2 The pleasures of counting. Additional Material

Problems

2.22. For $A_j^0, A_j^1 \subset X$, $j \in \mathbb{N}$, we have

$$\bigcup_{j\in\mathbb{N}} (A_j^0 \cap A_j^1) = \bigcap_{i=(i(k))_{k\in\mathbb{N}}\in\{0,1\}^{\mathbb{N}}} \bigcup_{k\in\mathbb{N}} A_k^{i(k)}.$$

Solution to 2.22: Since for $A, A', B, B' \subset X$ we have the 'multiplication rule'

$$(A \cap B) \cup (A' \cap B') = (A \cup A') \cap (A \cup B') \cap (B \cup A') \cap (B \cup B')$$

and since this rule carries over to the infinite case, we get the formula from the problem by 'multiplying out' the countable union

$$(A_1^0 \cap A_1^1) \cup (A_2^0 \cap A_2^1) \cup (A_3^0 \cap A_3^1) \cup (A_4^0 \cap A_4^1) \cup \cdots .$$

More formally, one argues as follows:

$$x \in \bigcup_{j\in\mathbb{N}} (A_j^0 \cap A_j^1) \iff \exists\, j_0 \,:\, x \in A_{j_0}^0 \cap A_{j_0}^1 \tag{*}$$

while

$$\begin{aligned} x \in & \bigcap_{i=(i(k))_{k\in\mathbb{N}}\in\{0,1\}^{\mathbb{N}}} \bigcup_{k\in\mathbb{N}} A_k^{i(k)} \\ & \iff \forall i = (i(k))_{k\in\mathbb{N}} \in \{0,1\}^{\mathbb{N}} \,:\, x \in \bigcup_{k\in\mathbb{N}} A_k^{i(k)} \\ & \iff \forall i = (i(k))_{k\in\mathbb{N}} \in \{0,1\}^{\mathbb{N}}\ \exists\, k_0 \in \mathbb{N} \,:\, x \in A_{k_0}^{i(k_0)} \qquad (**) \end{aligned}$$

Clearly, (*) implies (**). On the other hand, assume that (**) holds but that (*) is wrong, i.e. suppose that for every j we have that either $x \in A_j^0$ or $x \in A_j^1$ or x is in neither of A_j^0, A_j^1. Thus we can construct a uniquely defined sequence $i(j) \in \{0,1\}$, $j \in \mathbb{N}$, by setting

$$i(j) = \begin{cases} 0 & \text{if } x \in A_j^0; \\ 1 & \text{if } x \in A_j^1; \\ 0 & \text{if } x \notin A_j^0 \text{ and } x \notin A_j^1. \end{cases}$$

Define by $i'(j) := 1 - i(j)$ the 'complementary' 0-1-sequence. Then

$$x \in \bigcup_j A_j^{i(j)} \quad \text{but} \quad x \notin \bigcup_j A_j^{i'(j)}$$

contradicting our assumption (**).

4 Measures. Additional Material

Problems

4.16. Let $(X, \mathcal{A}, \mu)$ be a measure space, $\mathcal{F} \subset \mathcal{A}$ be a sub-σ-algebra and denote by $\mathcal{N} = \{N \in \mathcal{A} : \mu(N) = 0\}$ the collection of all μ-null sets. Then

$$\sigma(\mathcal{F}, \mathcal{N}) = \big\{F \triangle N : F \in \mathcal{F},\ N \in \mathcal{N}\big\}$$

where $F \triangle N := (F \setminus N) \cup (N \setminus F)$ denotes the symmetric difference, cf. also Problems 2.2, 2.6.

4.17. Let $\mathcal{A}^*$ denote the completion of $\mathcal{A}$ as in Problem 4.13 and write $\mathcal{N} := \{N \subset X : \exists\, M \in \mathcal{A},\ N \subset M,\ \mu(M) = 0\}$ for the family of all subsets of $\mathcal{A}$-measurable null sets. Show that

$$\mathcal{A}^* = \sigma(\mathcal{A}, \mathcal{N}) = \big\{A \triangle N : A \in \mathcal{A},\ N \in \mathcal{N}\big\}.$$

Conclude that for every set $A^* \in \mathcal{A}^*$ there is some $A \in \mathcal{A}$ such that $A \triangle A^* \in \mathcal{N}$.

4.18. Consider on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$ the Dirac measure δ_x for some fixed $x \in \mathbb{R}^n$. Find the completion of $\mathcal{B}(\mathbb{R}^n)$ with respect to δ_x.

4.19. **Regularity.** Let X be a metric space and μ be a finite measure on the Borel sets $\mathcal{B} = \mathcal{B}(X)$ and denote the open sets by $\mathcal{O}$ and the closed sets by $\mathcal{F}$. Define a family of sets

$$\Sigma := \{A \subset X : \forall \epsilon > 0\ \exists\, U \in \mathcal{O},\ F \in \mathcal{F} \text{ s.t. } F \subset A \subset U,\ \mu(U \setminus F) < \epsilon\}.$$

(i) Show that $A \in \Sigma \implies A^c \in \Sigma$ and that $\mathcal{F} \subset \Sigma$.

(ii) Show that Σ is stable under finite intersections.

(iii) Show that Σ is a σ-algebra containing the Borel sets $\mathcal{B}$.

(iv) Conclude that μ is *regular*, i.e. for all Borel sets $B \in \mathcal{B}$

$$\mu(B) = \sup_{F \subset B,\, F \in \mathcal{F}} \mu(F) = \inf_{U \supset B,\, U \in \mathcal{O}} \mu(U).$$

(v) Assume that there exists an increasing sequence of compact (cpt, for short) sets K_j such that $K_j \uparrow X$. Show that μ satisfies

$$\mu(B) = \sup_{F \subset B,\, K \text{ cpt}} \mu(K).$$

(vi) Extend $\mu(B) = \sup_{F \subset B,\, F \in \mathcal{F}} \mu(F)$ to a σ-finite measure μ.

4.20. **Regularity on Polish spaces.** A Polish space X is a complete metric space which has a countable dense subset $D \subset X$.

Let μ be a finite measure on $(X, \mathcal{B}(X))$. Then μ is regular in the sense that

$$\mu(B) = \sup_{K \subset B,\, K \text{ compact}} \mu(K) = \inf_{U \supset B,\, U \text{ open}} \mu(U).$$

[Hint: Use Problem 4.19 and show that the set $K := \bigcap_n \bigcup_{j=1}^{k(n)} K_{1/n}(d_{k(n)})$, $K_\epsilon(x)$ is a closed ball of radius ϵ and centre x and $\{d_k\}_k$ is an enumeration of D, is compact. Choosing $k(n)$ sufficiently large, we can achieve that $\mu(X \setminus K) < \epsilon$.]

4.21. Find a measure μ on $(\mathbb{R}, \mathcal{B}^1)$ which is σ-finite but assigns to every interval $[a, b)$ with $b - a > 2$, finite mass.

Solution to 4.16: Set

$$\Sigma := \big\{ F \triangle N \ :\ F \in \mathcal{F},\ N \in \mathcal{N} \big\}.$$

and denote, without further mentioning, by F, F_j resp. N, N_j sets from $\mathcal{F}$ resp. $\mathcal{N}$. Since $F \triangle \emptyset = F$, $\emptyset \triangle N = N$ and $F \triangle N \in \sigma(\mathcal{F}, \mathcal{N})$ we get

$$\mathcal{F},\ \mathcal{N} \subset \Sigma \subset \sigma(\mathcal{F}, \mathcal{N}) \tag{*}$$

and the first assertion follows if we can show that Σ is a σ-algebra. In this case, we can apply the σ-operation to the inclusions (*) and get

$$\sigma(\mathcal{F}, \mathcal{N}) \subset \sigma(\Sigma) \subset \sigma(\sigma(\mathcal{F}, \mathcal{N}))$$

which is just

$$\sigma(\mathcal{F}, \mathcal{N}) \subset \Sigma \subset \sigma(\mathcal{F}, \mathcal{N}).$$

To see that Σ is a σ-algebra, we check conditions (Σ_1)–(Σ_3).

(Σ_1): Clearly, $X \in \mathcal{F}$ and $N \in \mathcal{N}$ so that $X = X \triangle \emptyset \in \Sigma$;

(Σ_2): We have, using de Morgan's identities over and over again:

$$\begin{aligned}
[F \triangle N]^c &= [(F \setminus N) \cup (N \setminus F)]^c \\
&= (F \cap N^c)^c \cap (N \cap F^c)^c \\
&= (F^c \cup N) \cap (N^c \cup F) \\
&= (F^c \cap N^c) \cup (G^c \cap G) \cup (N \cap N^c) \cup (N \cap F) \\
&= (F^c \cap N^c) \cup (N \cap F) \\
&= (F^c \setminus N) \cup (N \setminus F^c) \\
&= \underbrace{F^c}_{\in \mathcal{F}} \triangle N \\
&\in \Sigma;
\end{aligned}$$

(Σ_3): We begin by a few simple observations, namely that for all $F \in \mathcal{F}$ and $N, N' \in \mathcal{N}$

$$F \cup N = F \triangle \underbrace{(N \setminus F)}_{\in \mathcal{N}} \in \Sigma; \tag{a}$$

$$F \setminus N = F \triangle \underbrace{(N \cap F)}_{\in \mathcal{N}} \in \Sigma; \tag{b}$$

$$N \setminus F = N \triangle \underbrace{(F \cap N)}_{\in \mathcal{N}} \in \Sigma; \tag{c}$$

$$\begin{aligned}(F \triangle N) \cup N' &= \big(F \triangle N\big) \triangle \big(N' \setminus (F \triangle N)\big) \\ &= F \triangle \underbrace{\Big(N \triangle (N' \setminus (F \triangle N))\Big)}_{\in \mathcal{N}} \in \Sigma,\end{aligned} \tag{d}$$

where we used Problem 2.6 and part (a) for (d).

Now let $(F_j)_{j\in\mathbb{N}} \subset \mathcal{F}$ and $(N_j)_{j\in\mathbb{N}} \subset \mathcal{N}$ and set $F := \bigcup_j F_j \in \mathcal{F}$ and, because of σ-subadditivity of measures $N := \bigcup_j N_j \in \mathcal{N}$. Then

$$F \setminus N = \bigcup_{j\in\mathbb{N}} (F_j \setminus N) \subset \bigcup_{j\in\mathbb{N}} (F_j \setminus N_j) \subset \bigcup_{j\in\mathbb{N}} F_j = F$$

as well as

$$\emptyset \subset \bigcup_{j\in\mathbb{N}} (N_j \setminus F_j) \subset \bigcup_{j\in\mathbb{N}} N_j = N$$

which shows that

$$F \setminus N \subset \bigcup_{j\in\mathbb{N}} (F_j \triangle N_j) \subset F \cup N. \tag{**}$$

Since $\mathcal{F}, \mathcal{N} \subset \mathcal{A}$, and consequently $\bigcup_{j\in\mathbb{N}} (F_j \triangle N_j) \in \mathcal{A}$, and since $\mathcal{A}$-measurable subsets of null sets are again in $\mathcal{N}$, the inclusions (**) show that there exists some $N' \in \mathcal{N}$ so that

$$\bigcup_{j\in\mathbb{N}} (F_j \triangle N_j) = \underbrace{(F \setminus N)}_{\in \Sigma,\text{ cf. (b)}} \cup N' \in \Sigma$$

where we used (d) for the last inclusion.

Solution to 4.17: By definition,

$$\mathcal{A}^* = \big\{A \cup N \;:\; A \in \mathcal{A},\; N \in \mathcal{N}\big\}.$$

Since

$$A \cup N = A \triangle \underbrace{(N \setminus A)}_{\in \mathcal{N}}$$

and since by an application of Problem 4.16 to $(X, \mathcal{A}^*, \bar{\mu}), \mathcal{A}, \mathcal{N}$ (instead of $(X, \mathcal{A}, \mu), \mathcal{G}, \mathcal{N}$) we get

$$\sigma(\mathcal{A}, \mathcal{N}) = \big\{A \triangle N \;:\; A \in \mathcal{A},\; N \in \mathcal{N}\big\}$$

and we conclude that

$$\mathcal{A}^* \subset \sigma(\mathcal{A}, \mathcal{N}).$$

On the other hand,

$$\mathcal{A} \subset \mathcal{A}^* \quad \text{and} \quad \mathcal{N} \subset \mathcal{A}^*$$

so that, since $\mathcal{A}^*$ is a σ-algebra,

$$\sigma(\mathcal{A}, \mathcal{N}) \subset \sigma(\mathcal{A}^*) = \mathcal{A}^* \subset \sigma(\mathcal{A}, \mathcal{N}).$$

Finally, assume that $A^* \in \mathcal{A}^*$ and $A \in \mathcal{A}$. Then $A = A^* \triangle N$ and we get

$$A^* \triangle A = A \triangle N \triangle A = (A \triangle A) \triangle N = N.$$

Note that this result would also follow directly from 4.13 since we know from there that $A^* = A \cup N$ so that

$$A^* \triangle A = (A \cup N) \triangle A = A \triangle (N \setminus A) \triangle A = N \setminus A$$

Solution to 4.18: Denote the completion by $\mathcal{B}^*$ and write $\mathcal{N}_x$ for all subsets of Borel null sets of δ_x. Clearly,

$$\mathcal{N}_x = \{A \subset \mathbb{R}^n \;:\; x \notin A\}.$$

Recall from Problem 4.13(i) that $\mathcal{B}^*$ contains all sets of the form $B \cup N$ with $B \in \mathcal{B}$ and $N \in \mathcal{N}_x$. Now let $C \subset \mathbb{R}^n$ be any set. If $x \in C$, then write

$$C = \underbrace{\{x\}}_{\in \mathcal{B}} \cup \underbrace{(C \setminus \{x\})}_{\in \mathcal{N}_x} \in \mathcal{B}^*;$$

Otherwise, $x \notin C$ and

$$C = C \setminus \{x\} = \underbrace{\emptyset}_{\in \mathcal{B}} \cup \underbrace{(C \setminus \{x\})}_{\in \mathcal{N}_x} \in \mathcal{B}^*.$$

This shows that $\mathcal{B}^* = \mathcal{P}(\mathbb{R}^n)$ is the power set of $\mathbb{R}^n$.

Solution to 4.19: (i) Fix $\epsilon > 0$ and choose for $A \in \Sigma$ sets $U \in \mathcal{O}$, $F \in \mathcal{F}$ such that $F \subset A \subset U$ and $\mu(U \setminus F) < \epsilon$. Set $U' := F^c \in \mathcal{O}$ and $F' := U^c \in \mathcal{F}$. Then we have

$$F' \subset A^c \subset U' \text{ and } U' \setminus F' = F^c \setminus U^c = F^c \cap U = U \setminus F$$

and so $\mu(U' \setminus F') = \mu(U \setminus F) < \epsilon$. This means that $A^c \in \Sigma$.

Denote by $d(x, y)$ the distance of two points $x, y \in X$ and write $B_{1/n}(0)$ for the open ball $\{y \in X : d(y, 0) < \frac{1}{n}\}$. As in the solution of Problem 3.12(ii) we see that $U_n := F + B_{1/n}(0)$ is a sequence of open sets such that $U_n \downarrow F$. Because of the continuity of measures we get $\mu(U_n \setminus F) \xrightarrow{n\to\infty} 0$ and since $\mathcal{F} \ni F \subset F \subset U_n \in \mathcal{O}$, this means that $\mathcal{F} \subset \Sigma$.

(ii) Fix $\epsilon > 0$ and pick for $A_j \in \Sigma, j = 1, 2$, open sets U_j and closed sets F_j such that $F_j \subset A \subset U_j$ and $\mu(U_j \setminus F_j) < \epsilon$. Then $F_1 \cap F_2$ and $U_1 \cap U_2$ are again closed resp. open, satisfy $F_1 \cap F_2 \subset A_1 \cap A_2 \subset U_1 \cap U_2$ as well as

$$\begin{aligned}\mu\big((U_1 \cap U_2) \setminus (F_1 \cap F_2)\big) &= \mu\big((U_1 \cap U_2) \cap (F_1^c \cup F_2^c)\big)\\ &= \mu\big([(U_1 \cap U_2) \setminus F_1] \cup [(U_1 \cap U_2) \setminus F_2]\big)\\ &\leqslant \mu\big((U_1 \cap U_2) \setminus F_1\big) + \mu\big((U_1 \cap U_2) \setminus F_2\big)\\ &< 2\epsilon.\end{aligned}$$

This shows that Σ is $\cap$-stable.

(iii) Fix ϵ and pick for a given sequence $(A_j)_{j\in\mathbb{N}} \subset \Sigma$ open sets U_j and closed sets F_j such that

$$F_j \subset A_j \subset U_j \text{ and } \mu(U_j \setminus F_j) < \epsilon 2^{-j}.$$

Set $A := \bigcup_j A_j$. Then $U := \bigcup_j U_j \supset A$ is an open set wile $F := \bigcup_j F_j$ is contained in A but it is only an increasing limit of closed sets $\Phi_n := F_1 \cup \ldots \cup F_n$. Using Problem 4.9 we get

$$\mu(U \setminus F) \leqslant \sum_j \mu(U_j \setminus F_j) \leqslant \sum_j \epsilon 2^{-j} \leqslant \epsilon.$$

Since $\Phi_n \subset A \subset U$ and $U \setminus \Phi_n \downarrow U \setminus F$, we can use the continuity of measures to conclude that $\inf_n \mu(U \setminus \Phi_n) = \mu(U \setminus F) \leqslant \epsilon$, i.e. $\mu(U \setminus \Phi_N) \leqslant 2\epsilon$ if $N = N_\epsilon$ is sufficiently large. This shows that Σ contains all countable unions of its members. Because of part (i) it is also stable under complementation and contains the empty set. Thus, Σ is a σ-algebra.

As $\mathcal{F} \subset \Sigma$ and $\mathcal{B} = \sigma(\mathcal{F})$, we have $\mathcal{B} \subset \Sigma$.

(iv) For any Borel set $B \in \Sigma$ and any $\epsilon > 0$ we can find open and closed sets U_ϵ and F_ϵ, respectively, such that $F_\epsilon \subset B \subset U_\epsilon$ and

$$\mu(B \setminus F_\epsilon) \leqslant \mu(U_\epsilon \setminus F_\epsilon) < \epsilon \implies \mu(B) \leqslant \epsilon + \mu(F_\epsilon),$$
$$\mu(U_\epsilon \setminus B) \leqslant \mu(U_\epsilon \setminus F_\epsilon) < \epsilon \implies \mu(B) \geqslant \mu(U_\epsilon) - \epsilon.$$

Thus,

$$\sup_{F\subset B,\, F\in\mathcal{F}} \mu(F) \leqslant \mu(B) \leqslant \epsilon + \mu(F_\epsilon) \leqslant \epsilon + \sup_{F\subset B,\, F\in\mathcal{F}} \mu(F)$$
$$\inf_{U\supset B,\, U\in\mathcal{O}} \mu(U) - \epsilon \leqslant \mu(U_\epsilon) - \epsilon \leqslant \mu(b) \leqslant \inf_{U\supset B,\, U\in\mathcal{O}} \mu(U).$$

(v) For every closed $F \in \mathcal{F}$ the intersections $K_j \cap F$, $j \in \mathbb{N}$, will be compact and $K_j \cap F \uparrow F$. By the continuity of measures we get

$$\mu(F) = \sup_j \mu(K_j \cap F) \leqslant \sup_{K\subset F, K \text{ cpt}} \mu(K) \leqslant \mu(F).$$

Thus,

$$\mu(F) = \sup_{K\subset F, K \text{ cpt}} \mu(K) \qquad \forall\, F \in \mathcal{F}. \tag{*}$$

Combining (iv) and (*) we get

$$\begin{aligned}
\mu(B) &\overset{\text{(iv)}}{=} \sup_{F\subset B,\, F\in\mathcal{F}} \mu(F) \\
&\overset{(*)}{=} \sup_{F\subset B,\, F\in\mathcal{F}} \sup_{K\subset F, K \text{ cpt}} \mu(K) \\
&\leqslant \sup_{F\subset B,\, F\in\mathcal{F}} \underbrace{\sup_{K\subset B, K \text{ cpt}} \mu(K)}_{\text{note: independent of } F\subset B} \\
&= \sup_{K\subset B, K \text{ cpt}} \mu(K)
\end{aligned}$$

and since $\mu(K) \leqslant \mu(B)$ for $K \subset B$ and $\sup_{K\subset B, K \text{ cpt}} \mu(K) \leqslant \mu(B)$ are obvious, we are finished.

(vi) Assume now that μ is σ-finite. Let $(B_n)_{n\in\mathbb{N}} \subset \mathcal{B}$ be an exhausting sequence for X such that $\mu(B_n) < \infty$. Then the measures $\mu_n(B) := \mu(B \cap B_n)$ defined on $\mathcal{B}$ are finite and regular according to part (iv). Since we may interchange any two suprema (cf. the solution of Problem 4.6) we get

$$\mu(B) = \sup_n \mu_n(B) = \sup_n \sup_{F\subset B,\, F\in\mathcal{F}} \mu_n(F)$$

$$= \sup_{F \subset B,\, F \in \mathcal{F}} \sup_n \mu_n(F)$$
$$= \sup_{F \subset B,\, F \in \mathcal{F}} \mu(F).$$

Solution to 4.20: First of all, Problem 4.19 (iv) shows that

$$\mu(B) = \sup_{F \subset B, F \text{ closed}} \mu(F). \tag{*}$$

Let $(d_k)_k$ be an enumeration of the dense set $D \subset X$ and write ρ for the metric in X and $K_r(x) := \{y \in X \,:\, \rho(x,y) \leqslant r\}$ for the closed ball with centre x and radius r.

Since, for any fixed $n \in \mathbb{N}$ the sets

$$K_{1/n}(d_1) \cup \cdots \cup K_{1/n}(d_m) \uparrow X \quad \text{for} \quad m \to \infty$$

we get from (*)

$$\forall \epsilon > 0 \quad \exists\, k(n) \in \mathbb{N} \,:\, \mu(F_n) + \frac{\epsilon}{2^n} \geqslant \mu(X)$$

if $F_n := K_{1/n}(d_1) \cup \cdots \cup K_{1/n}(d_{k(n)})$. Setting

$$K := K_\epsilon := \bigcap_n F_n$$

it is clear that K is closed. Moreover, since K is, for every $1/n$, covered by finitely many balls of radius $1/n$, to wit,

$$K \subset K_{1/n}(d_1) \cup \cdots \cup K_{1/n}(d_{k(n)}),$$

we see that K is compact. Indeed, if $(x_j)_j \subset K$ is a sequence, there is a subsequence $(x_j^n)_j$ which is completely contained in one of the balls $K_{1/n}(d_1), \dots, K_{1/n}(d_{k(n)})$. Passing iteratively to sub-sub-etc. sequences we find a subsequence $(y_j)_j \subset (x_j)_j$ which is contained in a sequence of closed balls $K_{1/n}(c_n)$ (c_n is a suitable element from D). Thus $(y_j)_j$ is a Cauchy sequence and converges, because of completeness, to an element x^* which is, as the F_n are closed, in every F_n, hence in K. Thus K is (sequentially) compact.

Since

$$\mu(X \setminus K) = \mu\Big(\bigcup_n X \setminus F_n\Big) \leqslant \sum_n \mu(X \setminus F_n) \leqslant \sum_n \frac{\epsilon}{2^n} = \epsilon,$$

we have found a sequence of compact sets K_n such that $\mu(K_n) \to \mu(X)$ (note that the K_n need not 'converge' X as a set!). Obviously, $K_n \cap F$ is compact for every closed F and we have $\mu(K_n \cap F) \to \mu(F)$, hence

$$\mu(F) = \sup_{K \subset F, K \text{ cpt}} \mu(K) \quad \forall\, F \in \mathcal{F}.$$

Now we can use the argument from the proof of Problem 4.20(v).

Solution to 4.21: Define a measure μ which assigns every point $n - \frac{1}{2k}$, $n \in \mathbb{Z}$, $k \in \mathbb{N}$ the mass $\frac{1}{2k}$:

$$\mu = \sum_{n \in \mathbb{Z}} \sum_{k \in \mathbb{N}} \frac{1}{2k}\, \delta_{n - \frac{1}{2k}}.$$

(Since $\mathbb{Z} \times \mathbb{N}$ is countable, Problem 4.6 shows that this object is indeed a measure!) Obviously, any interval $[a, b)$ of length $b - a > 2$ contains some integer, say $m \in [a, b)$ so that $[m - 1/2, m) \subset [a, b)$, thus

$$\mu[a, b) \geqslant \mu[m - 1/2, m) = \sum_{k \in \mathbb{N}} \frac{1}{2k} = \infty.$$

On the other hand, the sequence of sets

$$B_n := \bigcup_{k=-n}^{n} \left[k - 1, k - \tfrac{1}{2n}\right)$$

satisfies $\mu(B_n) < \infty$ and $\bigcup_n B_n = \mathbb{R}$.

7 Measurable mappings. Additional Material

Problems

7.12. Let $\mathcal{E}, \mathcal{F} \subset \mathcal{P}(X)$ be two families of subsets of X. One *usually* uses the notation (as we do in this book)

$$\mathcal{E} \cup \mathcal{F} = \{A : A \in \mathcal{E} \text{ or } A \in \mathcal{F}\} \quad \text{and} \quad \mathcal{E} \cap \mathcal{F} = \{A : A \in \mathcal{E} \text{ and } A \in \mathcal{F}\}.$$

Let us, for this problem, also introduce the families

$$\mathcal{E} \Cup \mathcal{F} = \{E \cup F : E \in \mathcal{E},\ F \in \mathcal{F}\} \quad \text{and} \quad \mathcal{E} \Cap \mathcal{F} = \{E \cap F : E \in \mathcal{E},\ F \in \mathcal{F}\}.$$

Assume now that $\mathcal{E}$ and $\mathcal{F}$ are σ-Algebras.

(i) Show that $\mathcal{E} \Cup \mathcal{F} \supset \mathcal{E} \cup \mathcal{F}$ and $\mathcal{E} \Cap \mathcal{F} \supset \mathcal{E} \cup \mathcal{F}$;

(ii) Show that, in general, we have no equality in (i);

(iii) Show that $\sigma(\mathcal{E} \Cup \mathcal{F}) = \sigma(\mathcal{E} \Cap \mathcal{F}) = \sigma(\mathcal{E} \cup \mathcal{F})$.

Solution to 7.12: (i) Since $\emptyset \in \mathcal{E}$ and $\emptyset \in \mathcal{F}$ we get

$$\forall E \in \mathcal{E} : E \cup \emptyset \in \mathcal{E} \Cup \mathcal{F} \implies \mathcal{E} \subset \mathcal{E} \Cup \mathcal{F}$$

and

$$\forall F \in \mathcal{F} : \emptyset \cup F \in \mathcal{E} \Cup \mathcal{F} \implies \mathcal{F} \subset \mathcal{E} \Cup \mathcal{F}$$

so that $\mathcal{E} \cup \mathcal{F} \subset \mathcal{E} \Cup \mathcal{F}$. A similar argument, using that $X \in \mathcal{E}$ and $X \in \mathcal{F}$, shows $\mathcal{E} \cup \mathcal{F} \subset \mathcal{E} \Cap \mathcal{F}$.

(ii) Let $A, B \subset X$ such that $A \cap B \neq \emptyset$, $A \cup B \neq X$ and that $A \not\subset B$, $B \not\subset A$. Then we find for $\mathcal{E} := \{\emptyset, A, A^c, X\}$ and $\mathcal{F} := \{\emptyset, B, B^c, X\}$ that

$$\mathcal{E} \cup \mathcal{F} = \{\emptyset, A, B, A^c, B^c, X\}$$

while

$$\mathcal{E} \Cup \mathcal{F} = \{\emptyset, A, B, A^c, B^c, A \cup B, A^c \cup B^c, A \cup B^c, A^c \cup B, X\}.$$

A similar example works for $\mathcal{E} \Cap \mathcal{F}$.

(iii) Part (i) shows immediately that

$$\sigma(\mathcal{E} \Cup \mathcal{F}) \supset \sigma(\mathcal{E} \cup \mathcal{F}) \quad \text{and} \quad \sigma(\mathcal{E} \Cap \mathcal{F}) \supset \sigma(\mathcal{E} \cup \mathcal{F}).$$

Conversely, it is obvious that

$$\mathcal{E} \Cup \mathcal{F} \subset \sigma(\mathcal{E} \cup \mathcal{F}) \quad \text{and} \quad \mathcal{E} \Cap \mathcal{F} \subset \sigma(\mathcal{E} \cup \mathcal{F})$$

so that

$$\sigma(\mathcal{E} \Cup \mathcal{F}) \subset \sigma(\mathcal{E} \cup \mathcal{F}) \quad \text{and} \quad \sigma(\mathcal{E} \Cap \mathcal{F}) \subset \sigma(\mathcal{E} \cup \mathcal{F})$$

which proves

$$\sigma(\mathcal{E} \Cup \mathcal{F}) = \sigma(\mathcal{E} \cup \mathcal{F}) = \sigma(\mathcal{E} \Cap \mathcal{F}).$$

9 Integration of positive functions. Additional Material

Problems

9.13. Let $(X, \mathcal{A}, \mu)$ be a measure space and $A_1, \dots, A_N \in \mathcal{A}$ such that $\mu(A_j) < \infty$. Then

$$\mu\Big(\bigcup_{j=1}^{N} A_j\Big) \geqslant \sum_{j=1}^{N} \mu(A_j) \;-\; \sum_{1\leqslant j<k\leqslant N} \mu(A_j \cap A_k).$$

[Hint: show first an inequality for indicator functions.]

9.14. Let $(X, \mathcal{A}, \mu)$ be a measure space. Show the following variant of Theorem 9.6: If $u_j \geqslant 0$ are measurable functions such that for some u we have

$$\exists\, K \in \mathbb{N} \,:\, \forall\, x \,:\, u_{j+K}(x) \uparrow u(x) \;\text{ as } j \to \infty,$$

then $u \geqslant 0$ is measurable and $\int u_j \,d\mu \uparrow \int u \,d\mu$.

Show that we cannot replace the above condition with

$$\forall\, x \;\; \exists\, K \in \mathbb{N} \;\; u_j(x) \uparrow u(x) \;\text{ as } j \to \infty.$$

Solution to 9.13: We use indicator functions. Note that any fixed x can be contained in $k \in \{0, 1, \dots, N\}$ of the sets A_j. Then x is contained in $A_1 \cup \dots \cup A_N$ as well as in $\binom{k}{2}$ of the pairs $A_j \cup A_k$ where $j < k$; as usual: $\binom{m}{n} = 0$ if $m < n$. This gives

$$\begin{aligned}\sum_j \mathbf{1}_{A_j} = k \leqslant 1 + \binom{k}{2} &= \mathbf{1}_{A_1\cup\cdots A_N} + \sum_{j<k} \mathbf{1}_{A_j}\mathbf{1}_{A_k} \\ &= \mathbf{1}_{A_1\cup\cdots A_N} + \sum_{j<k} \mathbf{1}_{A_j\cap A_k}.\end{aligned}$$

Integrating this inequality w.r.t. μ yields the result.

Solution to 9.14: The first part is trivial since it just says that the sequence becomes only from index K onwards. This K does not depend on x but is uniform for the whole sequence. Since we are anyway only interested in $u = \lim_{j\to\infty} u_j = \sup_{j\geqslant K} u_j$, we can neglect the elements $u_1, \dots, u_K$ and consider only the then increasing sequence $(u_{j+K})_j$. Then we can directly apply Beppo Levi's theorem, Theorem 9.6.

The other condition says that the sequence $u_{j+K}(x)$ is increasing for some $K = K(x)$. But since K may depend on x, we will never get some overall increasing behaviour of the sequence of functions. Take, for example, on $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda := \lambda^1)$,

$$u_j(x) = j^2(x + \tfrac{1}{j})\mathbf{1}_{(-1/j,0)}(x) - j^2(x - \tfrac{1}{j})\mathbf{1}_{(0,1/j)}(x).$$

This is a sequence of symmetric tent-like functions of tents with base $(-1/j, 1/j)$ and tip at j^2 (which we take out and replace by the value 0). Clearly:

$$u_j(x) \xrightarrow{j\to\infty} 0 \quad \text{and} \quad \int u_j(x)\,dx = 1 \quad \forall\, j.$$

Moreover, if $j \geqslant K = K(x)$ with $K(x)$ defined to be the smallest integer $> 1/|x|$, then $u_j(x) = 0$ so that the second condition is clearly satisfied, but $\int u_j(x)\,dx = 1$ cannot converge to $\int 0\,dx = \int u(x)\,dx = 0$.

10 Integrals of measurable functions and null sets. Additional Material

Problems

10.17. Prove Lemma 10.8.

Solution to 10.17: Clearly, ν is defined on $\mathcal{A}$ and takes values in $[0,\infty]$. Since $\mathbf{1}_\emptyset \equiv 0$ we have

$$\nu(\emptyset) = \int \mathbf{1}_\emptyset \cdot u \, d\mu = \int 0 \, d\mu = 0.$$

If $(A_j)_{j\in\mathbb{N}} \subset \mathcal{A}$ are pairwise disjoint measurable sets, we get

$$\begin{aligned}
\nu\Big(\dot{\bigcup_{j=1}^{\infty}} A_j\Big) &= \int \mathbf{1}_{\dot\bigcup_{j=1}^{\infty} A_j} \cdot u \, d\mu \\
&= \int \sum_{j=1}^{\infty} \mathbf{1}_{A_j} \cdot u \, d\mu \\
&= \sum_{j=1}^{\infty} \int \mathbf{1}_{A_j} \cdot u \, d\mu \;=\; \sum_{j=1}^{\infty} \nu(A_j)
\end{aligned}$$

which proves σ-additivity.

13 Product measures and Fubini's theorem Additional Material

Problems

13.15. Let $(\Omega, \mathcal{A}, P)$ be a probability space, i.e. a measure space such that $P(\Omega) = 1$. Show that for a measurable function $T : \Omega \to [0, \infty)$ and every $\lambda > 0$ the following formula holds:

$$\int e^{-\lambda T} \, dP = 1 - \lambda \int_0^\infty e^{-\lambda s} \, P(T \geqslant s) \, ds.$$

What happens if we also allow negative values of λ?

Solution to 13.15: Assume first that $\lambda \geqslant 0$. The point here is that Corollary 13.13 does not apply to the function $s \mapsto e^{-\lambda s}$ since this function is decreasing and has the value 1 for $s = 0$. Consider therefore $\phi(s) := 1 - e^{-\lambda s}$. This ϕ is admissible in 13.13 and we get

$$\int \phi(T) \, dP = \int \left(1 - e^{-\lambda T}\right) dP = \int_0^\infty \lambda e^{-\lambda s} \, P(T \geqslant s) \, ds.$$

Rearranging this equality then yields

$$\int e^{-\lambda T} \, dP = 1 - \lambda \int_0^\infty e^{-\lambda s} \, P(T \geqslant s) \, ds.$$

If $\lambda < 0$ the formula remains valid if we understand it in the sense that either both sides are finite or both sides are infinite. The above argument needs some small changes, though. First, $e^{-\lambda s}$ is now increasing (which is fine) but still takes the value 1 if $s = 0$. So we should change to $\phi(s) := e^{-\lambda s} - 1$. Now the same calculation as above goes through. If one side is finite, so is the other; and if one side is infinite, then the other is infinite, too. The last statement follows from Theorem 13.11 or Corollary 13.13.

14 Integrals with respect to image measures Additional Material

Problems

14.12. **A general Young inequality.** Generalize Young's inequality given in Problem 14.9 and show that

$$\|f_1 \star f_2 \star \cdots \star f_N\|_r \leqslant \prod_{j=1}^{N} \|f_j\|_p, \qquad p = \frac{Nr}{(N-1)r+1},$$

for all $N \in \mathbb{N}$, $r \in [1, \infty)$ and $f_j \in \mathcal{L}^p(\lambda^n)$.

Solution to 14.12: For $N = 1$ the inequality is trivial, for $N = 2$ it is in line with Problem 14.9 with $p = q$.

Let us, first of all, give a *heuristic derivation* of this result which explains how one arrives at the particular form for the value of $p = p(r, N)$. We may assume that $N \geqslant 2$. Set $F_j := f_j \star \ldots \star f_N$ for $j = 1, 2, \ldots N-1$. Then

$$\begin{aligned}
&\|f_1 \star \cdots \star f_N\|_r \\
&\quad\leqslant \|f_1\|_p \|F_2\|_{q_2} = \|f_1\|_p \|f_2 \star F_3\|_{q_2} \\
&\qquad \text{by Pr. 14.9 where } \tfrac{1}{r} + 1 = \tfrac{1}{p} + \tfrac{1}{q_2} = \left(\tfrac{1}{p} - 1\right) + \tfrac{1}{q_2} + 1 \\
&\quad\leqslant \|f_1\|_p \|f_2\|_p \|F_3\|_{q_3} = \|f_1\|_p \|f_2\|_p \|f_3 \star F_4\|_{q_3} \\
&\qquad \text{by Pr. 14.9 where } \tfrac{1}{r} + 1 = \left(\tfrac{1}{p} - 1\right) + \underbrace{\tfrac{1}{p} + \tfrac{1}{q_3}}_{= \frac{1}{q_2} + 1} = 2\left(\tfrac{1}{p} - 1\right) + 1 + \tfrac{1}{q_3}
\end{aligned}$$

and repeating this procedure $N - 2$ times we arrive at

$$\begin{aligned}
\|f_1 \star \cdots \star f_N\|_r &\leqslant \|f_1\|_p \cdots \|f_{N-2}\|_p \cdot \|f_{N-1} \star f_N\|_{q_{N-1}} \\
&\leqslant \|f_1\|_p \cdots \|f_{N-2}\|_p \cdot \|f_{N-1}\|_p \cdot \|f_N\|_{q_N}
\end{aligned}$$

with the condition

$$\frac{1}{r} + 1 = (N-2)\Big(\frac{1}{p} - 1\Big) + 1 + \frac{1}{q_{N-1}} = (N-2)\Big(\frac{1}{p} - 1\Big) + \frac{1}{p} + \frac{1}{q_N}$$

and since we need $q_N = p$ we get

$$\frac{1}{r} + 1 = (N-2)\Big(\frac{1}{p} - 1\Big) + \frac{2}{p} = \frac{N}{p} - N + 2$$

and rearranging this identity yields

$$p = \frac{Nr}{(N-1)r+1}\,.$$

If you do not like this derivation of if you got lost counting the repetitions, here's the *formal proof* using induction—but with the drawback that one needs a good educated guess what $p = p(N, r)$ should look like. The start of the induction $N = 2$ is done in Problem 14.9 (starting at $N = 1$ won't help much as we need Young's inequality for $N = 2$ anyway...).

The induction hypothesis is, of course,

$$\|f_1 \star \cdots \star f_M\|_t \leqslant \prod_{j=1}^{M} \|f_j\|_\tau \quad \text{for all} \quad M = 1, 2, \ldots, N-1$$

where $t > 0$ is arbitrary and $\tau = \frac{Mt}{(M-1)t+1}$.

The induction step uses Young's inequality:

$$\|f_1 \star f_2 \star \cdots \star f_N\|_r \leqslant \|f_1\|_p \cdot \|f_2 \star \cdots \star f_N\|_q$$

where $p = \frac{Nr}{(N-1)r+1}$ and q is given by

$$\frac{1}{r} + 1 = \frac{1}{q} + \frac{1}{q} = \frac{(N-1)r+1}{Nr} + \frac{1}{q} = 1 + \frac{1}{q} - \frac{1}{N} + \frac{1}{Nr}$$

so that

$$q = \frac{Nr}{N+r-1}\,.$$

Using the induction hypothesis we now get

$$\|f_1 \star \cdots \star f_N\|_r \leqslant \|f_1\|_p \cdot \|f_2 \star \cdots \star f_N\|_q \leqslant \|f_1\|_p \cdot \big(\|f_2\|_s \cdots \|f_N\|_s\big)$$

where s is, because of the induction assumption, given by

$$s = \frac{(N-1)q}{(N-2)q+1}$$

$$
\begin{aligned}
&= \frac{(N-1)\frac{Nr}{N+r-1}}{(N-2)\frac{Nr}{N+r-1}+1} \\
&= \frac{(N-1)Nr}{(N-2)Nr+N+r-1} \\
&= \frac{(N-1)Nr}{N^2r-2Nr+r+(N-1)} \\
&= \frac{(N-1)Nr}{(N-1)^2r+(N-1)} \\
&= \frac{Nr}{(N-1)r+1} = p
\end{aligned}
$$

and we are done.

16 Uniform integrability and Vitali's convergence theorem. Additional Material

*This is an **extended** version of Lemma 16.4, page 164 of the printed text*

16.4 Lemma. *Let* $(u_j)_{j\in\mathbb{N}} \subset \mathcal{L}^p(\mu)$, $p \in [1,\infty)$, *and* $(w_k)_{k\in\mathbb{N}} \subset \mathcal{M}(\mathcal{A})$. *Then*

(i) $\lim_{j\to\infty} \|u_j - u\|_p = 0$ *implies* $u_j \xrightarrow{\mu} u$;

(ii) $\lim_{k\to\infty} w_k(x) = w(x)$ *a.e. implies* $w_k \xrightarrow{\mu} w$.

Let $f : \mathbb{R} \to \mathbb{R}$ *be a continuous function. If* $\mu(\{|u| = \infty\}) = 0$, *then*

(iii) $u_j \xrightarrow{\mu} u$ *implies* $f \circ u_j \xrightarrow{\mu} f \circ u$.

Proof. **(i)** follows immediately from the Markov inequality P10.12,

$$\begin{aligned}\mu\left(\{|u_j - u| > \epsilon\} \cap A\right) \leqslant \mu\left(\{|u_j - u| > \epsilon\}\right) &= \mu\left(\{|u_j - u|^p > \epsilon^p\}\right)\\ &\leqslant \frac{1}{\epsilon^p}\, \|u_j - u\|_p^p.\end{aligned}$$

(ii) Observe that for all $\epsilon > 0$

$$\{|w_k - w| > \epsilon\} \subset \{\epsilon \wedge |w_k - w| \geqslant \epsilon\}.$$

An application of the Markov inequality P10.12 yields

$$\begin{aligned}\mu(\{|w_k - w| > \epsilon\} \cap A) &\leqslant \mu(\{\epsilon \wedge |w_k - w| \geqslant \epsilon\} \cap A)\\ &\leqslant \frac{1}{\epsilon} \int_A \epsilon \wedge |w_k - w|\, d\mu \;=\; \frac{1}{\epsilon} \int \big(\epsilon \wedge |w_k - w|\big) \mathbf{1}_A\, d\mu.\end{aligned}$$

If $\mu(A) < \infty$, the function $\epsilon\, \mathbf{1}_A \in \mathcal{L}^1_+(\mu)$ is integrable, dominates the integrand $\big(\epsilon \wedge |w_k - w|\big)\mathbf{1}_A$, and Lebesgue's dominated convergence theorem 11.2 implies that $\lim_{k\to\infty} \int_A (\epsilon \wedge |w_k - w|)\, d\mu = 0$.

(iii) Let $R > 0$ and $\epsilon, \delta \in (0,1)$. Clearly

$$\begin{aligned}&\{|f \circ u - f \circ u_j| > \epsilon\}\\ &\qquad \subset \big(\{|f \circ u - f \circ u_j| > \epsilon\} \cap \{|u - u_j| \leqslant \delta\} \cap \{|u| \leqslant R\}\big)\end{aligned}$$

$$\cup \{|u-u_j| > \delta\} \cup \{|u| > R\}.$$

Since $\{|u-u_j| \leqslant \delta\} \cap \{|u| \leqslant R\} \subset \{|u_j| \leqslant R+1\} \cap \{|u| \leqslant R\}$, and since f is uniformly continuous on the closed interval $[-R-1, R+1]$, we can choose $\delta = \delta_\epsilon$ so small that the first set on the right hand side is empty, i.e.

$$\{|f \circ u - f \circ u_j| > \epsilon\} \subset \{|u-u_j| > \delta\} \cup \{|u| > R\}.$$

This shows that for all $A \in \mathcal{A}$ with $\mu(A) < \infty$

$$\mu\big(\{|f \circ u - f \circ u_j| > \epsilon\} \cap A\big) \leqslant \mu\big(\{|u-u_j| > \delta\} \cap A\big) + \mu\big(\{|u| > R\} \cap A\big).$$

Using $u_j \xrightarrow{\mu} u$, we find

$$\limsup_{j\to\infty} \mu\big(\{|f \circ u - f \circ u_j| > \epsilon\} \cap A\big) \leqslant \mu\big(\{|u| > R\} \cap A\big) \xrightarrow{R\to\infty} 0$$

since $\mu(\{|u| = \infty\}) = 0$. ∎

This is an ***extended*** *version of the Proof of (iii)⇒(i) of Theorem 16.8, pages 165-6 of the printed text*

Proof (of Theorem 16.8). (iii)⇒(ii): Fix $\epsilon \in (0,1)$. We have

$$\begin{aligned}\int_{\{\epsilon|u_j|>|u|\}} |u_j|^p \, d\mu &\leqslant \int_{\{\epsilon|u_j|>|u|\}} |u|^p \, d\mu + \int_{\{\epsilon|u_j|>|u|\}} \big(|u_j|^p - |u|^p\big) \, d\mu \\ &= \int_{\{\epsilon|u_j|>|u|\}} |u|^p \, d\mu + \int \big(|u_j|^p - |u|^p\big) \, d\mu + \int_{\{\epsilon|u_j|\leqslant|u|\}} \big(|u_j|^p - |u|^p\big) \, d\mu.\end{aligned}$$

Denote the three integrals I_1, I_2 and I_3, respectively. Because of assumption (iii) we know that

$$I_2 \leqslant \epsilon^p, \quad \text{for all} \quad j \geqslant N_\epsilon$$

for some $N_\epsilon \in \mathbb{N}$. Moreover, (iii) shows that $\sup_{j\in\mathbb{N}} \int |u_j| \, d\mu \leqslant C < \infty$ which means that

$$I_1 \leqslant \int_{\{\epsilon|u_j|>|u|\}} |u|^p \, d\mu \leqslant \epsilon^p \int_{\{\epsilon|u_j|>|u|\}} |u_j|^p \, d\mu \leqslant \epsilon^p \int |u_j|^p \, d\mu \leqslant C\,\epsilon^p.$$

Finally, the inclusion

$$\big\{\epsilon|u_j| \leqslant |u|\big\} \subset \big\{||u_j|^p - |u|^p| \leqslant (\epsilon^{-p}+1)|u|^p\big\}$$

yields with $\kappa := \epsilon^{-p} + 1$ and $\Delta_j^p := \big||u_j|^p - |u|^p\big|$

$$|I_3| \leqslant \int_{\{\Delta_j^p \leqslant \kappa |u|^p\}} \Delta_j^p \, d\mu.$$

Splitting the integration domain into three parts,

$$\big\{\Delta_j^p \leqslant (\kappa|u|^p) \wedge \eta\big\}, \quad \big\{\eta \leqslant \Delta_j^p \leqslant \kappa|u|^p\big\} \cap \big\{|u|^p > R\big\},$$
$$\text{and} \quad \big\{\eta \leqslant \Delta_j^p \leqslant \kappa|u|^p\big\} \cap \big\{|u|^p \leqslant R\big\},$$

we get

$$|I_3| \leqslant \int \eta \wedge (\kappa|u|^p)\, d\mu + \int_{\{|u|^p > R\}} \kappa |u|^p \, d\mu + \kappa\, R\, \mu\big(\big\{\Delta_j^p \geqslant \eta\big\} \cap \big\{|u|^p \geqslant \tfrac{\eta}{\kappa}\big\}\big).$$

By dominated convergence, the two integrals on the right hand side tend to 0 as $\eta \to 0$ resp. $R \to \infty$. Thus, we may choose $\eta = \eta_\epsilon$ and $R = R_\epsilon$ such that

$$|I_3| \leqslant \epsilon^p + \kappa\, R\, \mu\big(\big\{\Delta_j^p \geqslant \eta\big\} \cap \big\{|u|^p \geqslant \tfrac{\eta}{\kappa}\big\}\big).$$

Since the set $\{|u|^p \geqslant \eta/\kappa\}$ has finite μ-measure[✓] and since $|u_j|^p \xrightarrow{\mu} |u|^p$, i.e. $\Delta_j^p \xrightarrow{\mu} 0$, see Lemma 16.4(iii), we can find some $M_\epsilon \geqslant N_\epsilon$ such that

$$|I_3| \leqslant 2\epsilon^p \quad \text{for all} \quad j \geqslant M_\epsilon.$$

Setting $w_\epsilon := \max\{|u_1|, \dots, |u_{M_\epsilon}|, |u|\}$ we have $w_\epsilon \in \mathcal{L}^p_+(\mu)$[✓] and see

$$\sup_{j\in\mathbb{N}} \int_{\left\{|u_j| \geqslant \frac{1}{\epsilon} w_\epsilon\right\}} |u_j| \, d\mu \leqslant \sup_{j \geqslant M_\epsilon} \int_{\{\epsilon|u_j| \leqslant |u|\}} |u_j| \, d\mu \leqslant (C+3)\, \epsilon^p.$$

Since

$$w_\epsilon \in \mathcal{L}^p_+(\mu) \iff w_\epsilon^p \in \mathcal{L}^1_+(\mu) \quad \text{and} \quad \big\{|u_j| > \tfrac{1}{\epsilon}\, w_\epsilon\big\} = \big\{|u_j|^p > \tfrac{1}{\epsilon^p}\, w_\epsilon^p\big\}, \tag{16.3}$$

we have established the uniform integrability of $(|u_j|^p)_{j\in\mathbb{N}}$. ∎

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*This is a **streamlined** version of the proof of Theorem 16.8(vi)⇒(vii), pages 171-3 of the printed text*

(vi)⇒(vii): For $u \in \mathcal{F}$ we set $\alpha_n := \alpha_n(u) := \mu(\{|u| > n\})$ and define

$$\Phi(t) := \int_{[0,t)} \phi(s)\, \lambda(ds), \qquad \phi(s) := \sum_{n=1}^\infty \gamma_n \, \mathbf{1}_{[n,n+1)}(s).$$

We will now determine the numbers $\gamma_1, \gamma_2, \gamma_3, \ldots$. Clearly,

$$\Phi(t) = \sum_{n=1}^{\infty} \gamma_n \int_{[0,t)} \mathbf{1}_{[n,n+1)}(s)\,\lambda(ds) = \sum_{n=1}^{\infty} \gamma_n \left[(t-n)^+ \wedge 1\right]$$

and

$$\int \Phi(|u|)\,d\mu = \sum_{n=1}^{\infty} \gamma_n \int \left[(|u|-n)^+ \wedge 1\right] d\mu \leqslant \sum_{n=1}^{\infty} \gamma_n\,\mu(|u|>n). \tag{16.6}$$

If we can construct $(\gamma_n)_{n\in\mathbb{N}}$ such that it increases to ∞ and (16.6) is finite (uniformly for all $u \in \mathcal{F}$), then we are done: $\phi(s)$ will increase to ∞, $\Phi(t)$ will be convex[1] and satisfy

$$\frac{\Phi(t)}{t} = \frac{1}{t}\int_{[0,t)} \phi(s)\,\lambda(ds) \geqslant \frac{1}{t}\int_{[t/2,t)} \phi(s)\,\lambda(ds) \geqslant \tfrac{1}{2}\,\phi\big(\tfrac{t}{2}\big) \uparrow \infty.$$

By assumption we can find an increasing sequence $(r_j)_{j\in\mathbb{N}} \subset \mathbb{R}$ such that $\lim_{j\to\infty} r_j = \infty$ and $\int_{\{r_j<|u|\}} |u|\,d\mu \leqslant 2^{-j}$. Now[2]

$$\sum_{k=r_j}^{\infty} \mathbf{1}_{\{k<|u|\}} \leqslant \sum_{k=r_j}^{[|u|]} \mathbf{1}_{\{k<|u|\}} \leqslant \sum_{k=r_j}^{[|u|]} \mathbf{1}_{\{r_j<|u|\}} \leqslant |u|\,\mathbf{1}_{\{r_j<|u|\}}.$$

If we integrate both sides of this inequality and use Beppo Levi's theorem in the form of Corollary 9.9 on the left-hand side, we get

$$\sum_{k=r_j}^{\infty} \mu(\{|u|>k\}) = \sum_{k=r_j}^{\infty} \int \mathbf{1}_{\{k<|u|\}}\,d\mu \leqslant \int |u|\,\mathbf{1}_{\{r_j<|u|\}}\,d\mu \leqslant 2^{-j}.$$

---

[1]Usually one argues that $\Phi'' \geqslant 0$ a.e., but for this we need to know that the monotone function $\phi = \Phi'$ is almost everywhere differentiable—and this requires Lebesgue's differentiation theorem 19.20. Here is an alternative elementary argument: it is not hard to see that $\Phi : (a,b) \to \mathbb{R}$ is convex if, and only, if $\frac{\Phi(y)-\Phi(x)}{y-x} \leqslant \frac{\Phi(z)-\Phi(x)}{z-x}$ holds for all $a < x < y < z < b$, use e.g. the technique of the proof of Lemma 12.13. Since $\Phi(x) = \int_0^x \phi(s)\,ds$ (by L13.12 and T11.8), this is the same as

$$\frac{1}{y-x}\int_x^y \phi(s)\,ds \leqslant \frac{1}{z-x}\int_x^z \phi(s)\,ds \iff \frac{1}{y-x}\int_x^y \phi(s)\,ds \leqslant \frac{1}{z-y}\int_y^z \phi(s)\,ds$$
$$\iff \int_0^1 \phi(s(y-x)+x)\,ds \leqslant \int_0^1 \phi(s(z-y)+y)\,ds.$$

The latter inequality follows from the fact that ϕ is increasing and $s(y-x)+x \in [x,y]$ while $s(z-y)+y \in [y,z]$ for $0 \leqslant s \leqslant 1$.

[2]for $x \in \mathbb{R}$ we will denote by $[x]$ the Gauß bracket, i.e. the largest integer $\leqslant x$.

Summing over $j \in \mathbb{N}$ finally yields

$$\sum_{j=1}^{\infty}\sum_{k=r_j}^{\infty}\mu(\{|u|>k\}) \leqslant \sum_{j=1}^{\infty}2^{-j} = 1.$$

and a simple interchange in the order of the summation gives

$$\sum_{j=1}^{\infty}\sum_{k=r_j}^{\infty}\mu(\{|u|>k\}) = \sum_{k=1}^{\infty}\underbrace{\left(\sum_{j=1}^{\infty}\mathbf{1}_{[1,k]}(r_j)\right)}_{=:\ \gamma_k}\mu(\{|u|>k\}) \leqslant 1.$$

This finishes the construction of the sequence $(\gamma_k)_{k\in\mathbb{N}}$.

Problems

16.15. Let $(f_i)_{i\in I}$ be a family of uniformly integrable functions and let $(u_i)_{i\in I} \subset L^1(\mu)$ be some further family such that $|u_i| \leqslant |f_i|$ for every $i \in I$. Then $(u_i)_{i\in I}$ is uniformly integrable. In particular, every family of functions $(u_i)_{i\in I}$ with $|u_i| \leqslant g$ for some $g \in L^1_+$ is uniformly integrable.

16.16. Let $(X, \mathcal{A}, \mu)$ be an arbitrary measure space. Show that a family $\mathcal{F} \subset \mathcal{L}^1(\mu)$ is uniformly integrable if, and only if the following condition holds:

$$\forall\,\epsilon > 0 \quad \exists\, g_\epsilon \in \mathcal{L}^1_+(\mu)\ :\ \sup_{u\in\mathcal{F}}\int (|u| - g_\epsilon \wedge |u|)\,d\mu < \epsilon.$$

Give a simplified version of this equivalence for finite measure spaces.

Solution to 16.15: Fix $\epsilon > 0$. By assumption there is some $w = w_\epsilon \in L^1_+$ such that

$$\sup_i \int_{\{|f_i|>w\}} |f_i|\,d\mu \leqslant \epsilon.$$

Consider now

$$\begin{aligned}
\int_{\{|u_i|>2w\}} |u_i|\,d\mu &\leqslant \int_{\{|u_i|>2w\}} |f_i|\,d\mu \\
&= \int_{\{|u_i|>2w\}\cap\{|f_i|\leqslant w\}} |f_i|\,d\mu + \int_{\{|u_i|>2w\}\cap\{|f_i|>w\}} |f_i|\,d\mu \\
&\leqslant \int_{\{|u_i|>2w\}\cap\{|f_i|\leqslant w\}} \frac{1}{2}|u_i|\,d\mu + \int_{\{|u_i|>2w\}\cap\{|f_i|>w\}} |f_i|\,d\mu \\
&\leqslant \int_{\{|u_i|>2w\}} \frac{1}{2}|u_i|\,d\mu + \int_{\{|f_i|>w\}} |f_i|\,d\mu
\end{aligned}$$

Thus,

$$\begin{aligned}\frac{1}{2}\int_{\{|u_i|>2w\}}|u_i|\,d\mu &\leqslant \int_{\{|f_i|>w\}}|f_i|\,d\mu\\ &\leqslant \sup_i\int_{\{|f_i|>w\}}|f_i|\,d\mu \ \leqslant\ \epsilon\end{aligned}$$

uniformly for all $i \in I$.

Solution to 16.16: Let $g \in \mathcal{L}^1_+(\mu)$. Then

$$0 \leqslant \int(|u| - g\wedge|u|)\,d\mu = \int_{\{|u|\geqslant g\}}(|u|-g)\,d\mu \leqslant \int_{\{|u|\geqslant g\}}|u|\,d\mu.$$

This implies that uniform integrability of the family $\mathfrak{F}$ implies that the condition of Problem 16.16 holds. On the other hand,

$$\begin{aligned}\int_{\{|u|\geqslant g\}}|u|\,d\mu &= \int_{\{|u|\geqslant g\}}(2|u|-|u|)\,d\mu\\ &\leqslant \int_{\{|u|\geqslant g\}}(2|u|-g)\,d\mu\\ &\leqslant \int_{\{2|u|\geqslant g\}}(2|u|-g)\,d\mu\\ &= 2\int_{\{|u|\geqslant\frac{1}{2}g\}}\big(|u|-\tfrac{1}{2}g\big)\,d\mu\\ &= 2\int_{\{|u|\geqslant\frac{1}{2}g\}}\big(|u|-\big[\tfrac{1}{2}g\big]\wedge|u|\big)\,d\mu\end{aligned}$$

and since $g \in \mathcal{L}^1$ if, and only if, $\frac{1}{2}g \in \mathcal{L}^1$, we see that the condition given in Problem 16.16 entails uniform integrability.

In finite measure spaces this conditions is simpler: constants are integrable functions in finite measure spaces; thus we can replace the condition given in Problem 16.16 by

$$\lim_{R\to\infty}\sup_{u\in\mathfrak{F}}\int(|u|-R\wedge|u|)\,d\mu = 0.$$

18 Martingale convergence theorems. Additional Material

*This is a **streamlined** version of **(i)⇒(ii)** of the proof of 18.6, pages 194-5 of the printed text*

Proof. (i)⇒(ii): Since $\mu|_{\mathcal{A}_0}$ is σ-finite, we can fix an exhausting sequence $(A_k)_{k\in\mathbb{N}} \subset \mathcal{A}_0$ with $A_k \uparrow X$ and $\mu(A_k) < \infty$. It is not hard to see that the function $w := \sum_{k=1}^{\infty} 2^{-k}\,(1+\mu(A_k))^{-1}\,\mathbf{1}_{A_k}$ is strictly positive $w > 0$ and integrable $w \in \mathcal{L}^1(\mathcal{A}_0,\mu)$. Example 17.3(iv) shows that $(u_j^+)_{j\in\mathbb{N}\cup\{\infty\}}$ is still a submartingale, so that for every $L > 0$

$$\begin{aligned}\int_{\{u_j^+>Lw\}} u_j^+\,d\mu &\leqslant \int_{\{u_j^+>Lw\}} u_\infty^+\,d\mu\\ &\leqslant \int_{\left\{u_j^+>Lw\right\}\cap\left\{u_\infty^+>\frac12 Lw\right\}} u_\infty^+\,d\mu + \int_{\left\{u_j^+>Lw\right\}\cap\left\{u_\infty^+\leqslant\frac12 Lw\right\}} u_\infty^+\,d\mu\\ &\leqslant \int_{\left\{u_\infty^+>\frac12 Lw\right\}} u_\infty^+\,d\mu + \frac12\int_{\left\{u_j^+>Lw\right\}\cap\left\{u_\infty^+\leqslant\frac12 Lw\right\}} u_j^+\,d\mu.\end{aligned}$$

For the last estimate we used that on the set $\{u_j^+ > Lw\}\cap\{u_\infty^+ \leqslant \frac12 Lw\}$ the integrand of the second integral satisfies $u_\infty^+ \leqslant \frac12\,Lw < \frac12\,u_j^+$. Now we subtract $\frac12\int_{\{u_j^+>Lw\}} u_j^+\,d\mu$ on both sides and get, uniformly for all $j \in \mathbb{N}$,

$$\frac12\int_{\{u_j^+>Lw\}} u_j^+\,d\mu \leqslant \int_{\left\{u_\infty^+>\frac12 Lw\right\}} u_\infty^+\,d\mu \xrightarrow{L\to\infty} 0.$$

This follows from the dominated convergence theorem, Theorem 11.2, since $|u_\infty| \in \mathcal{L}^1(\mathcal{A}_\infty,\mu)$ dominates $u_\infty^+\mathbf{1}_{\{u_\infty^+>\frac12 Lw\}} \xrightarrow{L\to\infty} 0$. Thus, $(u_j^+)_{j\in\mathbb{N}}$ is uniformly integrable. From $\lim_{j\to\infty} u_j = u_\infty$ a.e., we conclude $\lim_{j\to\infty} u_j^+ = u_\infty^+$, and Vitali's convergence theorem 16.6 shows that $\lim_{j\to\infty}\int u_j^+\,d\mu = \int u_\infty^+\,d\mu$. Thus

$$\int |u_j|\,d\mu = \int (2u_j^+ - u_j)\,d\mu \xrightarrow{j\to\infty} \int (2u_\infty^+ - u_\infty)\,d\mu = \int |u_\infty|\,d\mu,$$

and another application of Vitali's theorem proves that $(u_j)_{j\in\mathbb{N}}$ is uniformly integrable. ∎

*This is a **streamlined** version of the second part of Example 18.8, page 199 of the printed text*

18.8 Example (continued). We can now continue with the proof of the *necessity* part of Kolmogorov's strong law of large numbers. Since the a.e. limit exists, we get

$$\frac{X_n}{n} = \frac{S_n}{n} - \frac{n-1}{n}\,\frac{S_{n-1}}{n-1} \xrightarrow{n\to\infty} 0,$$

which shows that $\omega \in A_n := \{|X_n| \geqslant n\}$ happens only for finitely many n. In other words, $P(\sum_{j=1}^{\infty} \mathbf{1}_{A_j} = \infty) = 0$; since the A_n are all independent, the Borel-Cantelli lemma T18.9 shows that $\sum_{j=1}^{\infty} P(A_j) < \infty$. Thus[1]

$$|X_1| \leqslant \big[|X_1|\big] + 1 = 1 + \sum_{n=1}^{|X_1|} 1 = 1 + \sum_{n=1}^{\infty} \mathbf{1}_{\{n \leqslant |X_1|\}}.$$

If we integrate this inequality w.r.t. P and use Beppo Levi's theorem, we arrive at

$$\int |X_1|\,dP \leqslant 1 + \sum_{n=1}^{\infty} P(\{|X_1| \geqslant n\}) = 1 + \sum_{n=1}^{\infty} P(\{|X_n| \geqslant n\})$$

since X_1 and X_n have the same distribution. ∎

We will see more applications of the martingale convergence theorems in the following chapters.

[1] for $x \in \mathbb{R}$ we will denote by $[x]$ the Gauß bracket, i.e. the largest integer $\leqslant x$.

19 The Radon-Nikodým Theorem and Other Applications of Martingales. Additional Material

Problems

19.19. Let μ and ν be measures on the measurable space $(X, \mathcal{A})$. Show that the absolute continuity condition (19.1) is equivalent to

$$\mu(A \triangle B) = 0 \quad \Longrightarrow \quad \nu(A) = \nu(B) \qquad \text{for all } A, B \in \mathcal{A}.$$

Solution to 19.19: "$\Rightarrow$": Assume first that (19.1) holds, i.e. that $\nu \ll \mu$. If $\mu(A \triangle B) = 0$ for some $A, B \in \mathcal{A}$ we get $\nu(A \triangle B) = 0$. By definition,

$$\nu(A \triangle B) = \nu(A \setminus B) + \nu(B \setminus A) = \nu(A \setminus (A \cap B)) + \nu(B \setminus (A \cap B)) = 0$$

so that

$$\nu(A \setminus (A \cap B)) = \nu(B \setminus (A \cap B)) = 0.$$

Assume that $\nu(A) < \infty$. Then $\nu(A \cap B) \leqslant \nu(A) < \infty$ and we see that

$$\nu(A) = \nu(A \cap B) \ \text{ and } \ \nu(B) = \nu(A \cap B)$$

which means that $\nu(A) = \nu(B)$.

If $\nu(A) = \infty$ the condition $\nu(A \setminus (A \cap B)) = 0$ shows that $\nu(A \cap B) = \infty$, otherwise $0 = \nu(A \setminus (A \cap B)) = \nu(A) - \nu(A \cap B) = \infty$ which is impossible. Again we have $\nu(A) = \infty = \nu(B)$.

"$\Leftarrow$": Assume now that the condition stated in the problem is satisfied. If $N \in \mathcal{A}$ is any μ-null set, we choose $A := N$ and $B := \emptyset$ and observe that $A \triangle B = N$. Thus,

$$\mu(N) = \mu(A \triangle B) = 0 \quad \Longrightarrow \quad \nu(A) = \nu(B)$$

but this is just $\nu(N) = \nu(A) = \nu(\emptyset) = 0$. Condition (19.1) follows.

22 Conditional expectations in L^2 Additional Material

Problems

22.4. Let $(X, \mathcal{A}, \mu)$ be a measure space and $\mathcal{G} \subset \mathcal{A}$ be a sub-σ-algebra. such that $\mu|_{\mathcal{G}}$ is σ-finite. Let $p, q \in (1, \infty)$ be conjugate numbers, i.e. $1/p + 1/q = 1$, and let $u, w \in \bigcap_{p\in[1,\infty]} L^p(\mathcal{A}, \mu)$. Show that

$$\left|\mathbf{E}^{\mathcal{G}}(uw)\right| \leqslant \left[\mathbf{E}^{\mathcal{G}}(|u|^p)\right]^{1/p}\left[\mathbf{E}^{\mathcal{G}}(|w|^q)\right]^{1/q}.$$

[Hint: use Young's inequality 12.1 with $A = |u|/\left[\mathbf{E}^{\mathcal{G}}(|u|^p)\right]^{1/p}$ and $B = |w|/\left[\mathbf{E}^{\mathcal{G}}(|w|^q)\right]^{1/q}$ whenever the numerator is not 0.]

Solution to 22.4: Let $G_u := \{\mathbf{E}^{\mathcal{G}}|u|^p > 0\}$, $G_w := \{\mathbf{E}^{\mathcal{G}}|w|^q > 0\}$ and $G := G_u \cup G_w$. Following the hint we get

$$\frac{|u|}{\left[\mathbf{E}^{\mathcal{G}}(|u|^p)\right]^{1/p}}\frac{|w|}{\left[\mathbf{E}^{\mathcal{G}}(|w|^q)\right]^{1/q}}\mathbf{1}_G \leqslant \frac{|u|^p}{p\,\mathbf{E}^{\mathcal{G}}(|u|^p)}\mathbf{1}_G + \frac{|u|^q}{q\,\mathbf{E}^{\mathcal{G}}(|w|^q)}\mathbf{1}_G$$

Since $\mathbf{1}_G$ is bounded and $\mathcal{G}$-measurable, we can apply $\mathbf{E}^{\mathcal{G}}$ on both sides of the above inequality and get

$$\frac{\mathbf{E}^{\mathcal{G}}(|u||w|)}{\left[\mathbf{E}^{\mathcal{G}}(|u|^p)\right]^{1/p}\left[\mathbf{E}^{\mathcal{G}}(|w|^q)\right]^{1/q}}\mathbf{1}_G \leqslant \frac{\mathbf{E}^{\mathcal{G}}(|u|^p)}{p\,\mathbf{E}^{\mathcal{G}}(|u|^p)}\mathbf{1}_G + \frac{\mathbf{E}^{\mathcal{G}}(|u|^q)}{q\,\mathbf{E}^{\mathcal{G}}(|w|^q)}\mathbf{1}_G = \mathbf{1}_G$$

or

$$\begin{aligned}\mathbf{E}^{\mathcal{G}}(|u||w|)\,\mathbf{1}_G &\leqslant \left[\mathbf{E}^{\mathcal{G}}(|u|^p)\right]^{1/p}\left[\mathbf{E}^{\mathcal{G}}(|w|^q)\right]^{1/q}\mathbf{1}_G \\ &\leqslant \left[\mathbf{E}^{\mathcal{G}}(|u|^p)\right]^{1/p}\left[\mathbf{E}^{\mathcal{G}}(|w|^q)\right]^{1/q}.\end{aligned}$$

Denote by G_n an exhaustion of X such that $G_n \in \mathcal{G}$, $G_n \uparrow X$ and $\mu(G_n) < \infty$. Then

$$\begin{aligned}\int_{G_u^c} |u|^p\,d\mu &= \sup_n \int_{G_u^c \cap G_n} |u|^p\,d\mu \\ &= \sup_n \langle \mathbf{1}_{G_u^c \cap G_n}, |u|^p\rangle \\ &= \sup_n \langle \mathbf{E}^{\mathcal{G}}\mathbf{1}_{G_u^c \cap G_n}, |u|^p\rangle\end{aligned}$$

$$= \sup_n \langle \mathbf{1}_{G_u^c \cap G_n}, \mathsf{E}^{\mathcal{G}}(|u|^p) \rangle$$
$$= 0$$

which means that $\mathbf{1}_{G_u} u = u$ almost everywhere. Thus,

$$\mathsf{E}^{\mathcal{G}}(|u||w|)\, \mathbf{1}_G = \mathsf{E}^{\mathcal{G}}(|u||w|\mathbf{1}_G) = \mathsf{E}^{\mathcal{G}}(|u|\mathbf{1}_{G_u}|w|\mathbf{1}_{G_w}) = \mathsf{E}^{\mathcal{G}}(|u||w|)$$

and the inequality follows since

$$\left|\mathsf{E}^{\mathcal{G}}(uw)\right| \leqslant \mathsf{E}^{\mathcal{G}}(|uw|).$$

23 Conditional expectations in L^p Additional Material

Problems

23.13. Let $(X, \mathcal{A}, \mathcal{A}_j, \mu)$ be a finite filtered measure space, let $u_j \in L^1(\mathcal{A}_j)$ be a sequence of measurable functions such that for some $f \in L^1(\mathcal{A})$

$$|u_j| \leqslant E^{\mathcal{A}_j} f$$

holds true. Show that the family $(u_j)_{j\in\mathbb{N}}$ is uniformly integrable.

Solution to 23.13: (Compare this problem with Problem 16.15.) Recall that in finite measure spaces uniform integrability follows from (and is actually equivalent to)

$$\lim_{R\to\infty} \sup_j \int_{\{|u_j|>R\}} |u_j|\, d\mu = 0;$$

this is true since in a finite measure space the constant function $w \equiv R$ is integrable.

Observe now that

$$\begin{aligned}
\int_{\{|u_j|>R\}} |u_j|\, d\mu &\leqslant \int_{\{|u_j|>R\}} E^{\mathcal{A}_j} f\, d\mu \\
&= \int_{\{|u_j|>R\}} f\, d\mu \\
&= \int_{\{|u_j|>R\}\cap\{f\leqslant R/2\}} f\, d\mu + \int_{\{|u_j|>R\}\cap\{f>R/2\}} f\, d\mu \\
&\leqslant \int_{\{|u_j|>R\}\cap\{f\leqslant R/2\}} \frac{1}{2}|u_j|\, d\mu + \int_{\{|u_j|>R\}\cap\{f>R/2\}} f\, d\mu \\
&\leqslant \int_{\{|u_j|>R\}} \frac{1}{2}|u_j|\, d\mu + \int_{\{f>R/2\}} f\, d\mu
\end{aligned}$$

This shows that

$$\frac{1}{2}\int_{\{|u_j|>R\}} |u_j|\, d\mu \leqslant \int_{\{f>R/2\}} f\, d\mu \xrightarrow[\text{uniformly for all } j]{R\to\infty} 0.$$

24 Orthonormal systems and their convergence behaviour Additional Material

*This is a **streamlined and corrected** version of the Epilogue 24.29, pages 309-12 of the printed text*

24.29 Epilogue. The combination of martingale methods and orthogonal expansions opens up a whole new world. Let us illustrate this by a rapid construction of one of the most prominent stochastic processes: the *Wiener process* or *Brownian motion*.

Choose in Theorem 24.27 $(X, \mathcal{A}, P) = ([0,1], \mathcal{B}[0,1], \lambda)$ where λ is one-dimensional Lebesgue measure on $[0,1]$; denoting points in $[0,1]$ by ω, we will often write $d\omega$ instead of $\lambda(d\omega)$. Assume that the independent, identically distributed random variables e_j are all standard normal Gaussian random variables, i.e.

$$P(e_j \in B) = \frac{1}{\sqrt{2\pi}} \int_B e^{-x^2/2}\,dx, \qquad B \in \mathcal{B}(\mathbb{R}),$$

and consider the series expansion

$$W_t(\omega) := \sum_{n=0}^{\infty} e_n(\omega) \langle \mathbf{1}_{[0,t]}, H_n \rangle, \qquad \omega \in [0,1].$$

Here $t \in [0,1]$ is a parameter, $\langle u, v \rangle = \int_0^1 u(x)v(x)\,dx$, and H_n, $n = 2^k + j$, $0 \leqslant j < 2^k$, denote the lexicographically ordered Haar functions (24.16). A short calculation confirms for $n \geqslant 1$

$$\langle \mathbf{1}_{[0,t]}, H_n \rangle = \int_0^t H_n(x)\,dx = 2^{k/2} \int_0^t H_1(2^k x - j)\,dx = F_n(t),$$

where $F_1(t) = \int_0^t H_1(x)\,dx\,\mathbf{1}_{[0,1]}(t) = t\mathbf{1}_{[0,\frac{1}{2})}(t) - (t-1)\mathbf{1}_{[\frac{1}{2},1]}(t)$ and $F_n(t) = 2^{-k/2} F_1(2^k t - j)$ are tent-functions which have the same supports as the Haar functions. Since the Haar functions are a complete orthonormal system, cf. Theorem 24.17, we may apply Bessel's inequality, 21.11(iii), to get

$$\sum_{n=0}^{\infty} \langle \mathbf{1}_{[0,t]}, H_n \rangle^2 \leqslant \langle \mathbf{1}_{[0,t]}, \mathbf{1}_{[0,t]} \rangle = t \leqslant 1.$$

Thus, Theorem 24.27(ii) guarantees that $W_t(\omega)$ exists, for each $t \in [0,1]$, both in $L^2(d\omega)$-sense and $\lambda(d\omega)$-almost everywhere.

More is true. Since the e_n are independent Gaussian random variables, so are their finite linear combinations (e.g. Bauer [**?**, §24]) and, in particular, the partial sums

$$S_N(t;\omega) := \sum_{n=0}^{N} e_n(\omega)\langle \mathbf{1}_{[0,t]}, H_n\rangle.$$

Gaussianity is preserved under L^2-limits;[6] we conclude that $W_t(\omega)$ has a Gaussian distribution for each t. The mean is given by

$$\int_0^1 W_t(\omega)\,d\omega = \sum_{n=0}^{\infty}\int_0^1 e_n(\omega)\,d\omega\,\langle \mathbf{1}_{[0,t]}, H_n\rangle = 0$$

(to change integration and summation use that $L^2(d\omega)$-convergence entails $L^1(d\omega)$-convergence on a finite measure space). Since $\int e_n e_m\,d\omega = 0$ or 1 according to $n \neq m$ or $n = m$, we can calculate for $0 \leqslant s < t \leqslant 1$ the variance by

$$\begin{aligned}
&\int_0^1 (W_t(\omega) - W_s(\omega))^2\,d\omega \\
&\quad= \sum_{n,m=0}^{\infty}\int_0^1 e_n(\omega)e_m(\omega)\,d\omega\,\langle \mathbf{1}_{[0,t]} - \mathbf{1}_{[0,s]}, H_n\rangle\langle \mathbf{1}_{[0,t]} - \mathbf{1}_{[0,s]}, H_m\rangle \\
&\quad= \sum_{n=0}^{\infty}\langle \mathbf{1}_{(s,t]}, H_n\rangle^2 \overset{24.17,21.13}{=} \langle \mathbf{1}_{(s,t]}, \mathbf{1}_{(s,t]}\rangle = t-s.
\end{aligned}$$

In particular, the increment $W_t - W_s$ has the same probability distribution as W_{t-s}. In the same vein we find for $0 \leqslant s < t \leqslant u < v \leqslant 1$ that

$$\int_0^1 (W_t(\omega) - W_s(\omega))(W_v(\omega) - W_u(\omega))\,d\omega = \langle \mathbf{1}_{(s,t]}, \mathbf{1}_{(u,v]}\rangle = 0.$$

Since $W_t - W_s$ is Gaussian, this proves already the independence of the two increments $W_t - W_s$ and $W_v - W_u$, cf. [**?**, §24]. By induction, we conclude that

$$W_{t_n} - W_{t_{n-1}}, \dots, W_{t_1} - W_{t_0},$$

[6] (cf. [**?**, §§23, 24]) if X_n is normal distributed with mean 0 and variance σ_n^2, its Fourier transform is $\int e^{i\xi X_n}\,dP = e^{-\sigma_n^2\xi^2/2}$. If $X_n \xrightarrow{n\to\infty} X$ in L^2-sense, we have $\sigma_n^2 \to \sigma^2$ and, by dominated convergence, $\int e^{i\xi X}\,dP = \lim_n \int e^{i\xi X_n}\,dP = \lim_n e^{-\sigma_n^2\xi^2/2} = e^{-\sigma^2\xi^2/2}$; the claim follows from the uniqueness of the Fourier transform.

are independent for all $0 \leqslant t_0 \leqslant \cdots \leqslant t_n \leqslant 1$.

Let us finally turn to the dependence of $W_t(\omega)$ on t. Note that for $m < n$

$$\begin{aligned}
&\int_0^1 \sup_{t\in[0,1]} \left|S_{2^n-1}(t;\omega) - S_{2^m-1}(t;\omega)\right|^4 d\omega \\
&\quad = \int_0^1 \sup_{t\in[0,1]} \left(\sum_{k=m}^{n-1}\sum_{j=0}^{2^k-1} e_{2^k+j}(\omega)\langle \mathbf{1}_{[0,t]}, H_{2^k+j}\rangle\right)^4 d\omega \\
&\quad = \int_0^1 \sup_{t\in[0,1]} \left(\sum_{k=m}^{n-1} 2^{-\frac{k}{8}}\left[\sum_{j=0}^{2^k-1} 2^{\frac{k}{8}}\, e_{2^k+j}(\omega)\langle \mathbf{1}_{[0,t]}, H_{2^k+j}\rangle\right]\right)^4 d\omega \\
&\quad \leqslant \int_0^1 \sup_{t\in[0,1]} \underbrace{\left[\sum_{k=m}^{n-1} 2^{-\frac{k}{6}}\right]}_{\leqslant 10} \cdot \sum_{k=m}^{n-1}\left[\sum_{j=0}^{2^k-1} 2^{\frac{k}{8}}\, e_{2^k+j}(\omega)\langle \mathbf{1}_{[0,t]}, H_{2^k+j}\rangle\right]^4 d\omega
\end{aligned}$$

where we used Hölder's inequality for the outer sum with $p = \frac{4}{3}$ and $q = 4$. Since the functions $F_{2^k+j}(t) = \langle \mathbf{1}_{[0,t]}, H_{2^k+j}\rangle$ with $0 \leqslant j < 2^k$ have disjoint supports and are bounded by $2^{-k/2}$, we find

$$\begin{aligned}
&\int_0^1 \sup_{t\in[0,1]} \left|S_{2^n-1}(t;\omega) - S_{2^m-1}(t;\omega)\right|^4 d\omega \\
&\qquad \leqslant 10 \int_0^1 \sup_{t\in[0,1]} \sum_{k=m}^{n-1}\sum_{j=0}^{2^k-1} 2^{\frac{k}{2}}\, e^4_{2^k+j}(\omega)\langle \mathbf{1}_{[0,t]}, H_{2^k+j}\rangle^4\, d\omega \\
&\qquad \leqslant 10 \sum_{k=m}^{n-1}\sum_{j=0}^{2^k-1} 2^{\frac{k}{2}} \underbrace{\int_0^1 e^4_{2^k+j}(\omega)\, d\omega}_{=(2\pi)^{-1/2}\int_{\mathbb{R}} y^4 e^{-y^2/2}\,dy \quad \forall j,k} 2^{-2k} \\
&\qquad = C \sum_{k=m}^{n-1} 2^{\frac{k}{2}} \cdot 2^k \cdot 2^{-2k} \;\leqslant\; 2C\, 2^{-\frac{m}{2}}
\end{aligned}$$

which means that the partial sums $S_{2^n-1}(t;\omega)$ of $W_t(\omega)$ converge in $L^4(d\omega)$ uniformly for all $t \in [0,1]$. By C12.8 we can extract a subsequence, which converges (uniformly in t) for $\lambda(d\omega)$-almost all ω to $W_t(\omega)$; since for fixed ω the partial sums $t \mapsto S_{2^n-1}(t;\omega)$ are continuous functions of t, this property is inherited by the a.e. limit $W_t(\omega)$.

The above construction is a variation of a theme by Lévy [**?**, Chap. I.1, pp. 15–20] and Ciesielski [**?**]. In one or another form it can be found in many probability textbooks, e.g. Bass [**?**, pp. 11–13] or Steele [**?**, pp. 35–39]. A related construction of Wiener, see Paley and Wiener [**?**, Chapter XI], using random Fourier series, is discussed in Kahane [**?**, §16.1–3].

Appendix E

A summary of the Riemann integral. Additional Material

This is a ***streamlined*** *and* ***extended*** *version of Theorem E.11, pages 344-5 of the printed text*

E.11 Theorem. *The Riemann integral is a positive linear form on the vector lattice $\mathcal{R}[a,b]$, that is, for all $\alpha, \beta \in \mathbb{R}$ and $u, w \in \mathcal{R}[a,b]$ one has*

(i) $\alpha u + \beta w \in \mathcal{R}[a,b]$ *and* $\displaystyle\int_a^b (\alpha u + \beta w)\,dt = \alpha \int_a^b u\,dt + \beta \int_a^b w\,dt$;

(ii) $u \leqslant w \implies \displaystyle\int_a^b u\,dt \leqslant \int_a^b w\,dt$;

(iii) $u \vee w,\ u \wedge w,\ u^+,\ u^-,\ |u| \in \mathcal{R}[a,b]$ *and* $\displaystyle\left|\int_a^b u\,dt\right| \leqslant \int_a^b |u|\,dt$;

(iv) $|u|^p,\ u\,w \in \mathcal{R}[a,b]$, $1 \leqslant p < \infty$.

(v) *If $\phi : \mathbb{R}^2 \to \mathbb{R}$ is a Lipschitz continuous function*[1] *then $\phi \circ (u, w) \in \mathcal{R}[a,b]$.*

Proof. (i) follows immediately from the linearity of the limit criterion in Theorem E.5(iv).

(ii): In view of (i) it is enough to show that $v := w - u \geqslant 0$ entails $\int_a^b v\,dt \geqslant 0$. This, however is clear since $v \in \mathcal{R}[a,b]$ and

$$0 \leqslant \int_{*a}^{b} v = \int_a^b v\,dt.$$

(iii), (iv) follow at once from (v) since the functions $(x,y) \mapsto x \vee y$, $(x,y) \mapsto x \wedge y$, $x \mapsto x \vee 0$, $x \mapsto (-x) \vee 0$ and $x \mapsto |x|$ are clearly Lipschitz continuous.

[1] i.e. for every ball $B_r(0) \subset \mathbb{R}^2$ there is a (so-called *Lipschitz*) constant $L = L(r) < \infty$ such that

$$|\phi(x,x') - \phi(y,y')| \leqslant L\big(|x-y| + |x'-y'|\big)$$

holds for all $(x,x'), (y,y') \in B_r(0)$.

The estimate in part (iii) can be derived from (i) and (ii) by observing that $\pm u \leqslant |u|$ entails $\pm \int_a^b u\,dt \leqslant \int_a^b |u|\,dt$ which implies $\left|\int_a^b u\,dt\right| \leqslant \int_a^b |u|\,dt$.

(v): Let $u, w \in \mathcal{R}[a,b]$. Since any Riemann integrable function is, by definition, bounded, there is some $r < \infty$ such that $-r \leqslant u(x), w(x) \leqslant r$; write $L = L(r)$ for the corresponding Lipschitz constant of $\phi|_{B_r(0)}$. Observe that for any partition $\pi = \{a = t_0 < t_1 < \cdots < t_k = b\}$ of $[a,b]$ we have

$$\begin{aligned}
S^\pi&[\phi \circ (u,w)] - S_\pi[\phi \circ (u,w)] \\
&= \sum_{j=1}^k \Big[\sup_{s\in[t_{j-1},t_j]} \phi(u(s),w(s)) - \inf_{t\in[t_{j-1},t_j]} \phi(u(t),w(t))\Big](t_j - t_{j-1}) \\
&= \sum_{j=1}^k \sup_{s,t\in[t_{j-1},t_j]} \big[\phi(u(s),w(s)) - \phi(u(t),w(t))\big](t_j - t_{j-1}) \\
&\leqslant L \sum_{j=1}^k \sup_{s,t\in[t_{j-1},t_j]} \big[|u(s)-u(t)| + |w(s)-w(t)|\big](t_j - t_{j-1}).
\end{aligned}$$

Because of the symmetric rôles of s and t,

$$\sup_{s,t} |u(s)-u(t)| = \sup_{s,t} \max\{u(s)-u(t), u(t)-u(s)\} = \sup_{s,t}(u(s)-u(t)),$$

and this proves

$$\begin{aligned}
S^\pi&[\phi \circ (u,w)] - S_\pi[\phi \circ (u,w)] \\
&\leqslant L \sum_{j=1}^k \sup_{s,t\in[t_{j-1},t_j]} \big(u(s)-u(t)\big)(t_j - t_{j-1}) \\
&\qquad + L \sum_{j=1}^k \sup_{s,t\in[t_{j-1},t_j]} \big(w(s)-w(t)\big)(t_j - t_{j-1}) \\
&= L\big(S^\pi[u] - S_\pi[u]\big) + L\big(S^\pi[w] - S_\pi[w]\big).
\end{aligned}$$

Thus, if both u and w are Riemann integrable, so is $\phi \circ (u,w)$. ∎