"LECTURE NOTES ON FOURIER SERIES"
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1. Conventions and first concepts

The purpose of these notes is to introduce the Fourier series of a function in $L^2([-\pi,\pi])$. We shall study convergence properties of the Fourier series. We also construct orthonormal bases for the Hilbert space $L^2([-\pi,\pi])$.

Let λ be Lebesgue measure on $[-\pi,\pi]$. We write dx for $d\lambda(x)$. The space $L^2([-\pi,\pi])$ consists of equivalence classes of measurable functions $f : [-\pi,\pi] \to \mathbb{C}$ such that

$$\int_{[-\pi,\pi]} |f|^2 d\lambda < \infty;$$

the usual convention is that we write $f \in L^2([-\pi,\pi])$ and remember that this means $[f] \in L^2([-\pi,\pi])$. Since functions in the same equivalence class have the same integral, there is no ambiguity in the convention as long as we compute integrals. However *we should avoid notation like* $[f](x_0)$ *for a specified point* x_0 because we can change the value of a representative for a class $[f]$ on a set of measure zero (a point $\{x_0\}$ has measure zero) without changing the class.

Recall that $L^2([-\pi,\pi])$ is a Hilbert space with inner-product given by

$$\langle f, g\rangle = \int_{-\pi}^{\pi} \overline{f(x)} g(x) dx. \tag{1}$$

The norm on $L^2([-\pi,\pi])$ comes from the inner product by setting $\|f\|_2 = \langle f, f\rangle^{1/2}$. Theorem 8.6 in [3] applied to $p = 2$ says that $L^2([-\pi,\pi])$ is complete for the norm $\|\cdot\|_2$.

Thus the space $L^2([-\pi,\pi])$ with the inner product defined in (1) is a Hilbert space and from now on we *denote it H*. For every $n \in \mathbb{Z}$ we define functions in H by

$$e_n(x) = \frac{1}{\sqrt{2\pi}} e^{inx} \text{ for } x \in [-\pi,\pi]. \tag{2}$$

Theorem 1.1. *The system $\{e_n\}_{n\in\mathbb{Z}}$ is an orthonormal basis in H.*

There are many ways to prove this result. The method we shall use (from [1] and [2]) will provide us with useful criteria for convergence of Fourier series.

We first note that $\{e_n\}_{n\in\mathbb{Z}}$ is an orthonormal system in H. (Proof: exercise. Since the functions e_n are continuous and the interval $[-\pi,\pi]$ is compact, the Lebesgue integral is the same as the Riemann integral.)

Definition 1.2. *Let $f \in H$. For every $n \in \mathbb{Z}$, the n'th Fourier coefficient of f is given by*

$$c_n(f) = \langle e_n, f\rangle = \frac{1}{\sqrt{2\pi}}\int_{-\pi}^{\pi} f(x)e^{-inx}dx.$$

The series $\sum_{n\in\mathbb{Z}} c_n(f)e_n$ is the Fourier series of f.

We note first that $\sum_{n\in\mathbb{Z}}\langle e_n, f\rangle e_n$ is a well-defined element in H by [3, Theorem 2.2].

Second, suppose that $f \in \mathcal{L}^1([-\pi,\pi])$ (or by our convention $f \in L^1([-\pi,\pi])$). Then $|f(x)e^{-inx}| = |f(x)|$ for all $x \in [-\pi,\pi]$, so the function $x \mapsto f(x)e^{-inx}$ is in $\mathcal{L}^1([-\pi,\pi])$ as well. Thus the Fourier coefficients $c_n(f)$, $n \in \mathbb{Z}$ are also defined for $f \in \mathcal{L}^1([-\pi,\pi])$, and we can talk of the Fourier series of such f. Moreover, it makes sense to define the Fourier series of $f \in L^1([-\pi,\pi])$ because $\sum_{n\in\mathbb{Z}}\langle e_n, f_1\rangle e_n$ and $\sum_{n\in\mathbb{Z}}\langle e_n, f_2\rangle e_n$ are the same when $[f_1] = [f_2] \in L^1([-\pi,\pi])$.

Note that the measure space $([-\pi,\pi],\lambda)$ is finite. We claim that $L^2([-\pi,\pi]) \subset L^1([-\pi,\pi])$. To prove this, note that the constant function 1 on $[-\pi,\pi]$ is in $L^2([-\pi,\pi])$ and has norm equal to $\sqrt{2\pi}$, so for any $f \in L^2([-\pi,\pi])$ we have $|\langle f, 1\rangle| \le \|f\|_2\|1\|_2 = \sqrt{2\pi}\|f\|_2$ by the Cauchy-Schwarz-Buniakowski inequality. Thus $\int_{[-\pi,\pi]}|f|d\lambda < \infty$ so $f \in L^1([-\pi,\pi])$.

Let's look at what convergence of $\sum_{n\in\mathbb{Z}}\langle e_n, f\rangle e_n$ in H means. It means convergence in the $\|\cdot\|_2$ norm, so there is a vector $g \in H$ such that $\|\sum_{n=-N}^{N}\langle e_n, f\rangle e_n - g\|_2 \to 0$ as $N \to \infty$. Thus

$$\lim_{N\to\infty}\int_{-\pi}^{\pi} |(\sum_{n=-N}^{N}\langle e_n, f\rangle e_n)(x) - g(x)|^2 dx = 0. \tag{3}$$

We let $s_N = \sum_{n=-N}^{N}\langle e_n, f\rangle e_n$ be the partial sums of the Fourier series of f, for $N \in \mathbb{N}$. Thus $s_N(x) = \sum_{n=-N}^{N} c_n(f)e_n(x)$ for $x \in [-\pi,\pi]$. Note that (3) only asserts convergence in the form

$$\lim_{N\to\infty}\int_{-\pi}^{\pi}|s_N(x) - g(x)|^2 dx = 0.$$

It *does not* assert that $\sum_{n\in\mathbb{Z}} c_n(f)e_n(x)$ is convergent pointwise at any x. Even if this series should be convergent at some x in $[-\pi,\pi]$, there is a priori no reason why the limit should be $f(x)$.

We point out that a deep result of L. Carleson from 1966 shows that for a function $f \in \mathcal{L}^2([-\pi,\pi])$ its Fourier series at x converges with sum $f(x)$ for almost all $x \in [-\pi,\pi]$. This is though not something we can prove here.

Note that after we prove Theorem 1.1, we will know that $g = f$ in H.

What we aim to achieve: for f with additional properties we will see that the Fourier series does converge pointwise (and it can do so even uniformly).

2. Pointwise convergence of Fourier series

The following is an important result.

Theorem 2.1. *(Riemann-Lebesgue lemma) Suppose $f \in L^1([-\pi,\pi])$. Then $c_n(f) \to 0$ as $n \to \pm\infty$.*

Proof. Assume first that $f \in L^2([-\pi,\pi])$. From Bessel's inequality for the index set $J = \mathbb{Z}$ (see (2.7) in [3]) we have

$$\sum_{n\in\mathbb{Z}} |c_n(f)|^2 \le \|f\|_2^2 < \infty.$$

Thus $\sum_{n\in\mathbb{Z}} |c_n(f)|^2$ is convergent and so $c_n(f) \to 0$ as $n \to \pm\infty$.

Now let $f \in \mathcal{L}^1([-\pi,\pi])$, and define

$$f_N(x) = \begin{cases} f(x), & \text{if } |f(x)| \le N \\ 0, & \text{if } |f(x)| > N \end{cases}$$

for each $N \in \mathbb{N}$. Then $f_N \in \mathcal{L}^2([-\pi,\pi])$, and $|f(x) - f_N(x)| \to 0$ as $N \to \infty$, for λ-almost all x in $[-\pi,\pi]$. But $|f - f_N| \le |f|$, so by the dominated convergence theorem we get

$$\lim_{N\to\infty} \int_{-\pi}^{\pi} |f(x) - f_N(x)|dx = 0.$$

Let $\varepsilon > 0$. Choose N big enough that $\int_{-\pi}^{\pi} |f(x) - f_N(x)|dx < \varepsilon\sqrt{\frac{\pi}{2}}$, and then choose N_0 such that $|c_n(f_N)| < \varepsilon/2$ for $|n| \ge N_0$ (by the first part of the proof, applied to f_N). Then

$$|c_n(f - f_N)| \le \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} |f(x) - f_N(x)|dx < \frac{\varepsilon}{2},$$

and therefore $|c_n(f)| \le |c_n(f - f_N)| + |c_n(f_N)| < \varepsilon$ for $|n| \ge N_0$. This proves the theorem. $\square$

We shall need to extend functions on $(-\pi, \pi)$ to 2π-periodic functions on $\mathbb{R}$. If f is defined on $(-\pi, \pi)$, we let $\bar{f}(x + 2n\pi) := f(x)$ be its extension to the interval $((2n-1)\pi, (2n+1)\pi)$ for $n \in \mathbb{Z}$, where $x \in (-\pi, \pi)$). So any function $f \in \mathcal{L}^1((-\pi, \pi))$ has en extension $\bar{f}$ to a 2π-periodic function on $\mathbb{R}$.

For a 2π-periodic function f on $\mathbb{R}$ which is integrable on $[-\pi, \pi]$ we have

$$\int_{-\pi}^{\pi} f(y)dy = \int_{-\pi+a}^{\pi+a} f(y)dy \tag{4}$$

for all $a \in \mathbb{R}$. Next we see that integration with respect to Lebesgue measure has a certain translation property.

Lemma 2.2. *Let λ be Lebesgue measure on the Borel σ-algebra $\mathcal{B}$. For $A \in \mathcal{B}$ and $a \in \mathbb{R}$ define $a + A = \{a + x \mid x \in A\}$. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is measurable and nonnegative or integrable, and define $g : \mathbb{R} \to \mathbb{R}$ by $g(x) = f(x + a)$ for all $x \in \mathbb{R}$. Then*

$$\int_A g d\lambda = \int_{a+A} f d\lambda.$$

When $A = \mathbb{R}$ we therefore have $\int f(x+a)dx = \int f(x)dx$ for all $a \in \mathbb{R}$.

(Proof: either from the example following Theorem 7.31 in [3] where $n = 1$ or prove directly as follows: first assume f is a simple nonnegative function. You will need to show that $\lambda(A) = \lambda(a + A)$, and this follows by first assuming that A is an interval, so that λ is interval length. For an arbitrary Borel subset A use that λ is the outer measure defined from interval length. Extend to measurable nonnegative functions by approximating with simple functions and using monotone convergence theorem. Then extend to integrable functions by linearity of the integral.)

Similar to Lemma 2.2 the following holds for a function $f : \mathbb{R} \to \mathbb{R}$ which is measurable and nonnegative or integrable and for $a \neq 0$:

$$\int f(ax)dx = \frac{1}{|a|} \int f(x)dx.$$

By Lemma 2.2, for a 2π-periodic function f on $\mathbb{R}$ which is integrable on $[-\pi, \pi]$ and for any $a \in \mathbb{R}$ we have

$$\int_{-\pi}^{\pi} f(y)dy = \int_{-\pi+a}^{\pi+a} f(y)dy = \int_{-\pi}^{\pi} f(y+a)dy. \tag{5}$$

For $N \in \mathbb{N}$, the *n'th Dirichlet kernel* is given by the quotient series

$$D_N(t) = \frac{1}{2\pi} \sum_{n=-N}^{N} e^{int}$$

for $t \in \mathbb{R}$. The function $t \to e^{it}$ is periodic with period 2π, so the same is true of D_N. Moreover, D_N is continuous. We have that

$$\sum_{n=-N}^{N} e^{int} = e^{-iNt} \sum_{n=0}^{2N} (e^{it})^n = \begin{cases} e^{-iNt} \frac{e^{i(2N+1)t}-1}{e^{it}-1}, & \text{if } t \notin 2\pi\mathbb{Z} \\ 2N+1, & \text{if } t \in 2\pi\mathbb{Z}. \end{cases}$$

Recall De Moivre's formula $e^{i\theta} = \cos\theta + i \sin\theta$ for a real angle θ. Then $\cos\theta = 1/2(e^{i\theta} + e^{-i\theta})$ and $\sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$. Therefore

$$D_N(t) = \begin{cases} \frac{\sin(N+\frac{1}{2})t}{2\pi \sin \frac{1}{2}t}, & \text{if } t \notin 2\pi\mathbb{Z} \\ \frac{2N+1}{2\pi}, & \text{if } t \in 2\pi\mathbb{Z}, \end{cases} \tag{6}$$

which shows that D_N is an even function (meaning $D_N(-t) = D_N(t)$ for all t). From the definition of D_N we have

$$\int_{-\pi}^{\pi} D_N(t)dt = 2\int_0^{\pi} D_N(t)dt = 1. \tag{7}$$

Theorem 2.3. *Given $f \in \mathcal{L}^1([-\pi,\pi])$, let $\bar{f}$ be its extension to a 2π-periodic function on $\mathbb{R}$. Suppose $x_0 \in \mathbb{R}$ and $s \in \mathbb{C}$ are such that the function*

$$g(x) = \frac{\bar{f}(x_0+x) + \bar{f}(x_0-x) - 2s}{x} \quad \textit{for } x \in (0,\infty)$$

is integrable on an interval $(0,\delta]$ for some $\delta > 0$. Then the Fourier series of f is convergent at x_0 with sum $\sum_{n\in\mathbb{Z}} c_n(f)e_n(x_0) = s$.

The assumption that g is integrable on $(0,\delta]$ for some $\delta > 0$ implies that g is integrable over $[\delta, a]$ for any $a > \delta$ because $|g(x)| \le \frac{1}{\delta}(|\bar{f}(x_0 + x)| + |\bar{f}(x_0 - x)| + 2|s|)$ for $x \in [\delta, a]$. Thus g will be integrable over any interval $(0,\delta]$ with $\delta > 0$.

Proof. We first compute that

$$\begin{aligned} s_N(x_0) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{n=-N}^{N} f(x) e^{in(x_0-x)} dx \\ &= \int_{-\pi}^{\pi} D_N(x_0-x) f(x) dx. \end{aligned}$$

Since D_N is 2π-periodic, (5) gives $s_N(x_0) = \int_{-\pi}^{\pi} D_N(x) f(x_0 + x)dx$. Since the Lebesgue integral is invariant under the transformation $y \to -y$ and since D_N is an even function, we get

$$\begin{aligned} s_N(x_0) &= \int_0^{\pi} D_N(x)f(x_0+x)dx + \int_{-\pi}^{0} D_N(x)f(x_0+x)dx \\ &= \int_0^{\pi} D_N(x)f(x_0+x)dx + \int_0^{\pi} D_N(-x)f(x_0-x)dx \\ &= \int_0^{\pi} D_N(x)(f(x_0+x)+f(x_0-x))dx. \end{aligned}$$

By (7) we can write $s = 2s\int_0^{\pi} D_N(x)dx$, and therefore

$$\begin{aligned} s_N(x_0) - s &= \int_0^{\pi} D_N(x)(f(x_0+x)+f(x_0-x)-2s)dx \\ &= \frac{1}{2\pi}\int_0^{\pi} \sin((N+\frac{1}{2})x)g(x)\frac{x}{\sin(\frac{x}{2})}dx. \end{aligned}$$

Since g is integrable on an interval $(0,\delta]$ with $\delta > 0$ and $x \to \frac{x}{\sin(\frac{x}{2})}$ is bounded, and therefore integrable on $(0,\pi]$, we can define a function h in $\mathcal{L}^1([-\pi,\pi])$ by setting $h(x) = g(x)\frac{x}{\sin(\frac{x}{2})}$ for $x \in (0,\delta]$ and $h(x) = 0$ for $x \in [-\pi, 0]$. Then also

$$h_1(x) = \frac{1}{2i\sqrt{2\pi}}h(x)e^{\frac{i}{2}x} \text{ and } h_2(x) = \frac{1}{2i\sqrt{2\pi}}h(x)e^{-\frac{i}{2}x}$$

belong to $\mathcal{L}^1([-\pi,\pi])$, so by Theorem 2.1 we have $c_{-N}(h_1) \to 0$ and $c_N(h_2) \to 0$ for $N \to \infty$.

Using that $\sin(\theta) = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$ we obtain

$$s_N(x_0) - s = \frac{1}{2\pi}\int_0^{\pi} \sin((N+\frac{1}{2})x)h(x)dx = c_{-N}(h_1) - c_N(h_2).$$

Hence $s_N(x_0) \to s$ when $N \to \infty$, as claimed. □

Definition 2.4. *A function f defined on an interval (a,b) is* piecewise continuous *if there is a finite set of points $a = t_0 < t_1 < \cdots t_{k-1} < t_k = b$ such that f is continuous at $x \in (a,b) \setminus \{t_1, \ldots, t_{k-1}\}$ and the limits*

$$\begin{aligned} f(t_i+) &= \lim_{\varepsilon\to 0,\varepsilon>0} f(t_i+\varepsilon) \ \textit{(right limit)} \\ f(t_j-) &= \lim_{\varepsilon\to 0,\varepsilon>0} f(t_j-\varepsilon) \ \textit{(left limit)} \end{aligned}$$

exist and are finite for all $i = 0, \ldots, k-1$ and all $j = 1, \ldots, k$.

In particular only the values $f(a+)$ and $f(b-)$ are known. Note that if we want to extend f to a periodic function $\bar{f}$ on $\mathbb{R}$ we must define

$$\bar{f}(a-) := f(b-) \text{ and } \bar{f}(b+) := f(a+).$$

This applies in particular when we start with a piecewise continuous function on $(-\pi, \pi)$ and we consider its 2π-periodic extension $\bar{f}$ to $\mathbb{R}$.

Corollary 2.5. *Suppose f is a 2π-periodic function on $\mathbb{R}$ which is piecewise continuous on $(-\pi, \pi)$, with points of discontinuity $t_0, t_1, \dots, t_k$ as in Definition 2.4. Assume f has derivatives from left and right at the point $x_0 \in \mathbb{R}$, in the sense that*

$$f'(x_0+) = \lim_{\varepsilon \to 0, \varepsilon > 0} \frac{f(x_0 + \varepsilon) - f(x_0+)}{\varepsilon} \tag{8}$$

$$f'(x_0-) = \lim_{\varepsilon \to 0, \varepsilon > 0} \frac{f(x_0 - \varepsilon) - f(x_0-)}{\varepsilon} \tag{9}$$

both exist and are finite. Then the Fourier series of f converges at the point x_0 and has sum $s = \frac{1}{2}(f(x_0+) + f(x_0-))$.

If (8) and (9) exist and are finite, and f is continuous at x_0, then the Fourier series of f is convergent with sum $f(x_0)$.

Proof. Since f is continuous and bounded on each of the intervals $(-\pi, t_1)$, $(t_1, t_2), \dots, (t_{k-1}, \pi)$, its restriction to $[-\pi, \pi]$ is a function in $\mathcal{L}^1([-\pi, \pi])$. We want to apply Theorem 2.3 (and we denote f both the function in $\mathcal{L}^1([-\pi, \pi])$ and its 2π-periodic extension.)

Take $s = \frac{1}{2}(f(x_0+) + f(x_0-))$. Then g from Theorem 2.3 is

$$g(x) = \frac{f(x_0 + x) - f(x_0+)}{x} + \frac{f(x_0 - x) - f(x_0-)}{x}.$$

By the assumption on f we have

$$\lim_{\varepsilon \to 0, \varepsilon > 0} g(\varepsilon) = f'(x_0+) + f'(x_0-) < \infty.$$

This shows that g is piecewise continuous, and therefore integrable, on an interval $(0, \delta]$ for some $\delta > 0$. The claim of the Corollary follows from Theorem 2.3. $\square$

Example 2.6. *Compute the Fourier series of the function*

$$f(x) = \begin{cases} 0, & \text{if } -\pi < x < 0 \\ 1, & \text{if } 0 \leq x < \pi. \end{cases}$$

We compute that $c_0(f) = \frac{1}{\sqrt{2\pi}} \int_0^\pi 1dx = \frac{\sqrt{\pi}}{\sqrt{2}}$. *For* $n \neq 0$ *we have*

$$\begin{aligned} c_n(f) &= \frac{1}{\sqrt{2\pi}} \int_0^\pi e^{-inx} dx = \frac{1}{\sqrt{2\pi}} \frac{e^{-inx}}{-in}\Big|_0^\pi \\ &= \frac{-1}{\sqrt{2\pi} in}[(-1)^n - 1] \\ &= \begin{cases} 0 & \text{if } n \text{ even} \\ \frac{\sqrt{2}}{\sqrt{\pi} in} & \text{if } n \text{ odd }. \end{cases} \end{aligned}$$

So the Fourier series of f *is* $c_0(f)e_0(x) + \sum_{n\in\mathbb{Z}, n\neq 0} c_n(f)e_n(x)$. *Since every* $n < 0$ *is of form* $-n$ *for* $n > 0$ *we can write the series as*

$$\frac{1}{2} + \sum_{n=1, n \text{ odd}}^{\infty} \frac{1}{\pi in} e^{inx} + \sum_{n=1, n \text{ odd}}^{\infty} \frac{-1}{\pi in} e^{-inx}.$$

Now we use that $\sin nx = \frac{1}{2\pi}(e^{inx} - e^{-inx})$ *to write the series in the final form*

$$\frac{1}{2} + \sum_{n=1, n \text{ odd}}^{\infty} \frac{2}{\pi n} \sin nx = \frac{1}{2} + \sum_{k=0}^{\infty} \frac{2}{\pi(2k+1)} \sin{(2k+1)x}.$$

The function f *is piecewise continuous on* $(-\pi, \pi)$ *with a discontinuity at* $x = 0$, *so its* 2π*-periodic extension to* $\mathbb{R}$ *is piecewise continuous with set of discontinuities equal to* $k\pi\mathbb{Z}$ *with* $k \in \mathbb{Z}$. *Also,* f *is differentiable from left and right at all points except* $x_0 \in \{k\pi\mathbb{Z}\}$ *for which we have* $f'(x_0+) = f'(x_0-) = 0$. *Corollary 2.5 implies that the Fourier series of* f *is pointwise convergent at* x_0 *with sum* $f(x_0)$ *for all* $x_0 \in (-\pi, \pi)$ *except* 0. *At* $x_0 = 0$ *(and at all points in* $\{k\pi\mathbb{Z}\}$*) the sum of the series is* $s = \frac{1}{2}(f(k\pi+) + f(k\pi-)) = \frac{1}{2}(1+0) = \frac{1}{2}$. *In all we have a convergent Fourier series with*

$$\frac{1}{2} + \sum_{k=0}^{\infty} \frac{2}{\pi(2k+1)} = \begin{cases} 0 & \text{if } -\pi < x < 0 \\ \frac{1}{2} & \text{if } x = -\pi \text{ or } x = 0 \\ 1 & \text{if } 0 < x < \pi. \end{cases}$$

3. Exercises

Exercise 1. Show that $\{e_n\}_{n\in\mathbb{Z}}$ is an orthonormal system in H.

Exercise 2. Show that the system

$$\{\frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}} \cos x, \frac{1}{\sqrt{\pi}} \sin x, \frac{1}{\sqrt{\pi}} \cos 2x, \frac{1}{\sqrt{\pi}} \sin 2x, \dots\}$$

is an orthonormal system in H. (You will need to use the trigonometric identities $\cos\alpha\cos\beta = \frac{1}{2}(\cos(\alpha-\beta)+\cos(\alpha+\beta))$, $\sin\alpha\sin\beta = \frac{1}{2}(\cos(\alpha-\beta)-\cos(\alpha+\beta))$ and $\sin\alpha\cos\beta = \frac{1}{2}(\sin(\alpha+\beta)+\sin(\alpha-\beta))$.)

Exercise 3. Compute the Fourier series of the function $f(x) = x$, $x \in [-\pi, \pi]$.

Exercise 4. Compute the Fourier series of the function

$$f(x) = \begin{cases} -1, & \text{if } -\pi < x < 0 \\ 1, & \text{if } 0 \le x < \pi. \end{cases}$$

Exercise 5. Compute the Fourier series of the function

$$f(x) = \begin{cases} 0, & \text{if } -\pi < x < 0 \\ \sin x, & \text{if } 0 \le x < \pi. \end{cases}$$

4. Uniform convergence

We make a new definition. A function f on (a, b) is *piecewise* C^1 if f and its derivative f' are piecewise continuous. We assume that the finite set of points $a = t_0 < t_1 < \cdots t_{k-1} < t_k = b$ at which there are discontinuities is the same for f and f'. If f is piecewise C^1 on $(-\pi, \pi)$ then its 2π-periodic extension to $\mathbb{R}$ is piecewise C^1 on any open interval.

Example: the function $f(x) = |x|$ defined on $(-\pi, \pi)$ is piecewise C^1 (draw its graph!). In fact f is continuous everywhere, and f' has a discontinuity at 0.

Note that since the Fourier coefficients are defined by integrals, and since the (Lebesgue) integral does not depend on values of f at a finite number of points, we can assign arbitrary values to $f(t_0), \ldots, f(t_k)$; their choice will not affect the Fourier coefficients.

Lemma 4.1. *Let f be a continuous 2π-periodic function on $\mathbb{R}$ which is piecewise C^1 on $[-\pi, \pi]$. Then $c_n(f') = inc_n(f)$ for all $n \in \mathbb{Z}$.*

Proof. Since f' is piecewise continuous on $[-\pi, \pi]$, it is integrable, so $c_n(f')$ is well-defined. Since f is continuous on $[-\pi, \pi]$, partial integration works in the usual form on $[-\pi, \pi]$, in the sense that we do not split up $[-\pi, \pi]$ at points $t_1, \ldots, t_{k-1}$ (this would give additional terms). So we get the claim from

$$\begin{aligned}
c_n(f') &= \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f'(x) e^{-inx} dx \\
&= \left[\frac{1}{\sqrt{2\pi}} f(x) e^{-inx} \right]_{-\pi}^{\pi} + \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} inf(x) e^{-inx} dx \\
&= inc_n(f).
\end{aligned}$$

□

Theorem 4.2. *Let f be a continuous 2π-periodic function on $\mathbb{R}$ which is piecewise C^1 on $[-\pi, \pi]$. Then the Fourier series of f is uniformly convergent on $\mathbb{R}$ with sum f.*

Proof. Lemma 4.1 implies that

$$\begin{aligned}
\sum_{n\in\mathbb{Z}} |c_n(f)| &= |c_0(f)| + \sum_{n\in\mathbb{N}} |c_n(f)| + \sum_{n\in\mathbb{N}} |c_{-n}(f)| \\
&= |c_0(f)| + \sum_{n\in\mathbb{N}} \frac{1}{n} |c_n(f')| + \sum_{n\in\mathbb{N}} \frac{1}{n} |c_{-n}(f')|.
\end{aligned}$$

Since f' is piecewise continuous, it is in $L^2([-\pi, \pi])$. Then $\sum_{n\in\mathbb{N}} \frac{1}{n} |c_n(f')|$ is the inner-product in $l^2(\mathbb{N})$ of the sequences $\{\frac{1}{n}\}_{n\in\mathbb{N}}$ and $\{c_n(f')\}_{n\in\mathbb{N}}$ (this last one is in $l^2(\mathbb{N})$ by Bessel's inequality applied to f'). It follows similarly that $\sum_{n\in\mathbb{N}} \frac{1}{n} |c_{-n}(f')| < \infty$. Then the series $\sum_{n\in\mathbb{N}} |c_n(f)|$ is convergent, and it follows that $\sum_{n\in\mathbb{N}} c_n(f)$ is uniformly convergent. Its limit must be f by the last assertion in Corollary 2.5. □

Example 4.3. *The function $f(x) = x^2$ on $[-\pi, \pi]$ has a uniformly convergent Fourier series.*

Proof of Theorem 1.1. Recall that $C_c^\infty((-\pi, \pi))$ is dense in $L^2((-\pi, \pi))$ (which is the same space as $L^2([-\pi, \pi])$). Then it is enough to prove that

$$C_c^\infty((-\pi, \pi)) \subset \overline{\operatorname{span}\{e_n \mid n \in \mathbb{Z}\}}.$$

Let $f \in C_c^\infty((-\pi, \pi))$, and let $\{s_N\}_{N\in\mathbb{N}}$ be the sequence of partial sums of the Fourier series of f. Then $s_N \in \operatorname{span}\{e_n \mid n \in \mathbb{Z}\}$. Extend f to a 2π-periodic function $\bar{f}$ on $\mathbb{R}$. Note that $\bar{f}$ will be continuous and piecewise C^1. Theorem 4.2 implies that $s_N - \bar{f} \to 0$ uniformly on $\mathbb{R}$ as $N \to \infty$. Restriction to $(-\pi, \pi)$ gives that $s_N - f \to 0$ uniformly as $N \to \infty$.

Let $\varepsilon > 0$. Choose N_0 such that for $N \geq N_0$ we have $|s_N(x) - f(x)| < \varepsilon/\sqrt{2\pi}$ for all $x \in (-\pi, \pi)$. Then

$$\|s_N - f\|_2^2 = \int_{-\pi}^{\pi} |s_N(x) - f(x)|^2 dx < \varepsilon^2,$$

for all $N \geq N_0$. This shows $\|s_N - f\|_2 \to 0$ as $N \to \infty$. The claim of the theorem is proved. □

Theorem 4.4. *The system*

$$\{\frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}} \cos x, \frac{1}{\sqrt{\pi}} \sin x, \frac{1}{\sqrt{\pi}} \cos 2x, \frac{1}{\sqrt{\pi}} \sin 2x, \dots\}$$

is an orthonormal basis in $L^2([-\pi, \pi])$.

Proof. Follows from Theorem 1.1 (How?). □

Remark 4.5. *Theorem 1.1 implies that for any* $f \in L^2([-\pi, \pi])$ *the following identity (Parseval's identity) holds:*

$$\sum_{n \in \mathbb{Z}} |c_n(f)|^2 = \int_{-\pi}^{\pi} |f(x)|^2 dx. \tag{10}$$

References

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