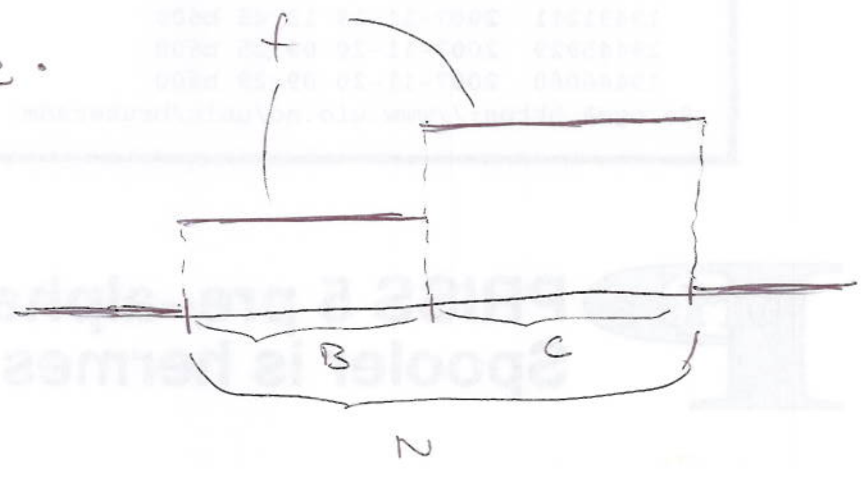
Then we have $f_E = \lim_{j\to\infty} (f_j)_E$. Since each $f_j \in \mathcal{M}$, we get from ~~Ex.~~ Exercise 3 that each $(f_j)_E \in \mathcal{M}_E$. Therefore we get that $f_E \in \mathcal{M}_E$ (since f_E is the limit of a sequence of measurable functions in $\mathcal{M}_E$).

Now, as we clearly have $f = \widetilde{(f_E)}$, it follows from Ex. Exercise 3 (with $h = f_E$) that $f \in \mathcal{M}$, as desired.

Extra-Exercise 5:

a) Assume $N \in \mathcal{A}$, $\mu(N) = 0$, $B \subset N$, $B \notin \mathcal{A}$, $C = N \setminus B$,

Set $f = 1_B + 2 \cdot 1_C$.



Then $\{x \in X \mid f(x) \neq 0\} = N$ and $\mu(N) = 0$ so $\underline{f = 0 \ \mu\text{-a.e.}}$

As $\{x \in X \mid f(x) = 1\} = B \notin \mathcal{A}$, we have $\underline{f \notin \mathcal{M}}$.

b) Assume $(X, \mathcal{A}, \mu)$ is complete, $f : X \to \mathbb{R}$, $g \in \mathcal{M}$, $f = g$ μ-a.e.

So there exists $N \in \mathcal{A} \mid \{x \in X \mid f(x) \neq g(x)\} \subset N$ and $\mu(N) = 0$.

Note that $f(x) = g(x)$ when $x \in N^c$.

Let $t \in \mathbb{R}$. Then

$$\{x \in X \mid f(x) \geq t\} = \underbrace{\Big(\underbrace{\{x \in X \mid g(x) \geq t\}}_{\substack{\in \mathcal{A} \\ \text{since } g \in \mathcal{M}}} \cap \underbrace{N^c}_{\substack{\in \mathcal{A} \\ \text{since} \\ N \in \mathcal{A}}}\Big)}_{\in \mathcal{A}} \cup \underbrace{\Big(\{x \in X \mid f(x) \geq t\} \cap N\Big)}_{B}$$

But B is a subset of N and $N \in \mathcal{N}_\mu$, so $\underline{B \in \mathcal{A}}$ since $(X, \mathcal{A}, \mu)$ is complete.

Hence $f^{-1}([t, \infty)) \in \mathcal{A}$ for all $t \in \mathbb{R}$, and it follows that $f \in \mathcal{M}$, as desired.