Solution to MATH4500, Fall 2009

Problem 1

The topologies are:

$\mathcal{T}_1 = \{\emptyset, X\}$ (the trivial topology)
$\mathcal{T}_1 = \{\emptyset, \{a\}, X\}$
$\mathcal{T}_1 = \{\emptyset, \{b\}, X\}$
$\mathcal{T}_1 = \{\emptyset, \{a\}, \{b\}, X\}$ (the discrete topology).

Only the last topology is Hausdorff.

Problem 2

Let $U = f^{-1}((-\infty, \frac{1}{3}))$, $V = f^{-1}((\frac{2}{3}, \infty))$, then U and V are open (as the inverse images of open sets), disjoint sets such that $x \in U$, $y \in V$. Hence X is Hausdorff.

Problem 3

Before one starts looking for examples, it is useful to understand why i_* is not likely to be injective or surjective in general. Observe that i_* will fail to be injective if there are loops in A which are homotopic in X but not in A. This is possible because there may be obstacles that we can move around in the big space X, but not in the smaller space A. Observe next that i_* will fail to be surjective if there are loops in X that are not homotopic to loops living inside A — this is possible if there are obstacles in X that makes it impossible to pull a curve back to A. With these observations in mind, it is not hard to find examples.

To show that i_* need not be injective, we take $X = \mathbb{R}^2$, $A = X - \{\mathbf{0}\}$, $x_0 = (1, 0)$. Then $\pi_1(X, x_0)$ is trivial, but $\pi_1(A, x_0)$ is isomorphic to $\mathbb{Z}$, and hence i_* cannot be injective.

To show that i_* need not be surjective, take $X = S^1$, let $A = S^1 - \{-1, 0\}$ and put $x_0 = \{1, 0\}$. Then $\pi_1(A, x_0)$ is trivial (as A is homeomorphic to the open interval $(-\pi, \pi)$), and since $\pi_1(X, x_0)$ is isomorphic to $\mathbb{Z}$, i_* cannot be surjective.

Problem 4

It suffices to prove that X is a deformation retract of the doubly punctured plane. Since all doubly punctured planes are homeomorphic, we may assume that the holes are at $(-2, 0)$ and $(2, 0)$. The retract process is very similar to the one described in Example 2 on page 362 in Munkres' book: Let be S the circle with radius 3 centered at the origin (this circle is just big enough to touch the two original circles on the outside). First retract all point outside S to S by pulling them radially inwards. Next push all points on and inside S (but outside the original circles) vertically toward the x-axis until they hit X. Finally, pull all points inside the original circles radially outwards till they hit the circle they were inside. This shows that X is a deformation retract of the doubly punctured plane.

Problem 5

a) Since any element $\mathbf{x} = (x_1, x_2, \ldots, x_n, \ldots) \in X$ clearly belongs to a set O_a (just take a to be the 1-tuple (x_1)), we only need to show that if $\mathbf{x} \in O_a \cap O_b$, then there is a c such that

$$\mathbf{x} \in O_c \subset O_a \cap O_b$$

It is easy to see that since $O_a \cap O_b \neq \emptyset$, one of the tuples a and b has to be a prolongation of the other. Choose c equal to the longer of the two tuples, then

$$\mathbf{x} \in O_c = O_a \cap O_b$$

b) Let $\mathbf{x}$ and $\mathbf{y}$ be two distinct elements i X. Since $\mathbf{x} \neq \mathbf{y}$, there must be an index k such that $x_k \neq y_k$. Choose $a = (x_1, \ldots, x_k)$, $b = (y_1, \ldots y_k)$; then $x \in O_a$, $y \in O_b$ and $O_a \cap O_b = \emptyset$, and hence X is Hausdorff.

c) Let n be the length of the tuple $a = (a_1, a_2, \ldots, a_n)$. There are only finitely many other tuples $b_1, b_2, \ldots, b_N$ of the same length, and we have $O_a^c = O_{b_1} \cup O_{b_2} \cup \ldots \cup O_{b_N}$. This means that O_a^c is open and O_a is closed.

Let A be a subset of X with more than one element, and let $\mathbf{x}$, $\mathbf{y}$ be two distinct elements of A. Since X is Hausdorff, there is a basis neighborhood O_a of $\mathbf{x}$ that does not contain $\mathbf{y}$. Since O_a^c is open, $O_a \cap A, O_a^c \cap A$ form a separation of A, and hence A is not connected.

d) Assume that $f : \mathbb{R} \to X$ is continuous. Since $\mathbb{R}$ is a connected set and the continuous image of a connected set is connected (see Munkres, Theorem 23.5), $f(\mathbb{R})$ is connected. By d) the only connected sets in X are the singletons, and hence f can take only one value, i.e. f is constant.

Remark: The topology in this problem is the product topology we get on $A^{\mathbb{N}}$ if we equip A with the discrete topology. This observation does not make the problem substantially simpler, but it would be useful if, e.g., we were to prove that X is compact.

Problem 6

a) Let K_1 and K_2 be two compact sets. Since the space is Hausdorff, K_1 and K_2 are closed. But then $K_1 \cap K_2$ is a closed subset of the compact set K_1 and hence compact.

b) Let us first show that $\mathcal{T}$ is a topology. Since $\emptyset$ and X are clearly in $\mathcal{T}$, it suffices to show that $\mathcal{T}$ is closed under arbitrary unions and finite intersections:

Unions: Assume that $\mathcal{T}_0$ is an arbitrary collection of subsets of X. If none of the elements in $\mathcal{T}_0$ contains an endpoint 0 or 1, then neither does the union, and hence $\bigcup_{T\in\mathcal{T}_0} T$ belongs to $\mathcal{T}$. On the other hand, if there is a set in $\mathcal{T}_0$ which contains an endpoint, then that set is cofinite and so is the larger set $\bigcup_{T\in\mathcal{T}_0} T$. Hence $\bigcup_{T\in\mathcal{T}_0} T \in \mathcal{T}$

Intersections: Assume that $T_1, T_2, \ldots, T_n \in \mathcal{T}$. To prove that $T_1 \cap T_2 \cap \ldots \cap T_n \in \mathcal{T}$, it suffices to show that if $T_1 \cap T_2 \cap \ldots \cap T_n$ contains one of the endpoints 0 or 1, then it is cofinite. But if $T_1 \cap T_2 \cap \ldots \cap T_n$ contains an endpoint, so does all the sets $T_1, T_2, \ldots, T_n$. This means that each T_i is cofinite, and hence $T_1 \cap T_2 \cap \ldots \cap T_n$ is cofinite. This proves that $T_1 \cap T_2 \cap \ldots \cap T_n \in \mathcal{T}$.

We next show that C_1 is compact (the argument for C_2 is similar). If $\mathcal{O}$ is an open covering of C_2, at least one set $O \in \mathcal{O}$ contains 0, and must be cofinite. The elements in X that are "missed" by O are covered by a finite number $O_1, O_2, \ldots, O_n$ of other elements in $\mathcal{O}$, and hence $\{O, O_1, \ldots, O_n\}$ is a finite subcovering.

Finally, we show that $C_1 \cap C_2 = (0,1)$ is not compact. This is easy — the collection of singletons $\{\{a\}\}_{a\in(0,1)}$ is an open covering with no finite subcovering.