Measures, Integrals and Martingales

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by

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List of misprints and smaller changes to the present text. Last update: October 27, 2008.

PAGE, LINE	READS	SHOULD READ
p 20, Problem 3.9	Is this still true for the family $\mathbb{B}' := \{B_r(x) : x \in \mathbb{Q}^n,\ r \in \mathbb{Q}^+\}$?	Denote by $B_r(x)$ an open ball in $\mathbb{R}^n$ with centre x and radius r. Show that the Borel sets $\mathcal{B}(\mathbb{R}^n)$ are generated by the family of open balls $\mathbb{B} := \{B_r(x) : x \in \mathbb{R}^n,\ r > 0\}$. Is this still true for the family $\mathbb{B}' := \{B_r(x) : x \in \mathbb{Q}^n,\ r \in \mathbb{Q}^+\}$?
p 36, Problem 5.3	for all $A, B \in \mathcal{D}$	for all $A, B \in \mathcal{D}$ with $A \subset B$
p 36, Problem 5.8	Mimic the proof of Theorem 5.8(I)...	Mimic the proof of Theorem 5.8(i)...
p 45, lines 10–11 below	side-lengths $\lim_{j\to\infty}\big(b_k^{(j)} - b_k^{(j)}\big) > 0$	side-lengths $\lim_{j\to\infty}\big(b_k^{(j)} - a_k^{(j)}\big) > 0$
p 54, Problem 7.5, line 13 below	is ‘take out‘ measurable if, and only if,...	is measurable if, and only if,...
p 54, Problem 7.9, line 2 below	Let μ be a measure on $(\mathbb{R}, \mathcal{B}^1)$.	Let μ be a measure on $(\mathbb{R}, \mathcal{B}^1)$ with $\mu[-n, n) < \infty$, $n \in \mathbb{N}$.

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PAGE, LINE	READS	SHOULD READ
p 65, Problem 8.8, hint	$\{f > \alpha\}$	$\{u > \alpha\}$
p 78, Theorem 10.4, proof (i)	*line missing, add* $\longrightarrow$	.... by 9.8(ii). To see the integral formula we assume that $\alpha \leqslant 0$. Then $(\alpha u)^{\pm} = (-\alpha)(-u)^{\pm} = -\alpha u^{\mp}$ and the formula follows directly from Definition 10.1. The case $\alpha \geqslant 0$ follows in a similar way.
p 78, Theorem 10.4, proof (ii)	*line missing, add* $\longrightarrow$	.... by 9.8(iii). For the integral formula observe that $(u+v)^+ - (u+v)^- = u+v = (u^+ + v^+) - (u^- + v^-)$. Thus, $(u+v)^+ + u^- + v^- = (u+v)^- + u^+ + v^+$ and we can integrate this equality and use 9.8(ii) to get $$\int (u+v)^+\,d\mu + \int u^-\,d\mu + \int v^-\,d\mu = \int (u+v)^-\,d\mu + \int u^+\,d\mu + \int v^+\,d\mu.$$ Since all terms are finite, we can rearrange this equality and use Definition 10.1 to see $$\int (u+v)\,d\mu = \int u\,d\mu + \int v\,d\mu$$

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PAGE, LINE	READS	SHOULD READ
p 85, Problem 10.9	$u \in \mathcal{L}^1(P) \iff \sum_{j=0}^{\infty} P(\{u \geqslant j\}) < \infty.$	$u \in \mathcal{L}^1(P) \iff \sum_{j=0}^{\infty} P(\{\lvert u\rvert \geqslant j\}) < \infty.$
p 86, Probl. 10.12(i)	$\mu_*(E) + \mu_*(F) \leqslant \mu_*(E \cup F)$	$\mu_*(E) + \mu_*(F) \leqslant \mu_*(E \,\dot\cup\, F)$
p 96, line 5 above	$\{\Sigma[f] = \sigma[f]\} \subset \{x \,:\, f(x) \text{ is continuous}\} \cup \Pi,$	$\{\Sigma[f] = \sigma[f]\} \supset \{x \,:\, f(x) \text{ is continuous}\},$
p 102, Problem 11.15	Let X be a random variable on …	Let X be a positive random variable on …
p 118, Problem 12.18	The conjugate index is given by $q := 1/(p-1) < 0.$	The conjugate index is given by $q := p/(p-1) < 0.$
p 133, Prob. 13.13(iv) line 14 below	$\Big[\phi(F(s)) - \phi(F(s-)) - \phi'(F(s))\Delta F(s)\Big]$	$\Big[\phi(F(s)) - \phi(F(s-)) - \phi'(F(s-))\Delta F(s)\Big]$
p 133, Problem 13.14, line 9 below	$\mu_f(\{f \geqslant t\})$	$\mu_f(t) := \mu(\{\lvert f\rvert \geqslant t\})$
pp 134-5, Thm 14.1		**The proof of Theorem 14.1 shows more than what is claimed. Add, therefore, the following line in the statement of Theorem 14.1 after formula (14.2):** *In particular, $u \circ T$ is μ-integrable if, and only if, u is $T(\mu)$-integrable.*
p 135, lines 3–5 below	Often we are … Theorem 14.1	**Delete lines 3–5 from below**
p 137, line 5 below	$\ldots = \alpha(\mu \times \nu).$	$\ldots = \alpha(\mu \times \nu)(B).$
p 139, line 13 above	such that $\lVert u - \phi_\epsilon \rVert$	such that $\lVert u - \phi_\epsilon \rVert_p$

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PAGE, LINE	READS	SHOULD READ
p 141, Problem 14.7, line 11 above	$u \in \mathcal{L}^1(\lambda^1)$	$u \in \mathcal{L}^1_+(\lambda^1)$
p 141, Problem 14.9, line 17 above	$\mathcal{L}^p$, $\mathcal{L}^q$	$\mathcal{L}^p(\lambda^n)$, $\mathcal{L}^q(\lambda^n)$
p 141, Prob. 14.10(iii)	$\ldots = \{y \,:\, \forall x \in \operatorname{supp} u \,:\, \lvert x-y\rvert \leqslant \epsilon\}$	$\ldots = \{y \,:\, \exists x \in \operatorname{supp} u \,:\, \lvert x-y\rvert \leqslant \epsilon\}$
p 141, Prob. 14.11(ii)	$u \star w(x) = \ldots$	$v \star w(x) = \ldots$
p 141, Prob. 14.11, line 2 below	…commutativity of the convolution which was used in …	…associativity of the convolution which is implicit in …
p 160, Prob. 15.6(i), line 5 below	$\det D\Phi(x)$	$\lvert D\Phi(x)\rvert^2$
p 160, Prob. 15.6(ii), line 4 below	$\det D\Phi(x)$	$\lvert D\Phi(x)\rvert$
p 161, Prob. 15.6(iv), line 5 above	$\det D\tilde\Phi(x,r) = 1 + (f'(x)^2) - r f''(x)$	$\det D\tilde\Phi(x,r) = \sqrt{1+(f'(x))^2} - \dfrac{r\, f''(x)}{1+(f'(x))^2}$
p 161, Prob. 15.6(vi), line 11 above	$\ldots \big\lvert \det D\tilde\Phi(x,r)\big\rvert\, \lambda^1(dr)$	$\ldots \big\lvert \det D\tilde\Phi(x,s)\big\rvert\, \lambda^1(ds)$
p 161, Prob. 15.6(vii), line 14 above	$\lim_{r\downarrow 0} \lambda^2(C(r)) = \int_{\Phi^{-1}(C)} \lvert \det D\Phi(x)\rvert\, \lambda^1(dx)$	$\lim_{r\downarrow 0} \frac{1}{2r}\lambda^2(C(r)) = \int_{\Phi^{-1}(C)} \lvert \det D\tilde\Phi(x,0)\rvert\, \lambda^1(dx)$
p 161, Prob. 15.7(ii), line 12 below	$\int_M u(r\,\xi)\, r^n\, d\lambda_M(\xi) = \ldots$	$\int_M u(r\,\xi)\, r^n\, d\lambda^n(\xi) = \ldots$
p 161, Prob. 15.7(iii), line 8 below	$= \int_{(0,\infty)} \int_{\{\lVert x\rVert=1\}} u(r\,x)\, \sigma(dx)\, \lambda^1(dr).$	$= \int_{(0,\infty)} \int_{\{\lVert x\rVert=1\}} r^{n-1}\, u(r\,x)\, \sigma(dx)\, \lambda^1(dr).$

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PAGE, LINE	READS	SHOULD READ
pp 165-6, Proof of Theorem 16.6		**a more detailed version of the proof of (iii)⇒(ii) is now available online**
p 173, Prob. 16.1, line 19 below	$f_j \xrightarrow{j\to\infty} 0$	$u_j \xrightarrow{j\to\infty} 0$
p 173, Prob. 16.3, line 13 below	be a measure space	be a σ-finite measure space
p 174, Prob. 16.8	ρ_μ, g_μ, d_μ *etc.*	ρ_P, g_P, d_P *etc.*
p 177, line 8 below	Therefore (ii) means	Therefore (i) means
p 188, Prob. 17.1	$u_0 = \int u_1\,d\mu$	$u_0 = \mu(X)^{-1} \int u_1\,d\mu$
p 188, Prob. 17.7	**delete the unneccessary** $u_0 := \int u_1\,d\mu$	
p 191, line 11 above	$(b-a)U([a,b];N)$	$(b-a)U([a,b];N) - (X_N - a)^-$
p 191, line 14 above	$(b-a)\int_A U([a,b];N)\,d\mu$	$(b-a)\int_A U([a,b];N)\,d\mu - \int_A (X_N - a)^-\,d\mu$
p 191, line 14 above	**delete the unneccessary** $\leqslant \int_A (X_N - a)^+\,d\mu$	
p 200, Prob. 18.6(ii)	Let $\mathcal{A}_n = \sigma(X_1, \dots, X_{2^n})$. Show that $A_n := \{X_{2^{n-1}+1} + \cdots + X_{2^n} = 0\}$ is for each $n \in \mathbb{N}$ contained in $\mathcal{A}_n$ and	Let $\mathcal{A}_n = \sigma(X_1, \dots, X_{n^2})$. Show that $A_n := \{X_{(n-1)^2+2} + \cdots + X_{n^2} = 0\}$ is for each $n \in \mathbb{N}, n \geqslant 2$ contained in $\mathcal{A}_n$ and
p 200, Prob. 18.6(iii)	$n \in \mathbb{N}_0$	$n \in \mathbb{N}, n \geqslant 2$

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PAGE, LINE	READS	SHOULD READ
p 200, Prob. 18.6(iii)	$M_0 := 0$, **(twice)** X_{2^n+1}	**(twice)** $M_2 := 0$, X_{n^2+1}
p 223, Prob. 19.9	$(X, \mathcal{A}, \mu)$	$(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mu)$
p 223, Prob. 19.9	**drop the hint**	
p 225, Prob. 19.14	A martingale $(u_j, \mathcal{A}_j)_{j\in\mathbb{N}}$ is called	A martingale $(u_j, \mathcal{A}_j)_{j\in\mathbb{N}}$ on a σ-finite filtered measure space is called
p 247, Prob. 21.13 (iv)–(vi)	**add the following** $\longrightarrow$	$u \in L^1(\mathcal{A}) \cap L^2(\mathcal{A})$
p 261, line 6 above	$\mathbf{1}_{\{0<\lvert E^{\mathcal{G}}u\rvert<\infty\}}$	$\mathbf{1}_{\{0<\lvert \mathbf{E}^{\mathcal{G}}u\rvert<\infty\}}$
p 261, line 14 below	$\lVert \mathbf{E}^{\mathcal{G}} u\, \mathbf{1}_{G_n} \rVert_1 = \langle \mathbf{E}^{\mathcal{G}} u, \mathbf{1}_{G_n} \rangle \overset{21.4(\text{iii}),(\text{ix})}{=} \langle u, \mathbf{1}_{G_n} \rangle \leqslant \lVert u \rVert_1$	$\lVert \mathbf{E}^{\mathcal{G}} u\, \mathbf{1}_{G_n} \rVert_1 \overset{22.4(\text{xii})}{\leqslant} \langle \mathbf{E}^{\mathcal{G}} \lvert u\rvert, \mathbf{1}_{G_n} \rangle \overset{22.4(\text{iii}),(\text{ix})}{=} \langle \lvert u\rvert, \mathbf{1}_{G_n} \rangle \leqslant \lVert u \rVert_1$
p 274, problem 23.4	$M(\mathcal{A})$	$M^+(\mathcal{A})$
p 274, problem 23.6	$L^1(\mathcal{A})$	$L^1_+(\mathcal{A})$
p 275, problem 23.11	**twice:** s_j	**twice:** u_j
p 281, lines 10–11 above	$C[0,1] \subset L^1[0,1] \subset L^2[0,1]$	$C[0,1] \subset L^2[0,1] \subset L^1[0,1]$

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PAGE, LINE	READS	SHOULD READ
p 283, (24.4)	... from the classical result that $\int_{(-\pi,\pi)} \cos kx\, \sin \ell x\, dx = \begin{cases} 0, & \text{if } k \neq \ell, \\ \pi, & \text{if } k = \ell \geqslant 1, \\ 2\pi, & \text{if } k = \ell = 0, \end{cases} \quad (24.4)$	... from the classical result that for $j, k \in \mathbb{N}_0$ and $\ell, m \in \mathbb{N}$ $\int_{(-\pi,\pi)} \cos jx\, \cos kx\, dx = \begin{cases} 0, & \text{if } k \neq j, \\ \pi, & \text{if } k = j \geqslant 1, \\ 2\pi, & \text{if } k = j = 0, \end{cases}$ $\int_{(-\pi,\pi)} \sin \ell x\, \sin mx\, dx = \begin{cases} 0, & \text{if } \ell \neq m, \\ \pi, & \text{if } \ell = m \end{cases} \quad (24.4)$ $\int_{(-\pi,\pi)} \cos kx\, \sin \ell x\, dx = 0 \ \text{ for all } \ k, \ell$
p 286–7, Lemma 24.11, Corollary 24.12	**replace <u>throughout</u>** "$C[-\pi,\pi]$" **with**	"$C_{\text{per}}[-\pi,\pi]$" (the set of 2π-periodic continuous functions $C_{\text{per}}[-\pi,\pi] = \{u \in C[-\pi,\pi] : u(\pi) = u(-\pi)\}$)
p 297, line 9 below	$\left(\mathbf{E}^{\mathcal{A}^{\Delta}_{-M}}\right)_{M\in\mathbb{N}}$	$\left(\mathbf{E}^{\mathcal{A}^{\Delta}_{-M}} u\right)_{M\in\mathbb{N}}$
p 309, line 5 below	most prominent stochastic process	most prominent stochastic processes
p 310, line 3 above	$B \in \mathcal{B}[0,1]$	$B \in \mathcal{B}(\mathbb{R})$

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PAGE, LINE	READS	SHOULD READ
p 310, lines 9-12 above	$$\langle \mathbf{1}_{[0,t]}, H_n\rangle = \int_0^t H_n(x)\,dx = \tfrac12\, 2^{k/2}\int_0^t H_1(2^kx-j)\,dx = \tfrac12\, 2^{-k/2}\,F_n(t),$$ where $F_1(t) = \int_0^t H_1(x)\,dx\,\mathbf{1}_{[0,1]}(t) = 2t\mathbf{1}_{[0,\frac12)}(t) - (2t-2)\mathbf{1}_{[\frac12,1]}(t)$ is a tent-function and $F_n(t) := F_1(2^kt-j)$. Since $0 \leqslant F_n \leqslant 1$, we see $$\sum_{n=0}^\infty \lvert\langle \mathbf{1}_{[0,t]}, H_n\rangle\rvert^2 \leqslant \frac14 \sum_{n=0}^\infty 2^{-k} = \frac12\,,$$	$$\langle \mathbf{1}_{[0,t]}, H_n\rangle = \int_0^t H_n(x)\,dx = 2^{k/2}\int_0^t H_1(2^kx-j)\,dx = 2^{-k/2}\,F_n(t),$$ where $F_1(t) = \int_0^t H_1(x)\,dx\,\mathbf{1}_{[0,1]}(t) = t\mathbf{1}_{[0,\frac12)}(t) - (t-1)\mathbf{1}_{[\frac12,1]}(t)$ is a tent-function and $F_n(t) := F_1(2^kt-j)$. Since $0 \leqslant F_n \leqslant \frac12$, we see $$\sum_{n=1}^\infty \lvert\langle \mathbf{1}_{[0,t]}, H_n\rangle\rvert^2 \leqslant \frac14 \sum_{n=1}^\infty 2^{-n} = \frac14\,,$$
p 310, footnote	$e^{\sigma_n^2\xi^2/2}$ (twice) resp. $e^{\sigma^2\xi^2/2}$	$e^{-\sigma_n^2\xi^2/2}$ (twice) resp. $e^{-\sigma^2\xi^2/2}$

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PAGE, LINE	READS	SHOULD READ
p 311, line 10-13 below	this paragraph is corrupted (the exponent 4 is missing) and should be corrected as follows (see also bonus material for an extended version):	Let us finally turn to the dependence of $W_t(\omega)$ on t. Note that for $2^m - 1 = M < N = 2^n - 1$ $$\int_0^1 \sup_{t\in[0,1]} \left\|S_N(t;\omega) - S_M(t;\omega)\right\|^4 d\omega$$ $$= \int_0^1 \sup_{t\in[0,1]} \left\| \sum_{k=m}^{n-1} \sum_{j=0}^{2^k-1} e_{2^k+j}(\omega)\langle \mathbf{1}_{[0,t]}, H_{2^k+j}\rangle \right\|^4 d\omega$$ $$= \int_0^1 \sup_{t\in[0,1]} \left\| \sum_{k=m}^{n-1} 2^{-\frac{k}{8}} \left[\sum_{j=0}^{2^k-1} 2^{\frac{k}{8}} e_{2^k+j}(\omega)\langle \mathbf{1}_{[0,t]}, H_{2^k+j}\rangle \right] \right\|^4 d\omega$$ $$\leqslant \int_0^1 \sup_{t\in[0,1]} \underbrace{\left[\sum_{k=m}^{n-1} 2^{-\frac{k}{6}} \right]}_{\leqslant 10} \cdot \sum_{k=m+1}^{n} \left[\sum_{j=0}^{2^k-1} 2^{\frac{k}{8}} e_{2^k+j}(\omega)\langle \mathbf{1}_{[0,t]}, H_{2^k+j}\rangle \right]^4 d\omega$$ where we used Hölder's inequality for the outer sum with $p = \frac{4}{3}$ and $q = 4$. Since the functions $t \mapsto \langle \mathbf{1}_{[0,1]}, H_{2^k+j}\rangle$ have for $j = 0, \ldots, 2^k-1$ disjoint supports and are bounded by $\frac{1}{2}\, 2^{-k/2}$, we find
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PAGE, LINE	READS	SHOULD READ
continues		$\int_0^1 \sup_{t\in[0,1]} \left\|S_N(t;\omega) - S_M(t;\omega)\right\|^4 d\omega \leqslant 10 \int_0^1 \sup_{t\in[0,1]} \sum_{k=m}^{n-1} \sum_{j=0}^{2^k-1} 2^{\frac{k}{2}} e^4_{2^k+j}(\omega) \langle \mathbf{1}_{[0,t]}, H_{2^k+j}\rangle^4 d\omega \leqslant 10 \sum_{k=m}^{n-1} \sum_{j=0}^{2^k-1} 2^{\frac{k}{2}} \underbrace{\int_0^1 e^4_{2^k+j}(\omega)\, d\omega}_{=C\ \forall j,k} 2^{4-2k} = 10C \sum_{k=m}^{n-1} 2^{\frac{k}{2}} \cdot 2^k \cdot 2^{4-2k} \leqslant \frac{5C}{4} 2^{-\frac{m}{2}}$
p 312, Prob. 24.5	$\sin^{k+1} x$	$\sin^{2k+1} x$
p 313, line 9 above	$[-\infty, +\infty)$ and $(-\infty, +\infty]$, respectively.	$[-\infty, +\infty]$.
p 320, Prop. B.6	**(ii)** Closed subsets of compact sets are closed.	**(ii)** Closed subsets of compact sets are compact.
p 365, Ref. [22]	Kaczmarz, S. and H. Steinhaus, *Theorie der Orthogonalreihen* (2nd corr. reprint), New York: Chelsea, 1951. First edition appeared under the same title with PWN, Warsaw: Monogr. Mat. Warszawa vol. VI, 1935.	Kaczmarz, S. and H. Steinhaus, *Theorie der Orthogonalreihen* (2nd corr. reprint), New York: Chelsea, 1951. First edition was published in Warsaw: PWN, Monogr. Mat. Warszawa vol. **VI**, 1935.
p 365, Ref. [34]	Paley, R. E. A. C. and N. Wiener, Providence (RI): *Fourier Transforms in the Complex Domain*, American Mathematical Society, Coll. Publ. vol. **19**, 1934.	Paley, R. E. A. C. and N. Wiener, *Fourier Transforms in the Complex Domain*, Providence (RI): American Mathematical Society, Coll. Publ. vol. **19**, 1934.

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PAGE, LINE	READS	SHOULD READ
p 372, index item *completion and inner/outer regularity*	160, (Pr 15.6)	160, (Pr 15.3)

I would like to thank the following readers for their comments and their contributions to this list:
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