Lecture Notes on Topology for MAT3500/4500 following J. R. Munkres' textbook

John Rognes

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Contents

Introduction

Topology (from Greek *topos* [place/location] and *logos* [discourse/reason/logic]) can be viewed as the study of continuous functions, also known as maps. Let X and Y be sets, and $f: X \to Y$ a function from X to Y. In order to make sense of the assertion that f is a continuous function, we need to specify some extra data. After all, continuity roughly asserts that if x and y are elements of X that are "close together" or "nearby", then the function values $f(x)$ and $f(y)$ are elements of Y that are also close together. Hence we need to give some sense to a notion of closeness for elements in X, and similarly for elements in Y.

In many cases this can be done by specifying a real number $d(x, y)$ for each pair of elements $x, y \in X$, called the distance between x and y, and saying at x and y are close together if $d(x, y)$ is sufficiently small. This leads to the notion of a *metric space* (X, d), when the distance function (or metric) d satisfies some reasonable properties.

The only information available about two elements x and y of a general set X is whether they are equal or not. Thus a set X appears as an unorganized collection of its elements, with no further structure. When (X, d) is equipped with a metric, however, it acquires a shape or form, which is why we call it a *space*, rather than just a set. Similarly, when (X, d) is a metric space we refer to the $x \in X$ as *points*, rather than just as elements.

However, metric spaces are somewhat special among all shapes that appear in Mathematics, and there are cases where one can usefully make sense of a notion of closeness, even if there does not exist a metric function that expresses this notion. An example of this is given by the notion of pointwise convergence for real functions. Recall that a sequence of functions f_n for $n = 1, 2, \dots$ converges pointwise to a function g if for each point t in the domain, the sequence $f_n(t)$ of real numbers converges to the number $g(t)$. There is no metric d on the set of real functions that expresses this notion of convergence.

To handle this, and many other more general examples, one can use a more general concept than that of metric spaces, namely *topological spaces.* Rather than specifying the distance between any two elements x and y of a set X, we shall instead give a meaning to which subsets $U \subset X$ are "open". Open sets will encode closeness as follows:

> If x lies within in U, and U is an open subset of X, then all other points y in X that are sufficiently close to x also lie within U.

The shape of X is thus defined not by a notion of distance, but by the specification of which subsets U of X are open. When this specification satisfies some reasonable conditions, we call X together with the collection of all its open subsets a "topological space". The collection of all open subsets will be called the *topology* on X, and is usually denoted $\mathscr{T}$.

As you can see, this approach to the study of shapes involves not just elements and functions, like the theory of metric spaces, but also subsets and even collections of subsets. In order to argue effectively about topological spaces, it is therefore necessary to have some familiarity with the basic notions of set theory. We shall therefore start the course with a summary of the fundamental concepts concerning sets and functions.

Having done this, we can reap some awards. For instance, the definition of what it means for a function $f\colon X \to Y$, from a topological space X to a topological space Y, to be continuous, is simply:

> For each open subset V in Y the preimage $f^{-1}(V)$ is open in X.

This may be compared with the (ϵ, δ)-definition for a function $f\colon X \to Y$, from a metric space (X, d) to another metric space (Y, d), to be continuous:

> For each point x in X and each $\epsilon > 0$ there exists a $\delta > 0$ such that for each point y in X with $d(x, y) < \delta$ we have $d(f(x), f(y)) < \epsilon$.

It may be worth commenting that the definition of a topological space may seem more abstract and difficult to fully comprehend than the subsequent definition on a continuous map. The situation is analogous to that in linear algebra, where we say that a function $f\colon V \to W$ between real vector spaces is linear if it satisfies

$$f(\lambda x + \mu y) = \lambda f(x) + \mu f(y)$$

for all vectors $x, y \in V$ and all real numbers λ and μ (the Greek letters lambda and mu).

However, to make sense of this, we must first give the abstract definition of a real vector space, as a set V of vectors, with a vector sum operation $+\colon V \times V \to V$ and a scalar multiplication $\cdot\colon \mathbb{R} \times V \to V$ satisfying a list of properties. That list in turn presupposes that the set of real numbers $\mathbb{R}$ is a *field*, which also involves the two operations addition and multiplication and about nine axioms, expressing associativity, commutativity, existence of neutral elements and inverses, for both sum and product, plus the distributive law relating the two operations.

The moral is that the axiomatization of the most fundamental objects, such as topological spaces and real vector spaces, may be so general as to make it difficult to immediately grasp their scope. However, it is often the *relations* between these objects that we are most interested in, such as the properties of continuous functions or linear transformations, and these will then often appear to be relatively concrete.

Once we have established the working definitions of topological spaces and continuous functions, or maps, we shall turn to some of the most useful properties that such topological spaces may satisfy, including being *connected* (not being a disjoint union of subspaces), *compact* (not having too many open subsets globally) or *Hausdorff* (having enough few open subsets locally). Then we discuss consequences of these properties, such as general forms of the intermediate value theorem, existence of maximal values, or uniqueness of limits, and many more.

These lecture notes are intended for the course MAT4500 at the University of Oslo, following James R. Munkres' textbook "Topology". The §-signs refer to the sections in that book.

Once the foundations of Topology have been set, as in this course, one may proceed to its proper study and its applications. A well-known example of a topological result is the classification of surfaces, or more precisely, of connected compact 2-dimensional manifolds. The answer is that two facts about a surface suffice to determine it up to topological equivalence, namely, whether the surface "can be oriented", and "how many handles it has". The number of handles is also known as the *genus*. A sphere has genus 0, while a torus has genus 1, and the surface of a mug with two handles has genus 2.

An interesting result about the relation between the global topological type of a surface and its local geometry is the Gauss–Bonnet theorem. For a surface F equipped with a so-called Riemannian metric, it is a formula

$$\int_F K\, dA = 2\pi \cdot \chi$$

expressing the integral of the locally defined curvature K of the surface in terms of the globally defined genus g, or more precisely in terms of the Euler characteristic $\chi = 2 - 2g$. For example, a sphere of radius r has curvature $1/r^2$ everywhere, and surface area $4\pi r^2$. The integral of the curvature over the whole surface is the product of these two quantities, i.e., 4π. This equals 2π times the Euler characteristic of the sphere, which is 2.

These results will be covered in the course MAT4510 Geometric Structures.

When considering surfaces given by algebraic equations among complex numbers, such as

$$x^2 + y^2 = 1$$

or

$$x^5 + y^2 = 1$$

(with $x, y \in \mathbb{C}$) there is also a subtle relationship between the topological type of the solution surface (known as an algebraic curve, since it has real dimension 2 but complex dimension 1) and the number of rational solutions to the equation (with $x, y \in \mathbb{Q}$). The first equation describes a sphere, and has infinitely many rational solutions, while the second equation describes a curve of genus 2, and has only finitely many rational solutions. It was conjectured by Mordell, and proved by Gerd Faltings in 1983, that any rationally defined algebraic curve of genus greater than one has only finitely many rational points. In other words, if the defining equation has rational coefficients, and a topological condition is satisfied, then there are only finitely many rational solutions.

For more on algebraic curves, see the courses in Algebraic Geometry.

Chapter 1

Set Theory and Logic

1.1 (§1) Fundamental Concepts

1.1.1 Membership

In naive set theory, a *set* is any collection of Mathematical objects, called its *elements*. We often use uppercase letters, like A or B, to denote sets, and lowercase letters, like x or y, to denote its elements.

If A is a set, and x is one of its elements, we write $x \in A$ and say that "x is an element of A". Otherwise, if x is not an element of A, we write $x \notin A$. The symbol "$\in$" thus denotes *membership* in a collection.

We can specify sets by listing its elements, as in the set of decimal digits:

$$D = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\},$$

or by selecting the elements of some previously given set that satisfy some well-defined condition:

$$P = \{n \in \mathbb{N} \mid n \text{ is a prime}\},$$

read as "the set of $n \in \mathbb{N}$ such that n is a prime". Here

$$\mathbb{N} = \{1, 2, 3, \dots\}$$

is the set of natural numbers (= positive integers). In this case $691 \in P$, while $693 \notin P$, since 691 is a prime, while $693 = 3 \cdot 3 \cdot 7 \cdot 11$ is not a prime. We may also specify the list of elements by means of an expression, as in the set

$$S = \{n^2 \mid n \in \mathbb{N}\}$$

of squares. We shall sometimes use informal notations, like $P = \{2, 3, 5, \dots\}$ and $S = \{1, 4, 9, \dots\}$, when it should be clear from the context what we really mean.

Note that for any object x, the *singleton set* $\{x\}$ is different from x itself, so $x \in \{x\}$ but $x \neq \{x\}$. Be careful with the braces!

The *empty set* $\varnothing = \{\}$ has no elements, so $x \notin \varnothing$ for all objects x. If a set A has one or more elements, so that $x \in A$ for some object x, then we say that A is *nonempty*.

1.1.2 Inclusion and equality

Let A and B be sets. We say that A is a *subset* of B, and write $A \subset B$, if each element of A is also an element of B. In logical terms, the condition is that $x \in A$ only if $x \in B$, so $(x \in A) \implies (x \in B)$. We might also say that A is contained in B. For example, $\{1\} \subset \{1, 2\}$.

Less commonly, we might say that A is a *superset* of B, or that A contains B, and write $A \supset B$, if each element of B is also an element of A. In logical terms, the condition is that $x \in A$ if $x \in B$, so $(x \in A) \Longleftarrow (x \in B)$. This is of course equivalent to $B \subset A$. For example, $\{1,2\} \supset \{2\}$.

We say that A is *equal* to B, written $A = B$, if $A \subset B$ and $B \subset A$. This means that $x \in A$ if and only if $x \in B$, so $(x \in A) \iff (x \in B)$. For example, $\{1,2\} = \{1,1,2\}$, since the notion of a set only captures whether an element is an element of a set, not how often it is listed.

If A is not a subset of B we might write $A \not\subset B$, and if A is not equal to B we write $A \neq B$. If $A \subset B$ but $A \neq B$, so that A is a *proper subset* of B, we write $A \subsetneq B$.

(In other texts you may find the alternate notations $A \subseteq B$ and $A \subset B$ for $A \subset B$ and $A \subsetneq B$, respectively.)

1.1.3 Intersection and union

Let A and B be sets. The *intersection* $A \cap B$ is the set of objects that are elements in A and in B:

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}\,.$$

Clearly $A \cap B \subset A$ and $A \cap B \subset B$. We say that A *meets* B if $A \cap B \neq \varnothing$ is nonempty, so that there exists an x with $x \in A \cap B$, or equivalently, with $x \in A$ and $x \in B$.

The *union* $A \cup B$ is the set of objects that are elements in A or in B:

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}\,.$$

Note that "or" in the mathematical sense does not exclude the possibility that both $x \in A$ and $x \in B$. Hence $A \subset A \cup B$ and $B \subset A \cup B$.

In addition to the commutative and associative laws, these operations satisfy the following two *distributive laws*, for all sets A, B and C:

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$
$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

1.1.4 Difference and complement

Let A and B be sets. The *difference* $A - B$ is the set of objects of A that are not elements in B:

$$A - B = \{x \in A \mid x \notin B\}\,.$$

Note that $A - B \subset A$, $(A - B) \cap B = \varnothing$ and $(A - B) \cup B = A \cup B$.

We shall also call $A - B$ the *complement* of B in A. (Some texts denote the difference set by $A \setminus B$, or introduce a notation like $\complement B$ for the complement of B in A, if $B \subset A$ and the containing set A is implicitly understood.)

The complement of a union is the intersection of the complements, and the complement of an intersection is the union of the complements. These rules are known as *De Morgan's laws*:

$$A - (B \cup C) = (A - B) \cap (A - C)$$
$$A - (B \cap C) = (A - B) \cup (A - C)$$

1.1.5 Collections of sets, the power set

A set is again a mathematical object, and may therefore be viewed as an element of another set. When considering a set whose elements are sets, we shall usually refer to it as a *collection* of sets, and denote it with a script letter like $\mathscr{A}$ or $\mathscr{B}$. (Sometimes sets of sets are called *families*.)

For example, each student at the university may be viewed as a mathematical object. We may consider the set of all students:

$$S = \{s \mid s \text{ is a student at UiO}\}\,.$$

Similarly, each course offered at the university may be viewed as another mathematical object. There is a set of courses

$$C = \{c \mid c \text{ is a course at UiO}\}\,.$$

For each course $c \in C$, we may consider the set E_c of students enrolled in that course:

$$E_c = \{s \in S \mid s \text{ is enrolled in } c\}\,.$$

Now we may consider the collection $\mathscr{E}$ of these sets of enrolled students:

$$\mathscr{E} = \{E_c \mid c \in C\}\,.$$

This $\mathscr{E}$ is a set of sets. Its elements are the sets of the form E_c, for some course $c \in C$. These sets in turn have elements, which are students at the university.

It may happen that no students are enrolled for a specific course c. In that case, $E_c = \varnothing$. If this is the case for two different courses, c and d, then both $E_c = \varnothing$ and $E_d = \varnothing$. Hence it may happen that $E_c = E_d$ in $\mathscr{E}$, even if $c \neq d$ in C.

For a given set A, the collection of all subsets $B \subset A$ is called the *power set* of A, and is denoted $\mathscr{P}(A)$:

$$\mathscr{P}(A) = \{B \mid B \subset A\}\,.$$

For example, the power set $\mathscr{P}(\{a, b\}) = \{\varnothing, \{a\}, \{b\}, \{a, b\}\}$ of the two-element set $\{a, b\}$ has four elements.

1.1.6 Arbitrary intersections and unions

Given a collection $\mathscr{A}$ of sets, the *intersection* of the elements of $\mathscr{A}$ is

$$\bigcap_{A \in \mathscr{A}} A = \{x \mid \text{for every } A \in \mathscr{A} \text{ we have } x \in A\}$$

and the *union* of the elements of $\mathscr{A}$ is

$$\bigcup_{A \in \mathscr{A}} A = \{x \mid \text{there exists an } A \in \mathscr{A} \text{ such that } x \in A\}\,.$$

When $\mathscr{A} = \{A, B\}$, these are the same as the previously defined sets $A \cap B$ and $A \cup B$, respectively. When $\mathscr{A} = \{A\}$ is a singleton set, both are equal to A.

When $\mathscr{A} = \varnothing$ is empty, the intersection $\bigcap_{A \in \mathscr{A}} A$ could be interpreted as the "set of all x", but this leads to set-theoretic difficulties. We shall therefore only use that notation if there is a fixed "universal set" U in the background, so that every other set considered is a subset of U, in which case $\bigcap_{A \in \varnothing} A = U$. There is no difficulty with the empty union: $\bigcup_{A \in \varnothing} A = \varnothing$.

Returning to the example of students and courses, the intersection

$$A = \bigcap_{E_c \in \mathscr{E}} E_c$$

is the set of those students that are enrolled to every single course at the university, which most likely is empty. The union

$$B = \bigcup_{E_c \in \mathscr{E}} E_c$$

is the subset of S consisting of those students that are enrolled to one or more courses. They might be referred to as the active students. The complement, $S - B$, is the set of students that are not enrolled to any courses. They might be referred to as the inactive students.

1.1.7 Cartesian products

Given two sets A and B, the *cartesian product* $A \times B$ consists of the ordered pairs of elements (x, y) with $x \in A$ and $y \in B$:

$$A \times B = \{(x, y) \mid x \in A \text{ and } y \in B\}\,.$$

The notion of an ordered pair (x, y) is different from that of the set $\{x, y\}$. For example, $(x, y) = (x', y')$ if and only if $x = x'$ and $y = y'$. (If desired, one can define ordered pairs in terms of sets by letting $(x, y) = \{\{x\}, \{x, y\}\}$.)

The cartesian product of two copies of the real numbers, $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$, is the set consisting of pairs (x, y) of real numbers. Thinking of these two numbers as the (horizontal and vertical) coordinates of a point in the plane, one is led to René Descartes' formulation of analytic geometry, as opposed to Euclid's synthetic approach to geometry.

1.2 (§2) Functions

1.2.1 Domain, range and graph

A *function* f from a set A to a set B is a rule that to each element $x \in A$ associates a unique element $f(x) \in B$. We call A the *domain* (or *source*) of f, and B the *range* (or *codomain*, or *target*) of f. We use notations like $f \colon A \to B$ or

$$A \xrightarrow{f} B$$

to indicate that f is a function with domain A and range B. Note that the sets A and B are part of the data in the definition of the function f, even though we usually emphasize the rule taking x to $f(x)$.

Sometimes that rule is defined by some explicit procedure or algorithm for computing $f(x)$ from x, such as in

$$f(x) = \begin{cases} 3x + 1 & \text{if } x \text{ is odd} \\ x/2 & \text{if } x \text{ is even} \end{cases}$$

for natural numbers x, but we make no such assumption in general.

In set theoretic terms, we can define a function $f \colon A \to B$ to be the subset of $A \times B$ given by its *graph*, i.e., the subset

$$\Gamma_f = \{(x, f(x)) \in A \times B \mid x \in A\}\,.$$

(Here Γ is the Greek letter Gamma.) The subsets $\Gamma \subset A \times B$ that arise in this way are characterized by the property that for each $x \in A$ there exists one and only one $y \in B$ such that $(x, y) \in \Gamma$.

Hence we can define a function f to be a triple of sets (A, B, Γ), with $\Gamma \subset A \times B$ having the property that for each element $x \in A$ there exists a unique $y \in B$ with $(x, y) \in \Gamma$. In this case we let $f(x) = y$ and call y the *value* of f at the *argument* x. We call A the domain and B the range of f.

1.2.2 Image, restriction, corestriction

Let $f: A \to B$ be a function. The *image* of f is the subset

$$f(A) = \{f(x) \in B \mid x \in A\}$$

of the range B, whose elements are all the values of f. The image may, or may not, be equal to the range. (Other texts may refer to the image of f as its range, in which case they usually have no name for the range/codomain/target.)

If $S \subset A$ is a subset of the domain, we define the *restriction* of f to S to be the function $f|S: S \to B$ given by $(f|S)(x) = f(x)$ for all $x \in S$. In terms of graphs, $f|S$ corresponds to the subset

$$\Gamma_f \cap (S \times B)$$

of $S \times B$, where $\Gamma_f \subset A \times B$ is the graph of f.

If $T \subset B$ is a subset of the range, with the property that $f(A) \subset T$, then there is also a well-defined function $g: A \to T$ given by $g(x) = f(x)$ for all $x \in T$. In terms of graphs, g corresponds to the subset

$$\Gamma_f \cap (A \times T)$$

of $A \times T$. There does not seem to be a standard notation for this "corestriction" of f. Note that this construction only makes sense when the new range T contains the image of f.

1.2.3 Injective, surjective, bijective

Let $f: A \to B$. We say that f is *injective* (or *one-to-one*) if $f(x) = f(y)$ only if $x = y$, for $x, y \in A$. We say that f is *surjective* (or *onto*) if for each $y \in B$ there exists a $x \in A$ with $f(x) = y$. Note that f is surjective if and only if its image equals its range, $f(A) = B$.

We say that f is *bijective* (or a *one-to-one correspondence*) if it is both injective and surjective, so that for each $y \in B$ there exists one and only one $x \in A$ with $f(x) = y$. A bijective function is also called a *bijection*.

When $f: A \to B$ is bijective, we can define a new function $f^{-1}: B \to A$, with the domain and range interchanged, by the rule that takes $y \in B$ to the unique $x \in A$ such that $f(x) = y$. Hence $f^{-1}(y) = x$ precisely when $y = f(x)$. We call f^{-1} the *inverse function* of f.

Note that the graph of f^{-1} is obtained from the graph of f by interchanging the two factors in the cartesian product $A \times B$. It is the subset

$$\Gamma_{f^{-1}} = \{(y, x) \in B \times A \mid (x, y) \in \Gamma_f\}$$

of $B \times A$.

The inverse function is not defined when f is not bijective. If the restriction $f|S$ of f to a subset $S \subset A$ is injective, and we let $T = f(S)$ be the image of the restricted function, then the resulting function $g: S \to T$ is bijective, and has an inverse function $g^{-1}: T \to S$. (Various abuses of notations are common here.)

1.2.4 Composition

Let $f: A \to B$ and $g: B \to C$ be functions, such that the range of f equals the domain of g. The *composite function* $g \circ f: A \to C$ is then defined by the rule

$$(g \circ f)(x) = g(f(x))$$

for all $x \in A$. We often abbreviate $g \circ f$ to gf. The graph Γ_{gf} of gf is the subset

$$\{(x, z) \in A \times C \mid \text{there exists a } y \in B \text{ with } (x, y) \in \Gamma_f \text{ and } (y, z) \in \Gamma_g\}$$

of $A \times C$.

When $f: A \to B$ is bijective, with inverse function $f^{-1}: B \to A$, the composite $f^{-1} \circ f: A \to A$ is defined and equals the identity function $id_A: A \to A$ taking $x \in A$ to $x \in A$. Furthermore, the composite $f \circ f^{-1}: B \to B$ is defined, and equals the identity function $id_B: B \to B$ taking $y \in B$ to $y \in B$.

Composition of functions is unital and associative, so that $f \circ id_A = f = id_B \circ f$ and $(h \circ g) \circ f = h \circ (g \circ f)$ (for $h: C \to D$), but hardly ever commutative. Even if $A = B = C$, so that both $g \circ f$ and $f \circ g$ are defined and have the same domains and ranges, it is usually not the case that $g(f(x)) = f(g(x))$ for all $x \in A$, so usually $g \circ f \neq f \circ g$. One exception is the case when $f = g$.

The composite $g \circ f: A \to C$ of two bijections $f: A \to B$ and $g: B \to C$ is again bijective, with inverse $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

Let $f: A \to B$ and $S \subset A$. Define the *inclusion function* $i: S \to A$ by the rule $i(x) = x$ for all $x \in S$. This is not the identity function, unless $S = A$. The composite $f \circ i: S \to B$ equals the restriction $f|S: S \to B$, since these functions have the same sources and ranges, and both map $x \in S$ to $f(x) \in B$.

Let $T \subset B$ and assume that $f(A) \subset T$. Let $j: T \to B$ be the inclusion function given by the rule $j(y) = y$ for all $y \in T$. Then the corestriction $g: A \to T$ of f is characterized by the property that $j \circ g = f$.

1.2.5 Images of subsets

Let $f: A \to B$ be a function. For each subset $S \subset A$ we let the *image* $f(S) \subset B$ of S under f be the set of values

$$f(S) = \{f(x) \in B \mid x \in S\}$$

of f, as the argument x ranges over S. When $S = A$, this agrees with the image of f.

The rule taking S to $f(S)$ defines a function

$$f: \mathscr{P}(A) \to \mathscr{P}(B).$$

Using the same notation for this function and the original function $f: A \to B$ is an abuse of notation. Since the two functions are defined on disjoint sets, namely for $x \in A$ and $S \in \mathscr{P}(A)$, respectively, one can usually avoid confusion by inspecting the argument of f, but some care is certainly appropriate.

The image function f respects inclusions and unions: If $S \subset T \subset A$, then

$$f(S) \subset f(T).$$

Similarly, if $S, T \subset A$, meaning that $S \subset A$ and $T \subset A$, then

$$f(S \cup T) = f(S) \cup f(T).$$

For intersections, we only have the inclusion

$$f(S \cap T) \subset f(S) \cap f(T)$$

in general. The inclusions $S \cap T \subset S$ and $S \cap T \subset T$ imply inclusions $f(S \cap T) \subset f(S)$ and $f(S \cap T) \subset f(T)$, and these imply the displayed inclusion. If f is injective, we have equality, but in general this can be a proper inclusion.

For complements, we have the inclusion

$$f(T) - f(S) \subset f(T - S)$$

for $S, T \subset A$, with equality if f is injective, but no equality in general.

1.2.6 Preimages of subsets

Let $f\colon A \to B$ as before. For each subset $T \subset B$ we let the *preimage* $f^{-1}(T) \subset A$ of T under f be the set of arguments

$$f^{-1}(T) = \{x \in A \mid f(x) \in T\}$$

for which f takes values in T. When $T = B$ this is all of A. (The preimage is also called the *inverse image*.)

The rule taking T to $f^{-1}(T)$ defines a function

$$f^{-1}\colon \mathscr{P}(B) \to \mathscr{P}(A)\,.$$

Note that we use this notation also in the cases where f is not bijective. So the use of the notation $f^{-1}(T)$ for the preimage of T under f does not imply that f is invertible.

In the special case when $f\colon A \to B$ is bijective, so that the inverse function $f^{-1}\colon B \to A$ is defined, we have the equality of sets

$$\{x \in A \mid f(x) \in T\} = \{f^{-1}(y) \in A \mid y \in T\}$$

so that the preimage $f^{-1}(T)$ of T under f is equal to the image $f^{-1}(T)$ of T under f^{-1}. Hence the potential conflict of notations does not lead to any difficulty.

The preimage function f^{-1} respects inclusions, unions, intersections and complements: If $S \subset T \subset B$, then

$$f^{-1}(S) \subset f^{-1}(T)\,.$$

If $S, T \subset B$, then

$$\begin{aligned} f^{-1}(S \cup T) &= f^{-1}(S) \cup f^{-1}(T)\,, \\ f^{-1}(S \cap T) &= f^{-1}(S) \cap f^{-1}(T) \end{aligned}$$

and

$$f^{-1}(S - T) = f^{-1}(S) - f^{-1}(T)\,.$$

The image and preimage constructions satisfy the relations

$$S \subset f^{-1}(f(S)) \quad \text{and} \quad f(f^{-1}(T)) \subset T$$

for $f\colon A \to B$, $S \subset A$ and $T \subset B$.

1.3 (§5) Cartesian Products

1.3.1 Indexed families

Let $\mathscr{A}$ be a collection of sets. An *indexing function* for $\mathscr{A}$ is a surjective function $f\colon J \to \mathscr{A}$ from some set J to $\mathscr{A}$. We call J the *index set*, and we call the collection $\mathscr{A}$ together with the indexing function f an *indexed family of sets*.

If

$$\mathscr{A} = \{A_1, A_2, \dots, A_n\} = \{A_i\}_{i=1}^n$$

is a finite collection, with $n \geq 0$, we may let $J = \{1, 2, \dots, n\}$ and let $f(i) = A_i$ for $1 \leq i \leq n$. If

$$\mathscr{A} = \{A_1, A_2, \dots\} = \{A_i\}_{i=1}^{\infty}$$

is a countably infinite sequence of sets, we may let $J = \mathbb{N}$ and let $f(i) = A_i$ for $i \in \mathbb{N}$. In general, we often use the notation $A_\alpha = f(\alpha)$, so that f is the rule taking α to A_α, and the indexed family is denoted

$$\{A_\alpha\}_{\alpha \in J} .$$

The surjectivity of f ensures that each set $A \in \mathscr{A}$ occurs as $A_\alpha = f(\alpha)$ for some $\alpha \in J$. We do not require that f is injective, so we may have $A_\alpha = A_\beta$ even if $\alpha \neq \beta$ in J.

1.3.2 General intersections and unions

We use the following alternate notations for general intersections and unions of sets. Let $\mathscr{A}$ be a collection of sets, with indexing function $f \colon J \to \mathscr{A}$ as above.

If $\mathscr{A}$ is nonempty (so J is nonempty, too), we define

$$\bigcap_{\alpha \in J} A_\alpha = \bigcap_{A \in \mathscr{A}} A$$

to be the set of x such that $x \in A_\alpha$ for all $\alpha \in J$, which is the same as the set of x such that $x \in A$ for all $A \in \mathscr{A}$.

In general, we define

$$\bigcup_{\alpha \in J} A_\alpha = \bigcup_{A \in \mathscr{A}} A$$

to be the set of x such that $x \in A_\alpha$ for some $\alpha \in J$, or equivalently, such that $x \in A$ for some $A \in \mathscr{A}$.

If $\mathscr{A} = \{A_1, A_2, \dots, A_n\}$ with $n \geq 1$ we also write

$$\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap \cdots \cap A_n$$

for the intersection of the sets in $\mathscr{A}$, and

$$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \cdots \cup A_n$$

for the union, and similarly in the countably infinite case.

1.3.3 Finite cartesian products

Let $n \geq 0$. Given a set X, an *n-tuple* of elements in X is a function

$$x \colon \{1, 2, \dots, n\} \to X .$$

Writing $x_i = x(i)$ for its value at $1 \leq i \leq n$, the function is determined by its list of values, which is the ordered n-tuple

$$(x_i)_{i=1}^{n} = (x_1, x_2, \dots, x_n) .$$

A family of sets $\mathscr{A} = \{A_1, A_2, \dots, A_n\}$ indexed by the set $J = \{1, 2, \dots, n\}$ is equivalent to an ordered n-tuple of sets $(A_1, A_2, \dots, A_n)$. Let

$$X = A_1 \cup A_2 \cup \cdots \cup A_n$$

be the union of the n sets. We define the *cartesian product* of this indexed family, denoted

$$\prod_{i=1}^{n} A_i = A_1 \times A_2 \times \cdots \times A_n \,,$$

to be the set of all n-tuples $(x_1, x_2, \dots, x_n)$ in X where $x_i \in A_i$ for each $1 \le i \le n$.

If all of the sets A_i are equal, so that each $A_i = X$, we write

$$X^n = X \times X \times \cdots \times X$$

(n copies of X) for the n-fold cartesian product of X. It is the set of all n-tuples in X.

1.3.4 Countable cartesian products

We use similar notation for *sequences* in X, which are functions

$$x \colon \mathbb{N} \to X$$

or ordered sequences

$$(x_i)_{i=1}^{\infty} = (x_1, x_2, \dots) \,.$$

(These are also called *ω-tuples*, where ω is the Greek letter omega.) Given a sequence of sets $\mathscr{A} = (A_i)_{i=1}^{\infty} = (A_1, A_2, \dots)$ we let $X = \bigcup_{i=1}^{\infty} A_i$ be the union, and define the cartesian product

$$\prod_{i=1}^{\infty} = A_1 \times A_2 \times \dots$$

to be the set of all sequences $(x_1, x_2, \dots)$ in X such that $x_i \in A_i$ for each $i \in \mathbb{N}$.

When all $A_i = X$, we write

$$X^{\omega} = \prod_{i=1}^{\infty} X$$

for the countably infinite product of copies of X. It is the set of sequences in X.

1.3.5 General cartesian products

Let J be any indexing set. Given a set X, a *J-tuple* of elements in X is a function

$$x \colon J \to X \,.$$

Writing $x_\alpha = x(\alpha)$ for its value at $\alpha \in J$, the α-th coordinate of x, we can also denote the J-tuple x by its values $(x_\alpha)_{\alpha \in J}$.

Let $\{A_\alpha\}_{\alpha \in J}$ be an indexed family of sets, with union $X = \bigcup_{\alpha \in J} A_\alpha$. Its *cartesian product*, denoted

$$\prod_{\alpha \in J} A_\alpha$$

is the set of all J-tuples $(x_\alpha)_{\alpha \in J}$ of elements in X such that $x_\alpha \in A_\alpha$ for all $\alpha \in J$. In other words, it is the set of functions

$$x \colon J \to \bigcup_{\alpha \in J} A_\alpha$$

such that $x(\alpha) \in A_\alpha$ for all $\alpha \in J$.

When all $A_\alpha = X$, we write

$$X^J = \prod_{\alpha \in J} X$$

for the J-fold product of copies of X. It is the set of functions $x \colon J \to X$.

1.4 (§6) Finite Sets

1.4.1 Cardinality

For each nonnegative integer n, the set

$$\{1, 2, \ldots, n\}$$

of natural numbers less than or equal to n is called a *section* of the natural numbers $\mathbb{N}$. For $n = 0$ this is the empty set $\varnothing$, for $n = 1$ it is the singleton set $\{1\}$.

Lemma 1.4.1. *If there exists an injective function*

$$f \colon \{1, 2, \ldots, m\} \to \{1, 2, \ldots, n\}$$

then $m \leq n$.

Proof. We prove this by induction on $n \geq 0$. For $n = 0$ this is clear, since there only exists a function $\{1, 2, \ldots, m\} \to \varnothing$ if $m = 0$. For the inductive step, let $n \geq 1$ and suppose that the lemma holds for $n - 1$. Let $f \colon \{1, 2, \ldots, m\} \to \{1, 2, \ldots, n\}$ be injective. Let $f(m) = k$. Then f restricts to an injective function $g \colon \{1, 2, \ldots, m - 1\} \to \{1, 2, \ldots, n\} - \{k\}$. Define a bijection $h \colon \{1, 2, \ldots, n\} - \{k\} \to \{1, 2, \ldots, n - 1\}$ by $h(x) = x$ for $x \neq n$, and $h(n) = k$. (If $k = n$ we let h be the identity.) The composite $h \circ g \colon \{1, 2, \ldots, m - 1\} \to \{1, 2, \ldots, n - 1\}$ is injective, so by the inductive hypothesis we know that $m - 1 \leq n - 1$. It follows that $m \leq n$, as desired. □

Corollary 1.4.2. *There does not exist an injective function $f \colon \{1, 2, \ldots, m\} \to \{1, 2, \ldots, n\}$ if $m > n$.*

Proposition 1.4.3. *If there exists a bijection $f \colon \{1, 2, \ldots, m\} \to \{1, 2, \ldots, n\}$ then $m = n$.*

Proof. Both f and its inverse f^{-1} are injective, so by the previous lemma we have $m \leq n$ and $n \leq m$. Thus $m = n$. □

Corollary 1.4.4. *There does not exist a bijective function $f \colon \{1, 2, \ldots, m\} \to \{1, 2, \ldots, n\}$ if $m \neq n$.*

Definition 1.4.5. A set A is *finite* if there exists a bijective function $f \colon A \to \{1, 2, \ldots, n\}$ for some $n \geq 0$. In this case, we say that A has *cardinality* n.

For example, the empty set is finite with cardinality 0, and each singleton set is finite with cardinality 1.

Lemma 1.4.6. *The cardinality of a finite set A is well-defined. That is, if there exists bijections $f \colon A \to \{1, 2, \ldots, n\}$ and $g \colon A \to \{1, 2, \ldots, m\}$ for some $m, n \geq 0$, then $m = n$.*

Proof. The composite

$$f \circ g^{-1} \colon \{1, 2, \ldots, m\} \to \{1, 2, \ldots, n\}$$

is a bijection, with inverse $g \circ f^{-1}$, so $m = n$ by the proposition above. □

1.4.2 Subsets

Lemma 1.4.7. *Let $A \subset \{1, 2, \dots, n\}$ be a subset. There exists a bijection $f : A \to \{1, 2, \dots, m\}$ for some $m \leq n$. Hence A is a finite set, of cardinality $\leq n$.*

Proof. We prove this by induction on $n \geq 0$. For $n = 0$ it is clear, since the only subset of $\varnothing$ is $\varnothing$. For the inductive step, let $n \geq 1$ and suppose that the lemma holds for $n - 1$. Let $A \subset \{1, 2, \dots, n\}$ be a subset. If $n \notin A$, then $A \subset \{1, 2, \dots, n-1\}$, so there exists a bijection $f : A \to \{1, 2, \dots, m\}$ for some $m \leq n-1$ by the inductive hypothesis. Hence $m \leq n$. Otherwise, we have $n \in A$. Let $B = A - \{n\}$. Then $B \subset \{1, 2, \dots, n-1\}$, and by the inductive hypothesis there exists a bijection $g : B \to \{1, 2, \dots, k\}$ for some $k \leq n-1$. Let $m = k+1$, so $m \leq n$. Define the bijection $f : A \to \{1, 2, \dots, m\}$ by $f(x) = g(x)$ if $x \in B$ and $f(n) = m$. This completes the inductive step. □

Proposition 1.4.8. *Let $A \subsetneq \{1, 2, \dots, n\}$ be a proper subset. Then the cardinality of A is strictly less than n, so there does not exists a bijection $f : A \to \{1, 2, \dots, n\}$.*

Proof. Since A is a proper subset, we can choose a $k \in \{1, 2, \dots, n\}$ with $k \notin A$. Define a bijection $h : \{1, 2, \dots, n\} \to \{1, 2, \dots, n\}$ by $h(k) = n$, $h(n) = k$ and $h(x) = x$ for the remaining x. Let $B = h(A)$, so that $h|A : A \to B$ is a bijection. Then $B \subset \{1, 2, \dots, n-1\}$, so by the lemma above there is a bijection $g : B \to \{1, 2, \dots, m\}$ with $m \leq n-1$. The composite $g \circ (h|A) : A \to \{1, 2, \dots, m\}$ is then a bijection. Hence the cardinality of A is $m \leq n-1 < n$. □

Theorem 1.4.9. *If A is a finite set, then there is no bijection of A with a proper subset of itself.*

Proof. Suppose that $B \subsetneq A$ is a proper subset, and that there exists a bijection $f : A \to B$. Since A is finite there is a bijection $g : A \to \{1, 2, \dots n\}$ for some $n \geq 0$, where n is the cardinality of A. Let $C = g(B) \subset \{1, 2, \dots, n\}$. Then $g|B : B \to C$ is a bijection, and C is a proper subset of $\{1, 2, \dots, n\}$. Hence there is a bijection $h : C \to \{1, 2, \dots, m\}$ for some $m < n$. If there exists a bijection $f : A \to B$, then the composite bijection $h \circ (g|B) \circ f : A \to \{1, 2, \dots, m\}$ would say that the cardinality of A is m, and not equal to n. This contradicts the fact that the cardinality is well-defined, hence no such bijection f exists. □

Corollary 1.4.10. *The set $\mathbb{N}$ of natural numbers is not finite.*

Proof. The function $f : \mathbb{N} \to \mathbb{N} - \{1\}$ defined by $f(x) = x + 1$ is a bijection of $\mathbb{N}$ with a proper subset of itself. □

Corollary 1.4.11. *Any subset B of a finite set A is finite. If B is a proper subset of A, then its cardinality is strictly less than the cardinality of A.*

1.4.3 Injections and surjections

Proposition 1.4.12. *Let A be a set. The following are equivalent:*

(1) A is finite.

(2) There exists a surjective function $\{1, 2, \dots, n\} \to A$ for some integer $n \geq 0$.

(3) There exists an injective function $A \to \{1, 2, \dots, n\}$ for some integer $n \geq 0$.

Proof. (1) $\implies$ (2): Since A is finite there exists a bijection $f\colon A \to \{1, 2, \dots, n\}$ for some integer $n \ge 0$. Then f^{-1} is surjective, as required.

(2) $\implies$ (3): Let $g\colon \{1, 2, \dots, n\} \to A$ be surjective. Define a function $h\colon A \to \{1, 2, \dots, n\}$ by letting $h(x)$ be the smallest element of the subset

$$g^{-1}(x) = \{i \mid g(i) = x\}\,.$$

of $\{1, 2, \dots, n\}$. This subset is nonempty since g is surjective. Then h is injective, since if $x \ne y$ then $g^{-1}(x)$ and $g^{-1}(y)$ are disjoint, so their smallest elements must be different.

(3) $\implies$ (1): If $f\colon A \to \{1, 2, \dots, n\}$ is injective, let $B = f(A) \subset \{1, 2, \dots, n\}$. The corestriction of f is then a bijection $g\colon A \to B$. By a previous lemma there is a bijective function $h\colon B \to \{1, 2, \dots, m\}$ for some $m \le n$. The composite bijection $h \circ g\colon A \to \{1, 2, \dots, m\}$ shows that A is finite. □

Proposition 1.4.13. *Finite unions and finite products of finite sets are finite.*

Proof. Let A and B be finite. Choose bijections $f\colon \{1, 2, \dots, m\} \to A$ and $g\colon \{1, 2, \dots, n\} \to B$ for suitable $m, n \ge 0$. We define a surjection

$$h\colon \{1, 2, \dots, m+n\} \to A \cup B$$

by

$$h(x) = \begin{cases} f(x) & \text{if } 1 \le x \le m \\ g(x-m) & \text{if } m+1 \le x \le m+n. \end{cases}$$

Here we regard A and B as subsets of $A \cup B$. Hence $A \cup B$ is finite by the proposition above.

By the case $n = 2$, the formula

$$A_1 \cup \dots \cup A_n = (A_1 \cup \dots \cup A_{n-1}) \cup A_n$$

and induction on n it follows that if $A_1, \dots, A_n$ are finite then $A_1 \cup \dots \cup A_n$ is finite, for all $n \ge 0$.

The cartesian product $A \times B$ is the union of the subsets $A \times \{y\}$ for all $y \in B$. Hence if A and B are finite this is a finite union of finite sets, so $A \times B$ is finite.

By the case $n = 2$, the formula

$$A_1 \times \dots \times A_n = (A_1 \times \dots \times A_{n-1}) \times A_n$$

and induction on n it follows that if $A_1, \dots, A_n$ are finite then $A_1 \times \dots \times A_n$ is finite, for all $n \ge 0$. □

Chapter 2

Topological Spaces and Continuous Functions

2.1 (§12) Topological Spaces

2.1.1 Open sets

Definition 2.1.1. Let X be a set. A *topology* on X is a collection $\mathscr{T}$ of subsets of X, such that:

(1) $\varnothing$ and X in $\mathscr{T}$.

(2) For any subcollection $\{U_\alpha\}_{\alpha\in J}$ of $\mathscr{T}$, the union $\bigcup_{\alpha\in J} U_\alpha$ is in $\mathscr{T}$.

(3) For any finite subcollection $\{U_1, \dots, U_n\}$ of $\mathscr{T}$ the intersection $U_1 \cap \dots \cap U_n$ is in $\mathscr{T}$.

A *topological space* $(X, \mathscr{T})$ is a set X with a chosen topology $\mathscr{T}$.

The subsets $U \subset X$ with $U \in \mathscr{T}$ are said to be *open*. Note that this *defines* the property of being open. With this terminology, the axioms above assert that:

(1) $\varnothing$ and X are open (as subsets of X).

(2) The union of any collection of open subsets of X is open.

(3) The intersection of any finite collection of open subsets of X is open.

With the convention that $\varnothing$ is the union of the empty collection of subsets of X, and X is the intersection of the empty collection of subsets of X, one may agree that (1) follows from (2) and (3), but condition (1) is usually included for clarity. We express (2) by saying that $\mathscr{T}$ is closed under (arbitrary) unions, and express (3) by saying that $\mathscr{T}$ is closed under finite intersections.

To check that $\mathscr{T}$ is closed under finite intersections, it suffices to prove that if $U_1, U_2 \in \mathscr{T}$ then $U_1 \cap U_2 \in \mathscr{T}$, since this implies the case of n-fold intersections by an inductive argument.

2.1.2 Discrete and trivial topologies

Let X be any set. Here are two extreme examples of topologies on X.

Definition 2.1.2. The *discrete topology* on X is the topology $\mathscr{T}_{\text{disc}}$ where *all* subsets $U \subset X$ are defined to be open. Hence the collection of open subsets equals the power set of X: $\mathscr{T}_{\text{disc}} = \mathscr{P}(X)$. It is clear that the axioms of a topology are satisfied, since it is so easy to be open in this topology. We call $(X, \mathscr{T}_{\text{disc}})$ a *discrete topological space*.

The terminology can be explained as follows. Note that for each point $x \in X$, the singleton set $\{x\}$ is a subset of X, hence is a open in the discrete topology. Thus all other points $y \neq x$ of X are separated away from x by this open set $\{x\}$. We therefore think of X with the discrete topology as a space of separate, isolated points, with no close interaction between different points. In this sense, the space is discrete.

Definition 2.1.3. The *trivial topology* on X is the topology $\mathscr{T}_{\text{triv}}$ where only the subsets $\varnothing$ and X are defined to be open. Hence $\mathscr{T}_{\text{triv}} = \{\varnothing, X\}$. It is clear that the axioms of a topology are satisfied, since there are so few collections of open subsets. We call $(X, \mathscr{T}_{\text{triv}})$ a *trivial topological space*. (Some texts call this the *indiscrete topology*.)

This terminology probably refers to the fact that the trivial topology is the minimal example of a topology on X, in the sense that only those subsets of X that axiom (1) demand to be open are open, and no others.

2.1.3 Finite topological spaces

Definition 2.1.4. If X is a finite set, and $\mathscr{T}$ is a topology on X, we call $(X, \mathscr{T})$ a *finite topological space*.

When X is finite, the power set $\mathscr{P}(X)$ and any topology $\mathscr{T} \subset \mathscr{P}(X)$ is finite, so the nuance between finite and arbitrary unions and intersections plays no role. Hence to check conditions (2) and (3) for a topology, it suffices to check that if $U_1, U_2 \in \mathscr{T}$ then $U_1 \cup U_2 \in \mathscr{T}$ and $U_1 \cap U_2 \in \mathscr{T}$.

In the case when X is empty, or a singleton set, the discrete topology on X is equal to the trivial topology on X, and these are the only possible topologies on X.

Example 2.1.5. Let $X = \{a, b\}$ be a 2-element set. There are four different possible topologies on X:

(1) The minimal possibility is the trivial topology $\mathscr{T}_{\text{triv}} = \{\varnothing, X\}$.

(2) An intermediate possibility is $\mathscr{T}_a = \{\varnothing, \{a\}, X\}$.

(3) Another intermediate possibility is $\mathscr{T}_b = \{\varnothing, \{b\}, X\}$.

(4) The maximal possibility is the discrete topology $\mathscr{T}_{\text{disc}} = \{\varnothing, \{a\}, \{b\}, X\}$.

We already explained why cases (1) and (4) are topologies. Examples (2) and (3) are known as *Sierpinski spaces*. To see that $\mathscr{T}_a$ is a topology on $\{a, b\}$, note that $\{b\}$ does not occur as the union or the intersection of any collection of sets in $\mathscr{T}_a$. Interchanging the role of a and b we also see that $\mathscr{T}_b$ is a topology.

In the Sierpinski space $(X = \{a, b\}, \mathscr{T}_a)$, the element a is separated away from the other point by the open set $\{a\}$, while the element b not separated away from the other point by any open set. For the only open set containing b is $\{a, b\}$, which also contains a. This means that a is "arbitrarily close" to b, even if b is not arbitrarily close to a. This kind of asymmetry of "closeness" in topological spaces is not seen in metric spaces.

In these examples each collection of subsets containing the trivial topology defined a topology. When X has cardinality 3 this is no longer true.

Example 2.1.6. Let $X = \{a, b, c\}$. There are 29 different topologies on X. Here are nine of them:

(1) The trivial topology $\mathscr{T}_1 = \mathscr{T}_{\text{triv}} = \{\varnothing, X\}$.

(2) $\mathscr{T}_2 = \{\varnothing, \{a\}, X\}$.

(3) $\mathscr{T}_3 = \{\varnothing, \{a, b\}, X\}$.

(4) $\mathscr{T}_4 = \{\varnothing, \{a\}, \{a, b\}, X\}$.

(5) $\mathscr{T}_5 = \{\varnothing, \{a, b\}, \{c\}, X\}$.

(6) $\mathscr{T}_6 = \{\varnothing, \{a\}, \{b\}, \{a, b\}, X\}$.

(7) $\mathscr{T}_7 = \{\varnothing, \{a\}, \{a, b\}, \{a, c\}, X\}$.

(8) $\mathscr{T}_8 = \{\varnothing, \{a\}, \{c\}, \{a, b\}, \{a, c\}, X\}$.

(9) The discrete topology $\mathscr{T}_9 = \mathscr{T}_{\text{disc}} = \mathscr{P}(X)$ (with 8 elements).

The reader should check that each of these is closed under unions and intersections. The remaining topologies on X arise by permuting the elements a, b and c.

Example 2.1.7. Let $X = \{a, b, c\}$. Here are some collections of subsets of X that are not topologies:

(1) $\{\{a\}, \{c\}, \{a, b\}, \{a, c\}\}$ does not contain $\varnothing$ and X.

(2) $\{\varnothing, \{a\}, \{b\}, X\}$ is not closed under unions.

(3) $\{\varnothing, \{a, b\}, \{a, c\}, X\}$ is not closed under intersections.

2.1.4 Coarser and finer topologies

Definition 2.1.8. Let $\mathscr{T}$ and $\mathscr{T}'$ be two topologies on the same set X. We say that $\mathscr{T}$ is *coarser* than $\mathscr{T}'$, or equivalently that $\mathscr{T}'$ is *finer* than $\mathscr{T}$, if $\mathscr{T} \subset \mathscr{T}'$. This means that each subset $U \subset X$ that is open in $(X, \mathscr{T})$ is also open in $(X, \mathscr{T}')$. (Other texts use "stronger" and "weaker" in place of "coarser" and "finer", but not in a consistent way.)

Lemma 2.1.9. *The trivial topology is coarser than any other topology, and the discrete topology is finer than any other topology.*

Proof. For any topology $\mathscr{T}$ on X we have

$$\mathscr{T}_{\text{triv}} = \{\varnothing, X\} \subset \mathscr{T} \subset \mathscr{P}(X) = \mathscr{T}_{\text{disc}}\,. \quad \square$$

The set of topologies on X becomes partially ordered by the "coarser than"-relation. Note that two topologies need not be comparable under this relation. For example, neither one of the two Sierpinski topologies $\mathscr{T}_a$ and $\mathscr{T}_b$ on $\{a, b\}$ is coarser (or finer) than the other.

2.1.5 The cofinite topology

Definition 2.1.10. Let X be a set. Let the *cofinite topology* $\mathscr{T}_{\text{cof}}$ be the collection of subsets $U \subset X$ whose complement $X - U$ is finite, together with the empty set $U = \varnothing$.

The word "cofinite" refers to the fact that complements of finite sets are open, since if $S \subset X$ is finite, then $U = X - S$ has complement $X - U = X - (X - S) = S$, which is finite. Calling $\mathscr{T}_{\text{cof}}$ a "topology" requires justification:

Lemma 2.1.11. *The collection $\mathscr{T}_{\text{cof}}$ is a topology on X.*

Proof. We check the three conditions for a topology.

(1): The subset $\varnothing$ is in $\mathscr{T}_{\text{cof}}$ by definition. The subset X is in $\mathscr{T}_{\text{cof}}$ since its complement $X - X = \varnothing$ is finite.

(2): Let $\{U_\alpha\}_{\alpha \in J}$ be a subcollection of $\mathscr{T}_{\text{cof}}$, so for each $\alpha \in J$ we have that $X - U_\alpha$ is finite, or $U_\alpha = \varnothing$. Let $V = \bigcup_{\alpha \in J} U_\alpha$. We must prove that $X - V$ is finite, or that $V = \varnothing$.

If each U_α is empty, then V is empty. Otherwise, there is a $\beta \in J$ such that $X - U_\beta$ is finite. Since $U_\beta \subset V$, the complements satisfy $X - V \subset X - U_\beta$. Hence $X - V$ is a subset of a finite set, and is therefore finite, as desired.

(3): Let $\{U_1, \dots, U_n\}$ be a finite subcollection of $\mathscr{T}_{\text{cof}}$, so for each $1 \le i \le n$ we have that $X - U_i$ is finite, or $U_i = \varnothing$. Let $W = U_1 \cap \dots \cap U_n$. We must prove that $X - W$ is finite, or $W = \varnothing$.

If some U_i is empty, the $W \subset U_i$ is empty. Otherwise, $X - U_i$ is finite for each $1 \le i \le n$. By De Morgan's law,

$$X - W = X - (U_1 \cap \dots \cap U_n) = (X - U_1) \cup \dots \cup (X - U_n) .$$

The right hand side is a finite union of finite sets, hence is again finite. Thus $X - W$ is finite, as desired. ☐

When X is a finite set, the condition that $X - U$ is finite is always satisfied, so in this case the cofinite topology equals the discrete topology: $\mathscr{T}_{\text{cof}} = \mathscr{T}_{\text{disc}}$.

When X is an infinite set, the cofinite topology is strictly coarser than the discrete topology: $\mathscr{T}_{\text{cof}} \subsetneq \mathscr{T}_{\text{disc}}$. For example, in this case each finite, nonempty subset $S \subset X$ is open in the discrete topology, but not open in the cofinite topology. To see this, note that for S to be open in $\mathscr{T}_{\text{cof}}$ its complement $X - S$ would have to be finite. Then $X = S \cup (X - S)$ would be the union of two finite sets, and therefore would be finite. This contradicts the assumption that X is infinite. Such finite, nonempty subsets $S \subset X$ exist. For example, each singleton set $S = \{x\}$ for $x \in X$ will do. Hence $\mathscr{T}_{\text{cof}} \neq \mathscr{T}_{\text{disc}}$ for infinite X.

Example 2.1.12. Let $X = \mathbb{N}$ be the set of natural numbers. The discrete topology $\mathscr{T}_{\text{disc}}$ on $\mathbb{N}$ is strictly finer than the cofinite topology $\mathscr{T}_{\text{cof}}$ on $\mathbb{N}$, which is strictly finer than the trivial topology $\mathscr{T}_{\text{triv}}$ on $\mathbb{N}$.

2.1.6 Metric spaces

Definition 2.1.13. A *metric* on a set X is a function $d \colon X \times X \to \mathbb{R}$ such that:

(1) $d(x, y) \ge 0$ for all $x, y \in X$, and $d(x, y) = 0$ if and only if $x = y$.

(2) $d(x, y) = d(y, x)$ for all $x, y \in X$.

(3) $d(x, z) \le d(x, y) + d(y, x)$ for all $x, y, z \in X$ (the triangle inequality).

A *metric space* (X, d) is a set X with a chosen metric d.

Definition 2.1.14. Let (X, d) be a metric space. For each point $x \in X$ and each positive real number $\epsilon > 0$, let

$$B_d(x, \epsilon) = \{y \in X \mid d(x, y) < \epsilon\}$$

be the *ϵ-ball* around x in (X, d).

Definition 2.1.15. Let (X, d) be a metric space. The *metric topology* $\mathscr{T}_d$ on X is the collection of subsets $U \subset X$ satisfying the property: for each $x \in U$ there exists an $\epsilon > 0$ such that $B_d(x, \epsilon) \subset U$.

Lemma 2.1.16. *The collection $\mathscr{T}_d$ is a topology on X.*

Proof. We check the three conditions for a topology.

(1): The subset $\varnothing$ is in $\mathscr{T}_d$ since there are no $x \in \varnothing$ for which anything needs to be checked. The subset X is in $\mathscr{T}$ since for each $x \in U$ we can take $\epsilon = 1$, since $B_d(x, 1) \subset X$.

(2): Let $\{U_\alpha\}_{\alpha \in J}$ be a subcollection of $\mathscr{T}_d$. Let $V = \bigcup_{\alpha \in J} U_\alpha$ and consider any $x \in V$. By the definition of the union there exists an $\alpha \in J$ with $x \in U_\alpha$. By the property satisfied by the U_α in $\mathscr{T}_d$, there exists an $\epsilon > 0$ such that $B_d(x, \epsilon) \subset U_\alpha$. Since $U_\alpha \subset V$ it follows that $B_d(x, \epsilon) \subset V$. Hence $V \in \mathscr{T}_d$.

(3): Let $\{U_1, \dots, U_n\}$ be a finite subcollection of $\mathscr{T}_d$. Let $W = U_1 \cap \cdots \cap U_n$ and consider any $x \in W$. For each $1 \leq i \leq n$ we have $W \subset U_i$ so $x \in U_i$. By the defining property of $\mathscr{T}_d$ there exists an $\epsilon_i > 0$ such that $B_d(x, \epsilon_i) \subset U_i$. Let $\epsilon = \min\{\epsilon_1, \dots, \epsilon_n\}$. This makes sense since n is finite, and $\epsilon > 0$. Then $B_d(x, \epsilon) \subset B_d(x, \epsilon_i) \subset U_i$ for each $1 \leq i \leq n$, which implies that $B_d(x, \epsilon) \subset W$. Hence $W \in \mathscr{T}_d$. □

2.2 (§13) Basis for a Topology

2.2.1 Bases

In the case of a metric space (X, d), knowledge of the collection $\mathscr{B}_d$ of ϵ-balls $B_d(x, \epsilon)$ (for all $x \in X$ and $\epsilon > 0$) suffices to determine the collection of open sets, i.e., the metric topology $\mathscr{T}_d$. More precisely, a subset $U \subset X$ is open if for each $x \in U$ there exists an ϵ-ball B in the collection $\mathscr{B}_d$ with $x \in B$ and $B \subset U$.

(Note that here $B = B_d(y, \epsilon)$ might not be a ball centered at x, so this condition may appear to be a little weaker than the one used in the definition of the metric topology. However, if $x \in B_d(y, \epsilon)$ we have $d(x, y) < \epsilon$, so letting $\delta = \epsilon - d(x, y)$ we have $\delta > 0$, and the triangle inequality implies that $B_d(x, \delta) \subset B_d(y, \epsilon)$. Hence we could actually have assumed that $B = B_d(x, \delta)$ was a ball around x.)

Also for more general topological spaces $(X, \mathscr{T})$, it is often more efficient to specify a smaller collection $\mathscr{B}$ of open sets that "generates" the topology, in a certain sense, than to specify the whole topology $\mathscr{T}$. Here we will say that a collection $\mathscr{B}$ of subsets of X generates the collection $\mathscr{T}$ of subsets $U \subset X$ such that:

for each $x \in U$ there exists a $B \in \mathscr{B}$ with $x \in B$ and $B \subset U$.

To make sure that the resulting collection $\mathscr{T}$ is a topology, we shall assume that $\mathscr{B}$ satisfies two reasonable conditions. This collection $\mathscr{B}$ will then be called a *basis* for the topology $\mathscr{T}$, and $\mathscr{T}$ is the topology *generated* by $\mathscr{B}$.

Definition 2.2.1. Let X be a set. A collection $\mathscr{B}$ of subsets of X is a *basis* (for a topology) if

(1) For each $x \in X$ there exists a $B \in \mathscr{B}$ with $x \in B$.

(2) If $B_1, B_2 \in \mathscr{B}$ and $x \in B_1 \cap B_2$ then there exists a $B_3 \in \mathscr{B}$ with $x \in B_3 \subset B_1 \cap B_2$.

The sets $B \in \mathscr{B}$ in the basis are called *basis elements*.

Note that the basis elements are elements in $\mathscr{B}$, but subsets of X.

Definition 2.2.2. Let $\mathscr{B}$ be a basis for a topology on X. The topology $\mathscr{T}$ *generated* by $\mathscr{B}$ is the collection of subsets $U \subset X$ such that for each $x \in U$ there exists a $B \in \mathscr{B}$ with $x \in B \subset U$.

Lemma 2.2.3. *The collection $\mathscr{T}$ generated by a basis $\mathscr{B}$ is a topology on X.*

[[See pages 79–80 in Munkres.]]

Example 2.2.4. Let $X = \mathbb{R}^2$ be the plane, and let $\mathscr{B}$ be the set of all open circular regions in the plane. This is the set of all ϵ-balls in $\mathbb{R}^2$ with respect to the Euclidean metric $d(x, y) = \|y - x\|$ in the plane. [[Discuss conditions (1) and (2) for a basis.]]

Example 2.2.5. Let $X = \mathbb{R}^2$ be the xy-plane, and let $\mathscr{B}'$ be the set of all open rectangular regions

$$(a, b) \times (c, d) \subset \mathbb{R} \times \mathbb{R}$$

with $a < b$ and $c < d$. This is the rectangle bounded by the vertical lines $x = a$ and $x = b$, and the horizontal lines $y = c$ and $y = d$. [[Discuss conditions (1) and (2) for a basis. Here the intersection of two basis elements is again a basis element, unless it is empty.]]

Example 2.2.6. Let X be a set, and let $\mathscr{B}''$ be the collection of singleton sets $\{x\}$ for $x \in X$. It is a basis for the discrete topology $\mathscr{T}_{\text{disc}}$ on X.

2.2.2 From bases to topologies and back

Lemma 2.2.7. *Let $\mathscr{B}$ be a basis for a topology $\mathscr{T}$ on X.*

(1) Each basis element $B \in B$ is open in X, so $\mathscr{B} \subset \mathscr{T}$,

(2) Each open subset $U \subset X$ is a union of basis elements, so $\mathscr{T}$ equals the collection of all unions of elements in $\mathscr{B}$.

Proof. (1): Let $B \in \mathscr{B}$. For each $x \in B$ we obviously have $x \in B$ and $B \subset B$. Hence B is open.

(2): Let $U \subset X$ be open. For each $x \in U$ there exists a $B_x \in \mathscr{B}$ with $x \in B_x$ and $B_x \subset U$. Then

$$U = \bigcup_{x \in U} B_x$$

is the union of the collection of basis elements $\{B_x \mid x \in U\}$. To check the displayed equality, not that for each $x \in U$ we have $x \in B_x$, so $x \in \bigcup_{x \in U} B_x$. Hence $U \subset \bigcup_{x \in U} B_x$. On the other hand, each $B_x \subset U$, so $\bigcup_{x \in U} B_x \subset U$. □

The trick in the proof of part (2), of gluing together local solutions B_x to a single solution $\bigcup_{x \in U} B_x$, will be used again later.

Lemma 2.2.8. *Let $(X, \mathscr{T})$ be a topological space. Suppose that $\mathscr{C}$ is a collection of open subsets of X such that for each open $U \in \mathscr{T}$ and point $x \in U$ there exists an element $C \in \mathscr{C}$ with $x \in C$ and $C \subset U$. Then $\mathscr{C}$ is a basis for the topology $\mathscr{T}$.*

Proof. We first check that $\mathscr{C}$ is a basis.

(1): The set X is open in itself, so for each $x \in X$ there exists a $C \in \mathscr{C}$ with $x \in C$.

(2): Let $C_1, C_2 \in \mathscr{C}$. Since C_1 and C_2 are open, so is the intersection $C_1 \cap C_2$. Hence, for each $x \in C_1 \cap C_2$ there exists a $C_3 \in \mathscr{C}$ with $x \in C_3$ and $C_3 \subset C_1 \cap C_2$.

Next we check that the topology $\mathscr{T}'$ generated by $\mathscr{C}$ equals $\mathscr{T}$.

$\mathscr{T} \subset \mathscr{T}'$: If $U \in \mathscr{T}$ and $x \in U$ there exists a $C \in \mathscr{C}$ with $x \in C$ and $C \subset U$, by hypothesis, so $U \in \mathscr{T}'$ by definition.

$\mathscr{T}' \subset \mathscr{T}$: If $U \in \mathscr{T}'$ then U is a union of elements of $\mathscr{C}$ by the lemma above. Each element of $\mathscr{C}$ is in $\mathscr{T}$, hence so is the union U. □

2.2.3 Comparing topologies using bases

Lemma 2.2.9. *Let $\mathscr{B}$ and $\mathscr{B}'$ be bases for the topologies $\mathscr{T}$ and $\mathscr{T}'$ on X, respectively. The following are equivalent:*

(1) $\mathscr{T}$ is coarser than $\mathscr{T}'$, so $\mathscr{T} \subset \mathscr{T}'$.

(2) For each basis element $B \in \mathscr{B}$ and each point $x \in B$ there is a basis element $B' \in \mathscr{B}'$ with $x \in B'$ and $B' \subset B$.

Proof. (1) $\implies$ (2): Let $B \in \mathscr{B}$ and $x \in B$. Since $B \in \mathscr{T}$ and $\mathscr{T} \subset \mathscr{T}'$ we have $B \in \mathscr{T}'$. Since $\mathscr{T}'$ is the topology generated by $\mathscr{B}'$ there exists a $B' \in \mathscr{B}'$ with $x \in B'$ and $B' \subset B$.

(2) $\implies$ (1): Let $U \in \mathscr{T}$. For each $x \in U$ there exists a $B \in \mathscr{B}$ with $x \in B$ and $B \subset U$, since $\mathscr{B}$ generates $\mathscr{T}$. By hypothesis there exists a $B' \in \mathscr{B}'$ with $x \in B'$ and $B' \subset B$. Hence $B' \subset U$. Since this holds for each $x \in U$, it follows that $U \in \mathscr{T}'$. □

Note that in order to have $\mathscr{T} \subset \mathscr{T}'$ it is not necessary to have $\mathscr{B} \subset \mathscr{B}'$, that is each basis element $B \in \mathscr{B}$ does not need to be a basis element in $\mathscr{B}'$, but for each $x \in B$ there should be some smaller basis element $B' \in \mathscr{B}'$ with $x \in B' \subset B$.

Corollary 2.2.10. *Two $\mathscr{B}$ and $\mathscr{B}'$ bases for topologies on X generate the same topology if and only if (1) for each $x \in B \in \mathscr{B}$ and each point $x \in B$ there is a basis element $B' \in \mathscr{B}'$ with $x \in B' \subset B$, and furthermore, (2) for each basis element $x \in B' \in \mathscr{B}'$ there is a basis element $B \in \mathscr{B}$ with $x \in B \subset B'$.*

Example 2.2.11. The basis $\mathscr{B}$ of open circular regions in the plane, and the basis $\mathscr{B}'$ of open rectangular regions, generate the same topology on $\mathbb{R}^2$, namely the metric topology.

Example 2.2.12. Let $\mathbb{R}$ be the real line. With the usual metric $d(x, y) = |y - x|$, the ϵ-neighborhoods

$$B_d(x, \epsilon) = (x - \epsilon, x + \epsilon)$$

are open intervals, and each open interval

$$(a, b) = \{x \in \mathbb{R} \mid a < x < b\}$$

with $a < b$ has this form for $x = (a+b)/2$, $\epsilon = (b-a)/2$.. Let $\mathscr{B}$ be the collection of all intervals $(a, b) \subset \mathbb{R}$ for $a < b$. It is a basis for the *standard topology* on $\mathbb{R}$, that is, the metric topology $\mathscr{T}_d$.

Definition 2.2.13. The collection $\mathscr{B}_\ell$ of all intervals

$$[a, b) = \{x \in \mathbb{R} \mid a \le x < b\}$$

for $a < b$ is a basis for a topology $\mathscr{T}_\ell$, called the *lower limit topology* on $\mathbb{R}$. Let $\mathbb{R}_\ell = (\mathbb{R}, \mathscr{T}_\ell)$ denote $\mathbb{R}$ with this topology.

Lemma 2.2.14. *The lower limit topology is strictly finer than the standard topology on $\mathbb{R}$, so $\mathscr{T}_d \subsetneq \mathscr{T}_\ell$.*

Proof. For each basis element $B = (a, b) \in \mathscr{B}$ for the standard topology, and each $x \in (a, b)$, we can take $B' = [x, b) \in \mathscr{B}_\ell$ in the basis for the lower limit topology, and have $x \in B' \subset B$. Hence the standard topology is coarser than the lower limit topology. To show that the two topologies are not equal, consider a basis element $B' = [a, b)$ in $\mathscr{B}'$ with $a < b$, and let $x = a \in B'$. Then there is no basis element $B = (c, d) \in \mathscr{B}$ with $x \in B$ and $B \subset B'$. For $x \in B$ would imply $c < x = a < d$, and then $y = (c + a)/2$ satisfies $y \in (c, d)$ but $y \notin [a, b)$. Hence $B \not\subset B'$. □

2.2.4 Subbases

Starting with any collection $\mathscr{S}$ of subsets of a set X, we can form a basis $\mathscr{B}$ for a topology by taking all finite intersections $B = S_1 \cap \cdots \cap S_n$ of elements in $\mathscr{S}$. The open sets in the topology $\mathscr{T}$ generated by $\mathscr{B}$ are then all unions of such basis elements B. Such a collection $\mathscr{S}$ is then called a subbasis. To avoid ambiguity about the intersection of an empty collection, we mildly restrict the collection $\mathscr{S}$ as follows:

Definition 2.2.15. A *subbasis* for a topology on X is a collection $\mathscr{S}$ of subsets of X, whose union equals X. The *basis* associated to $\mathscr{S}$ is the collection $\mathscr{B}$ consisting of all finite intersections

$$B = S_1 \cap \cdots \cap S_n$$

of elements $S_1, \dots, S_n \in \mathscr{S}$, for $n \geq 1$. By the topology $\mathscr{T}$ generated by the subbasis $\mathscr{S}$ we mean the topology generated by the associated basis $\mathscr{B}$.

Clearly $\mathscr{S} \subset \mathscr{B} \subset \mathscr{T}$.

Lemma 2.2.16. *Let $\mathscr{S}$ be a subbasis on X. The associated collection $\mathscr{B}$ is a basis for a topology.*

Proof. (1): Each $x \in X$ lies in some $S \in \mathscr{S}$, hence is an element of the basis element $B = S \in \mathscr{B}$.

(2): Suppose that $B_1 = S_1 \cap \cdots \cap S_n$ and $B_2 = S_{n+1} \cap \cdots \cap S_{n+m}$ are basis elements, and $x \in B_1 \cap B_2$. Let $B_3 = B_1 \cap B_2 = S_1 \cap \cdots \cap S_{m+n}$. Then B_3 is a basis element, and $x \in B_3 \subset B_1 \cap B_2$. □

2.3 (§15) The Product Topology on $X \times Y$

2.3.1 A basis for the product topology

Definition 2.3.1. Let X and Y be topological spaces. The *product topology* on $X \times Y$ is the topology generated by the basis

$$\mathscr{B} = \{U \times V \mid U \text{ open in } X \text{ and } V \text{ open in } Y\}$$

consisting of all sets $U \times V \subset X \times Y$, where U ranges over all open subsets of X and V ranges over all open subsets of Y.

Lemma 2.3.2. *The collection $\mathscr{B}$ (as above) is a basis for a topology on $X \times Y$.*

Proof. (1): $X \times Y$ is itself a basis element.

(2): Let $B_1 = U_1 \times V_1$ and $B_2 = U_2 \times V_2$ be two basis elements. In view of the identity

$$(U_1 \times V_1) \cap (U_2 \times V_2) = (U_1 \cap U_2) \times (V_1 \cap V_2)$$

we have that $B_1 \cap B_2 = B_3$, where $B_3 = U_3 \times V_3$ is the basis element given by the product of the two open sets $U_3 = U_1 \cap U_2$ and $V_3 = V_1 \cap V_2$. □

The union of two basis elements

$$(U_1 \times V_1) \cup (U_2 \times V_2)$$

is usually not a basis element. The open sets in the product topology on $X \times Y$ are the unions

$$\bigcup_{\alpha \in J} (U_\alpha \times V_\alpha)$$

of arbitrary collections $\{B_\alpha = U_\alpha \times V_\alpha\}_{\alpha \in J}$ of basis elements.

Theorem 2.3.3. *Let X have the topology generated by a basis $\mathscr{B}$ and let Y have the topology generated by a basis $\mathscr{C}$. Then the collection*

$$\mathscr{D} = \{B \times C \mid B \in \mathscr{B} \text{ and } C \in \mathscr{C}\}$$

is a basis for the product topology on $X \times Y$.

Proof. We apply Lemma 2.2.8. The elements $B \times C$ of the collection $\mathscr{D}$ are open in the product topology, since each $B \in \mathscr{B}$ is open in X and each $C \in \mathscr{C}$ is open in Y, so $B \times C$ is one of the basis elements for the product topology.

Let $(x, y) \in W \subset X \times Y$ where W is open in the product topology. By definition of the topology generated by a basis, there exists a basis element $U \times V$ for the product topology, such that $(x, y) \in U \times V \subset W$. Since $x \in U$, U is open in X and $\mathscr{B}$ is a basis for the topology on X, there exists a basis element $B \in \mathscr{B}$ such that $x \in B \subset U$. Similarly, there exists a basis element $C \in \mathscr{C}$ such that $y \in C \subset V$. Then $B \times C$ is in the collection $\mathscr{D}$, and $(x, y) \in B \times C \subset U \times V \subset W$. □

Example 2.3.4. The collection $\mathscr{B}'$ of open rectangular regions

$$(a, b) \times (c, d)$$

for $a < b$ and $c < d$ is a basis for the product topology on $\mathbb{R} \times \mathbb{R}$, since the collection of open intervals (a, b) for $a < b$ is a basis for the standard topology on $\mathbb{R}$. As previously noted, this product topology is the same as the metric topology.

2.3.2 A subbasis for the product topology

Definition 2.3.5. Let $\pi_1 \colon X \times Y \to X$ denote the (first) projection $\pi_1(x, y) = x$, and let $\pi_2 \colon X \times Y \to Y$ denote the (second) projection $\pi_2(x, y) = y$, for $x \in X$ and $y \in Y$.

Lemma 2.3.6. *The preimage of $U \subset X$ under $\pi_1 \colon X \times Y \to X$ equals*

$$\pi_1^{-1}(U) = U \times Y\,.$$

Similarly, the preimage of $V \subset Y$ under $\pi_2 \colon X \times Y \to Y$ equals

$$\pi_2^{-1}(V) = X \times V\,.$$

Proof. An element (x, y) lies in $\pi_1^{-1}(U)$ if and only if $x = \pi_1(x, y)$ lies in U, which for $y \in Y$ is equivalent to asking that (x, y) lies in $U \times Y$. The second case is similar. □

Note the identity

$$(U \times Y) \cap (X \times V) = U \times V$$

of subsets of $X \times Y$. Hence each basis element $U \times V$ for the product topology on $X \times Y$ is the intersection of two subsets of the form $\pi_1^{-1}(U) = U \times Y$ and $\pi_2^{-1}(V) = X \times V$. It follows that the basis for the product topology is generated by a smaller subbasis:

Definition 2.3.7. Let

$$\begin{aligned}\mathscr{S} &= \{U \times Y \mid U \subset X \text{ open}\} \cup \{X \times V \mid V \subset Y \text{ open}\} \\ &= \{\pi_1^{-1}(U) \mid U \subset X \text{ open}\} \cup \{\pi_2^{-1}(V) \mid V \subset Y \text{ open}\}\,.\end{aligned}$$

Lemma 2.3.8. *The collection $\mathscr{S}$ (as above) is a subbasis for the product topology on $X \times Y$.*

Proof. The finite intersections of elements in the subbasis are all of the form

$$\pi_1^{-1}(U) \cap \pi_2^{-1}(V) = U \times V$$

for U open in X and V open in Y, hence the subbasis generates the usual basis for the product topology. □

2.4 (§16) The Subspace Topology

2.4.1 Subspaces

Definition 2.4.1. Let $(X, \mathscr{T})$ be a topological space, and let $Y \subset X$ be a subset. The collection

$$\mathscr{T}_Y = \{Y \cap U \mid U \in \mathscr{T}\}$$

of subsets of Y is called the *subspace topology* on Y. With this topology, $(Y, \mathscr{T}_Y)$ is called a *subspace* of X.

Lemma 2.4.2. *The collection $\mathscr{T}_Y$ (as above) is a topology on Y.*

Proof. (1): $\varnothing = Y \cap \varnothing$ and $Y = Y \cap X$ are in $\mathscr{T}_Y$, since $\varnothing$ and X are in $\mathscr{T}$.

(2): Each subcollection of $\mathscr{T}_Y$ can be indexed as $\{Y \cap U_\alpha\}_{\alpha\in J}$ for some subcollection $\{U_\alpha\}_{\alpha\in J}$ of $\mathscr{T}$. Then

$$\bigcup_{\alpha\in J}(Y \cap U_\alpha) = Y \cap \bigcup_{\alpha\in J} U_\alpha$$

by the distributive law, and $\bigcup_{\alpha\in J} U_\alpha$ is in $\mathscr{T}$, hence this union is in $\mathscr{T}_Y$.

(3): Each finite subcollection of $\mathscr{T}_Y$ can be indexed as $\{Y \cap U_1, \dots, Y \cap U_n\}$ for some finite subcollection $\{U_1, \dots, U_n\}$ of $\mathscr{T}$. Then

$$(Y \cap U_1) \cap \cdots \cap (Y \cap U_n) = Y \cap (U_1 \cap \cdots \cap U_n)$$

and $U_1 \cap \cdots \cap U_n$ is in $\mathscr{T}$, hence this intersection is in $\mathscr{T}_Y$. □

Remark 2.4.3. When $(Y, \mathscr{T}_Y)$ is a subspace of $(X, \mathscr{T})$, and $V \subset Y \subset X$, there are two possible meanings of the assertion "V is open", namely $V \in \mathscr{T}$ or $V \in \mathscr{T}_Y$. In general, these two meanings are different. We therefore say that "V is open in X", or that "V is an open subset of X", to indicate that $V \in \mathscr{T}$, while we say that "V is open in Y", or that "V is an open subset of Y", to indicate that $V \in \mathscr{T}_Y$. The latter means that $V = Y \cap U$ for some U that is open in X.

Lemma 2.4.4. *If $\mathscr{B}$ is a basis for a topology $\mathscr{T}$ on X, and $Y \subset X$, then the collection*

$$\mathscr{B}_Y = \{Y \cap B \mid B \in \mathscr{B}\}$$

is a basis for the subspace topology $\mathscr{T}_Y$ on Y.

Proof. We apply Lemma 2.2.8 for the topological space $(Y, \mathscr{T}_Y)$ and the collection $\mathscr{B}_Y$. Each subset $Y \cap B$ in $\mathscr{B}_Y$ is open in Y, since each basis element $B \in \mathscr{B}$ is open in X. Furthermore, each open subset of Y has the form $Y \cap U$ for some open subset U of X. If $x \in Y \cap U$ is any point, then $x \in U$, so since $\mathscr{B}$ is a basis for the topology $\mathscr{T}$ there exists a $B \in \mathscr{B}$ with $x \in B$ and $B \subset U$. Then $Y \cap B \in \mathscr{B}_Y$, $x \in Y \cap B$, and $Y \cap B \subset Y \cap U$. By the cited lemma, $\mathscr{B}_Y$ is a basis for the topology $\mathscr{T}_Y$. □

Example 2.4.5. Give $X = \mathbb{R}$ the standard topology generated by the open intervals (a, b), and let $A = [0, 1)$. The subspace topology on A has a basis consisting of the intersections $[0, 1) \cap (a, b)$, i.e., the subsets $[0, b)$ and (a, b) for $0 < a < b \leq 1$. For instance, $[0, 1/2) = [0, 1) \cap (-1/2, 1/2)$ and $(0, 1/2) = [0, 1) \cap (0, 1/2)$ are both open subsets of $[0, 1)$ in the subspace topology.

Example 2.4.6. Let (X, d) be a metric space, with basis $\mathscr{B} = \{B_d(x, \epsilon) \mid x \in X, \epsilon > 0\}$ for the metric topology $\mathscr{T} = \mathscr{T}_d$. Here

$$B_d(x, \epsilon) = \{y \in X \mid d(x, y) < \epsilon\}.$$

Let $Y \subset X$ be any subset, with metric $d' = d|Y \times Y$ given by $d'(x, y) = d(x, y)$ for all $x, y \in Y$. The metric space (Y, d') has basis $\mathscr{B}' = \{B_{d'}(x, \epsilon) \mid x \in Y, \epsilon > 0\}$ for the metric topology $\mathscr{T}' = \mathscr{T}_{d'}$, where

$$B_{d'}(x, \epsilon) = \{y \in Y \mid d'(x, y) < \epsilon\}.$$

Note that

$$B_{d'}(x, \epsilon) = Y \cap B_d(x, \epsilon)$$

for all $x \in Y$. Hence $\mathscr{B}' \subset \mathscr{B}_Y$ and $\mathscr{T}' \subset \mathscr{T}_Y$, where $\mathscr{T}_Y$ is the subspace topology on Y.

To prove that the two topologies are equal, so that $\mathscr{T}_Y = \mathscr{T}'$, we use Lemma 2.2.9 to check that $\mathscr{T}_Y \subset \mathscr{T}'$. Thus consider any basis element $Y \cap B_d(x, \epsilon)$ in $\mathscr{T}_Y$ and any element $y \in Y \cap B_d(x, \epsilon)$. Then $\delta = \epsilon - d(x, y)$ is positive, and $B_{d'}(y, \delta)$ is a basis element in $\mathscr{B}'$, $y \in B_{d'}(y, \delta)$ and $B_{d'}(y, \delta) \subset Y \cap B_d(x, \epsilon)$.

Lemma 2.4.7. *Let Y be an open subspace of X. Then a subset $V \subset Y$ is open in Y if and only if it is open in X.*

Proof. Suppose first that V is open in Y, in the subspace topology. Then $V = Y \cap U$ for some U that is open in X. Since Y is open in X, it follows that the intersection, $V = Y \cap U$ is open in X.

Conversely, suppose that $V \subset Y$ is open in X. Then $V = Y \cap V$, so V is also open in Y. □

2.4.2 Products vs. subspaces

Lemma 2.4.8. *Let X and Y be topological spaces, with subspaces A and B, respectively. Then the product topology on $A \times B$ is the same as the subspace topology on $A \times B$ as a subset of $X \times Y$.*

Proof. The subspace topology on A is generated by the basis with elements $A \cap U$, where U ranges over all open subsets of X. Likewise, the subspace topology on B is generated by the intersections $B \cap V$, where V ranges over all open subsets of Y. Hence the product topology on $A \times B$ is generated by the basis with elements

$$(A \cap U) \times (B \cap V)$$

where U and V range over the open subsets of X and Y, respectively.

On the other hand, the collection of products $U \times V$ is a basis for the product topology on $X \times Y$, so the collection of intersections

$$(A \times B) \cap (U \times V)$$

is a basis for the subspace topology on $A \times B$, where U and V still range over the open subsets of X and Y. In view of the identity

$$(A \cap U) \times (B \cap V) = (A \times B) \cap (U \times V)$$

these two bases are in fact equal, hence they generate the same topology. □

2.5 (§17) Closed Sets and Limit Points

2.5.1 Closed subsets

Definition 2.5.1. A subset A of a topological space X is said to be *closed* if (and only if) the complement $X - A$ is open. In other words, the closed subsets of X are the subsets of the form $X - U$ where U is open.

Example 2.5.2. The interval $[a, b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}$ is closed in $\mathbb{R}$ (with the standard topology), since the complement $\mathbb{R} - [a, b] = (-\infty, a) \cup (b, \infty)$ is open.

Example 2.5.3. In the discrete topology $\mathscr{T}_{\text{disc}}$ on a set X, every subset is closed. In the trivial topology $\mathscr{T}_{\text{triv}}$, only the subsets $\varnothing$ and X are closed.

Example 2.5.4. In the cofinite topology $\mathscr{T}_{\text{cof}}$ on a set X, the closed subsets are the finite subsets $S \subset X$, together with X itself.

Theorem 2.5.5. *Let X be a topological space.*

(1) $\varnothing$ and X are closed (as subsets of X).

(2) The intersection of any collection of closed subsets of X is closed.

(3) The union of any finite collection of closed subsets of X is closed.

Proof. (1): $\varnothing = X - X$ and $X = X - \varnothing$ are closed.

(2): If $\{A_\alpha\}_{\alpha \in J}$ is any collection of closed subsets of X, then the complements $U_\alpha = X - A_\alpha$ are all open, so that $\{U_\alpha\}_{\alpha \in J}$ is a collection of open subsets of X. To prove that the intersection $\bigcap_{\alpha \in J} A_\alpha$ is closed, we must check that its complement is open. By De Morgan's law

$$X - \bigcap_{\alpha \in J} A_\alpha = \bigcup_{\alpha \in J} (X - A_\alpha) = \bigcup_{\alpha \in J} U_\alpha$$

is a union of open sets, hence is open, as desired.

(3): If $\{A_1, \dots, A_n\}$ is a finite collection of closed subsets of X, then the complements $U_i = X - A_i$ are all open, so that $\{U_1, \dots, U_n\}$ is a finite collection of open subsets of X. To prove that the union $A_1 \cup \dots \cup A_n$ is closed, we must check that its complement is open. By De Morgan's law

$$X - (A_1 \cup \dots \cup A_n) = (X - A_1) \cap \dots \cap (X - A_n) = U_1 \cap \dots \cap U_n$$

is a finite intersection of open sets, hence is open, as desired. □

Clearly the collection $\mathscr{C} = \{X - U \mid U \in \mathscr{T}\}$ of closed subsets of a topological space $(X, \mathscr{T})$ uniquely determine the topology $\mathscr{T}$, and any collection $\mathscr{C}$ of subsets, called closed subsets, satisfying the three conditions of the theorem above, will determine a topology in this way.

Remark 2.5.6. When $(Y, \mathscr{T}_Y)$ is a subspace of $(X, \mathscr{T})$, and $A \subset Y \subset X$, there are two possible meanings of the assertion "A is closed", namely $X - A \in \mathscr{T}$ or $X - A \in \mathscr{T}_Y$. In general, these two meanings are different. We therefore say that "A is open in Y", or that "A is a closed subset of Y", if A is a subset of Y and A is closed in the subspace topology on Y, so that $Y - A$ is open in the subspace topology on Y.

Theorem 2.5.7. *Let Y be a subspace of X. A subset $A \subset Y$ is closed in Y if and only if there exists a closed subset $B \subset X$ with $A = Y \cap B$.*

Proof. If A is closed in Y, then $V = Y - A$ is open in Y, so there exists an open $U \subset X$ with $V = Y \cap U$. Then $B = X - U$ is closed in X, and

$$Y \cap B = Y \cap (X - U) = Y - (Y \cap U) = Y - V = A\,.$$

Conversely, if B is closed in X and $A = Y \cap B$, then $U = X - B$ is open in X so $V = Y \cap U$ is open in Y. Now

$$Y - A = Y - Y \cap B = Y \cap (X - B) = Y \cap U = V\,,$$

so A is closed in Y. □

Lemma 2.5.8. *Let Y be a closed subspace of X. Then a subset $A \subset Y$ is closed in Y if and only if it is closed in X.*

Proof. Suppose first that A is closed in Y, in the subspace topology. Then $A = Y \cap B$ for some B that is closed in X. Since Y is closed in X, it follows that the intersection, $A = Y \cap B$ is open in X. Conversely, suppose that $A \subset Y$ is closed in X. Then $A = Y \cap A$, so A is also closed in Y. □

2.5.2 Closure and interior

Definition 2.5.9. Let X be a topological space and $A \subset X$ a subset. The *closure* $\operatorname{Cl} A = \bar{A}$ of A is the intersection of all the closed subsets of X that contain A. The *interior* $\operatorname{Int} A$ of A is the union of all the open subsets of X that are contained in A.

Since unions of open sets are open, and intersections of closed sets are closed, the following lemmas are clear.

Lemma 2.5.10. *(1) The closure $\operatorname{Cl} A$ is a closed subset of X.*

(2) $A \subset \operatorname{Cl} A$.

(3) If $A \subset K \subset X$ with K closed, then $\operatorname{Cl} A \subset K$.

Lemma 2.5.11. *(1) The interior $\operatorname{Int} A$ is an open subset of X.*

(2) $\operatorname{Int} A \subset A$.

(3) If $U \subset A \subset X$ with U open, then $U \subset \operatorname{Int} A$.

Example 2.5.12. Let $X = \mathbb{R}$ and $A = [a, b)$ with $a < b$. The closure of A is the closed interval $[a, b]$, and the interior of A is the open interval (a, b). The closure cannot be smaller, since $[a, b)$ is not closed, and the interior cannot be larger, since $[a, b)$ is not open.

Example 2.5.13. If X has the discrete topology, $\operatorname{Int} A = A = \operatorname{Cl} A$ for each $A \subset X$, since each A is both open and closed.

If X has the indiscrete topology, and $A \subset X$ is a proper, non-empty subset, then $\operatorname{Int} A = \varnothing$ and $\operatorname{Cl} A = X$.

If $X = \{a, b\}$ has the Sierpinski topology $\mathscr{T}_a = \{\varnothing, \{a\}, X\}$, where $\{a\}$ is open and $\{b\}$ is closed, then $\operatorname{Cl}\{a\} = X$ but $\operatorname{Cl}\{b\} = \{b\}$.

Lemma 2.5.14. *The complement of the closure is the interior of the complement, and the complement if the interior is the closure of the complement:*

$$X - \operatorname{Cl} A = \operatorname{Int}(X - A)$$
$$X - \operatorname{Int} A = \operatorname{Cl}(X - A)$$

Proof. Since $\operatorname{Cl} A = \bigcap\{K \mid A \subset K,\, K \text{ closed in } X\}$,

$$\begin{aligned} X - \operatorname{Cl} A &= \bigcup\{X - K \mid A \subset K,\, K \text{ closed in } X\} \\ &= \bigcup\{U \mid U \subset X - A,\, U \text{ open in } X\} = \operatorname{Int}(X - A)\,, \end{aligned}$$

since $A \subset K$ is equivalent to $X - K \subset X - A$, where we substitute U for $X - K$.

The other assertion follows by considering $X - A$ in place of A. □

Example 2.5.15. Let $X = \mathbb{R}$ and $A = \mathbb{Q}$, the set of rational numbers. The closure of A equals $\mathbb{R}$, while the interior of A is empty. For every ϵ-ball $(x - \epsilon, x + \epsilon)$ in $\mathbb{R}$ contains both rational and irrational numbers, so the interiors of A and $X - A$ are both empty.

2.5.3 Closure in subspaces

Theorem 2.5.16. *Let X be a topological space, $Y \subset X$ a subspace, and $A \subset Y$ a subset. Let $\bar{A}$ denote the closure of A in X. Then the closure of A in Y equals $Y \cap \bar{A}$.*

Proof. Let B denote the closure of A in Y.

To see that $B \subset Y \cap \bar{A}$, note that $\bar{A}$ is closed in X, so $Y \cap \bar{A}$ is closed in Y and contains A. Hence it contains the closure B of A in Y.

To prove the opposite inclusion, note that B is closed in Y, hence has the form $B = Y \cap C$ for some C that is closed in X. Then $A \subset B \subset C$, so C is closed in X and contains A. Hence $\bar{A} \subset C$ and $Y \cap \bar{A} \subset Y \cap C = B$. □

Example 2.5.17. Let $X = \mathbb{R}$, $Y = \mathbb{Q}$ and $A = \mathbb{Q} \cap [0, \pi)$. The closure of A in X is $[0, \pi]$, and the closure of A in Y is $\mathbb{Q} \cap [0, \pi] = \mathbb{Q} \cap [0, \pi)$ (since π is not rational). Hence A is closed in Y.

2.5.4 Neighborhoods

Definition 2.5.18. Let X be a topological space, $U \subset X$ a subset and $x \in X$ a point. We say that U *is a neighborhood of* x (norsk: "U er en omegn om x") if $x \in U$ and U is open in X.

We say that a set A *meets*, or *intersects*, a set B if $A \cap B$ is not empty. Here is a criterion for detecting which points $x \in X$ lie in the closure of A:

Theorem 2.5.19. *Let A be a subset of a topological space X. A point $x \in X$ lies in the closure $\bar{A}$ if and only if A meets every open set U in X that contains x. Equivalently, $x \in \bar{A}$ if and only if A meets U for each neighborhood U of x.*

Proof. Consider the complement $X - A$ and its interior. We have $x \in \operatorname{Int}(X - A)$ if and only if there exists an open U with $x \in U$ such that $U \subset X - A$. The negation of $x \in \operatorname{Int}(X - A)$ is $x \in X - \operatorname{Int}(X - A) = \bar{A}$. The negation of $U \subset X - A$ is $A \cap U \neq \varnothing$. The negation of "there exists an open U with $x \in U$ such that $U \subset X - A$" is therefore "for each open U with $x \in U$ we have $A \cap U \neq \varnothing$". Hence x is in the closure of A if and only if A meets each neighborhood U of x. □

It suffices to check this for neighborhoods in a basis:

Theorem 2.5.20. *Let $\mathscr{B}$ be a basis for a topology on X, and let $A \subset X$. A point $x \in X$ lies in $\bar{A}$ if and only if A meets each basis element $B \in \mathscr{B}$ with $x \in B$.*

Proof. If there exists an open U with $x \in U$ such that $U \subset X - A$ then there exists a basis element $B \in \mathscr{B}$ with $x \in B$ such that $B \subset X - A$, and conversely. Hence we may replace "an open U" by "a basis element B" in the previous proof. □

Example 2.5.21. Let $A = \{1/n \mid n \in \mathbb{N}\} \subset \mathbb{R}$. Then $0 \in \bar{A}$, since each basis element (a, b) for the standard topology on $\mathbb{R}$ with $0 \in (a, b)$ contains $(-\epsilon, \epsilon)$ for some $\epsilon > 0$, hence also contains $1/n \in A$ for each $n > 1/\epsilon$. The closure of A is $\bar{A} = \{0\} \cup A$. This is a closed subset of $\mathbb{R}$, since the complement is the union of the open sets $(-\infty, 0)$, $(1/(n+1), 1/n)$ for $n \in \mathbb{N}$, and $(1, \infty)$.

2.5.5 Limit points

Definition 2.5.22. Let A be a subset of a topological space X. A point $x \in X$ is a *limit point* of A if each neighborhood U of x contains a point of A other than x, or equivalently, of x belongs to the closure of $A - \{x\}$. (If $x \notin A$, recall that $A - \{x\} = A$.)

The set of limit points of A is often denoted A', and is called the *derived set* of A in X.

Example 2.5.23. Let $A = \{1/n \mid n \in \mathbb{N}\} \subset \mathbb{R}$. Then $0 \in \mathbb{R}$ is a limit point of A. In this case there are no other limit points of A.

Theorem 2.5.24. *Let A be a subset of a topological space X, with closure $\bar{A}$ and set of limit points A'. Then*

$$\bar{A} = A \cup A' .$$

Proof. $A \cup A' \subset \bar{A}$: Clearly $A \subset \bar{A}$. If $x \in A'$ then every neighborhood U of x meets $A - \{x\}$, hence it also meets A, so $x \in \bar{A}$.

$\bar{A} \subset A \cup A'$: Let $x \in \bar{A}$. If $x \in A$, then $x \in A \cup A'$. Otherwise, $x \notin A$, so $A - \{x\} = A$. Since $x \in \bar{A}$, every neighborhood U of x meets $A - \{x\} = A$, so $x \in A'$ is a limit point of A. □

Corollary 2.5.25. *A subset A of a topological space is closed if and only if it contains all its limits points.*

Proof. We have $A = \bar{A}$ if and only if $A = A \cup A'$, which holds if and only if $A \supset A'$. □

2.5.6 Convergence to a limit

Definition 2.5.26. Let $(x_1, x_2, \dots) = (x_n)_{n=1}^{\infty}$ be a sequence of points in a topological space X, so $x_n \in X$ for each $n \in \mathbb{N}$. We say that $(x_n)_{n=1}^{\infty}$ *converges* to a point $y \in X$ if for each neighborhood U of y there is an $N \in \mathbb{N}$ such that $x_n \in U$ for all $n \geq N$. In this case we call y *a limit* of the sequence $(x_n)_{n=1}^{\infty}$, and may write $x_n \to y$ as $n \to \infty$.

[[Suffices to check this for U in a basis for the topology on X.]]

Example 2.5.27. Consider the Sierpinski space $X = \{a, b\}$ with the topology $\mathscr{T}_a = \{\varnothing, \{a\}, X\}$. The constant sequence $(x_n)_{n=1}^{\infty}$ with $x_n = a$ for all $n \in \mathbb{N}$ converges to a, since the only neighborhoods of a are $\{a\}$ and $\{a, b\}$, both of which contain x_n for all n. Hence a is a limit of $(a, a, \dots)$.

However, the same sequence also converges to b, since the only neighborhood of b is $\{a, b\}$, which also contains x_n for all n. Hence b is also a limit for $(a, a, \dots)$.

On the other hand, the constant sequence $(y_n)_{n=1}^{\infty}$ with $y_n = b$ for all $n \in \mathbb{N}$ converges to b, since the only neighborhood $\{a, b\}$ of b contains y_n for all n.

This constant sequence does not converge to a, since the neighborhood $\{a\}$ of a does not contain y_n for any n, hence there is no $N \in \mathbb{N}$ such that $y_n \in \{a\}$ for all $n \geq N$.

2.5.7 Hausdorff spaces

To obtain unique limits for convergent sequences, and be able to talk about *the limit* of a sequence, we must assume that the topology is sufficiently fine to separate the individual points. Such additional hypotheses are called *separation axioms* (German: Trennungsaxiome). The most common separation axiom is known as the Hausdorff property.

Definition 2.5.28. A topological space X is called a *Hausdorff space* if for each pair of points $x, y \in X$, with $x \neq y$, there exist open sets $U, V \subset X$ with $x \in U$, $y \in V$ and $U \cap V = \varnothing$. In other words, there exist neighborhood U and V of x and y, respectively, that are disjoint.

Example 2.5.29. The set $X = \{a, b\}$ with the discrete topology is a Hausdorff space, since the only pair of distinct points is a and b, and the open subsets $\{a\}$ and $\{b\}$ are neighborhoods of a and b, respectively, with empty intersection.

The set $X = \{a, b\}$ with the Sierpinski topology $\mathscr{T}_a = \{\varnothing, \{a\}, X\}$ is not a Hausdorff space, since the only neighborhood of b is $V = X$, and no neighborhood U of a can be disjoint from X.

Lemma 2.5.30. *Each metric space (X, d) is Hausdorff.*

Proof. Let $x, y \in X$ be two distinct points. Then $\delta = d(x, y) > 0$. Consider the neighborhoods $U = B_d(x, \delta/2)$ and $V = B_d(y, \delta/2)$ of x and y, respectively. Then $U \cap V = \varnothing$ by the triangle inequality. □

Remark 2.5.31. If $(X, \mathscr{T})$ is Hausdorff, clearly $(X, \mathscr{T}')$ is also Hausdorff if $\mathscr{T}'$ is a finer topology than $\mathscr{T}$. In rough terms, the Hausdorff property asserts that there are "enough" open sets, locally in X.

2.5.8 Uniqueness of limits in Hausdorff spaces

Theorem 2.5.32. *If X is a Hausdorff space, then a sequence $(x_n)_{n=1}^\infty$ of points in X converges to at most one point in X.*

Proof. Suppose that $(x_n)_{n=1}^\infty$ converges to y and z. We must prove that $y = z$.

Suppose, to achieve a contradiction, that $y \neq z$. Then there exist neighborhoods U of y and V of z with $U \cap V = \varnothing$. Since $(x_n)_{n=1}^\infty$ converges to y there exists an $N \in \mathbb{N}$ such that $x_n \in U$ for all $n \geq N$. Since $(x_n)_{n=1}^\infty$ converges to z there exists an $M \in \mathbb{N}$ such that $x_n \in V$ for all $n \geq M$. Hence, for $n \geq \max\{N, M\}$ we have $x_n \in U \cap V$, which is impossible, since this intersection is empty. □

Definition 2.5.33. If X is a Hausdorff space, and a sequence $(x_n)_{n=1}^\infty$ of points in X converges to a point $y \in X$, we say that y is *the limit* of $(x_n)_{n=1}^\infty$, and write

$$y = \lim_{n\to\infty} x_n \,.$$

2.5.9 Closed sets and limits points in Hausdorff spaces

Theorem 2.5.34. *Each finite subset $A \subset X$ in a Hausdorff space is closed.*

Proof. The set A is the union of a finite collection of singleton sets $\{x\}$, so it suffices to prove that each singleton set $\{x\}$ is closed in X.

Consider any other point $y \in X$, with $x \neq y$. By the Hausdorff property there are open subsets $U, V \subset X$ with $x \in U$ and $y \in V$, such that $U \cap V = \varnothing$. Then $x \notin V$, so $X - V$ is a closed set that contains $\{x\}$. Hence $\mathrm{Cl}\{x\} \subset X - V$, so $y \notin \mathrm{Cl}\{x\}$. Since $\mathrm{Cl}\{x\}$ cannot contain any other points than x, it follows that $\{x\} = \mathrm{Cl}\{x\}$ and $\{x\}$ is closed. □

[We omit the T_1 version of the following.]

Theorem 2.5.35. *Let A be a subset of a Hausdorff space X. A point $x \in X$ is a limit point of A if and only if each neighborhood U of x meets A in infinitely many points.*

Proof. If $U \cap A$ consists of infinitely many points, then it certainly contains other points than x, so U meets $A - \{x\}$.

Conversely, if $U \cap A$ is finite, then then $U \cap (A - \{x\}) = \{x_1, \dots, x_n\}$ is closed, so

$$V = U - \{x_1, \dots, x_n\} = U \cap (X - \{x_1, \dots, x_n\})$$

is open. Then $x \in V$, V is open, and $V \cap (A - \{x\}) = \varnothing$, so x is not a limit point of A. □

2.5.10 Products of Hausdorff spaces

Lemma 2.5.36. *If X and Y are Hausdorff spaces, then so is $X \times Y$.*

Proof. Let (x, y) and (x', y') be distinct points in $X \times Y$. Then $x \neq x'$ or $y \neq y'$. If $x \neq x'$ there are open subsets $U, V \subset X$ with $x \in U$, $x' \in V$ and $U \cap V = \varnothing$. Then $U \times Y, V \times Y$ are open subsets of $X \times Y$, with $(x, y) \in U \times Y$, $(x', y') \in V \times Y$ and $(U \times Y) \cap (V \times Y) = \varnothing$. The argument if $y \neq y'$ is the same. □

2.6 (§18) Continuous Functions

2.6.1 Continuity in terms of preimages

Definition 2.6.1. Let X and Y be topological spaces. A function $f\colon X \to Y$ is said to be *continuous* if for each open set V in Y the preimage

$$f^{-1}(V) = \{x \in X \mid f(x) \in V\}$$

is open in X. A continuous function is also called a *map*.

Lemma 2.6.2. *Let X, Y and Z be topological spaces. If $f\colon X \to Y$ and $g\colon Y \to Z$ are continuous functions, then the composite $g \circ f\colon X \to Z$ is continuous.*

Proof. Let $W \subset Z$ be open. Then $g^{-1}(W) \subset Y$ is open since g is continuous, and $f^{-1}(g^{-1}(W)) \subset X$ is open since f is continuous. But this set equals $(g \circ f)^{-1}(W)$, so $g \circ f$ is continuous. □

Lemma 2.6.3. *Let X and Y be topological spaces, and suppose that $\mathscr{B}$ is a basis for the topology on Y. Then a function $f\colon X \to Y$ is continuous if and only if for each basis element $B \in \mathscr{B}$ the preimage $f^{-1}(B)$ is open in X.*

Proof. Each basis element B is open in Y, so if f is continuous then $f^{-1}(B)$ is open in X.

Conversely, each open $V \subset Y$ is a union $V = \bigcup_{\alpha \in J} B_\alpha$ of basis elements, and

$$f^{-1}(V) = \bigcup_{\alpha \in J} f^{-1}(B_\alpha)\,,$$

so if each $f^{-1}(B_\alpha)$ is open in X, so is $f^{-1}(V)$. □

Lemma 2.6.4. *Let X and Y be topological spaces, and suppose that $\mathscr{S}$ is a subbasis for the topology on Y. Then a function $f\colon X \to Y$ is continuous if and only if for each basis element $S \in \mathscr{S}$ the preimage $f^{-1}(S)$ is open in X.*

Proof. We build on the previous lemma.

Each subbasis element S is a basis element in the associated basis $\mathscr{B}$, so if f is continuous then $f^{-1}(S)$ is open in X.

Conversely, each basis element $B \in \mathscr{B}$ is a finite intersection $B = S_1 \cap \cdots \cap S_n$ of subbasis elements, and

$$f^{-1}(B) = f^{-1}(S_1) \cap \cdots \cap f^{-1}(S_n)\,,$$

so if each $f^{-1}(S_i)$ is open in X, so is $f^{-1}(B)$. □

Example 2.6.5. If (X, d) and (Y, d') are metric spaces, then $f\colon X \to Y$ is continuous if and only if for each $x \in X$ and $\epsilon > 0$ there is a $\delta > 0$ such that $B_d(x, \delta) \subset f^{-1}(B_{d'}(f(x), \epsilon))$, i.e., such that for all $y \in X$ with $d(x, y) < \delta$ we have $d'(f(x), f(y)) < \epsilon$.

Example 2.6.6. Let $\mathbb{R}$ and $\mathbb{R}_\ell$ be the real numbers with the standard topology and the lower limit topology, respectively. The identity function

$$id\colon \mathbb{R} \to \mathbb{R}_\ell$$

(given by $id(x) = x$) is not continuous, since $[a, b)$ is not open in the standard topology for $a < b$. However, the identity function

$$id\colon \mathbb{R}_\ell \to \mathbb{R}$$

(given by $id(x) = x$) is continuous, since (a, b) is open in the lower limit topology.

2.6.2 Continuity at a point

Theorem 2.6.7. *Let X and Y be topological spaces, and $f\colon X \to Y$ a function. Then f is continuous if and only if for each $x \in X$ and each neighborhood V of $f(x)$ there is a neighborhood U of x with $f(U) \subset V$.*

Definition 2.6.8. We say that f is *continuous at x* if for each neighborhood V of $f(x)$ there is a neighborhood U of x with $f(U) \subset V$. Hence $f\colon X \to Y$ is continuous if and only if it is continuous at each $x \in X$.

Proof. If f is continuous, $x \in X$ and V is a neighborhood of $f(x)$, then $U = f^{-1}(V)$ is a neighborhood of x with $f(U) \subset V$.

Conversely, if V is open in Y and $x \in f^{-1}(V)$ then V is a neighborhood of $f(x)$, so by hypothesis there is a neighborhood U_x of x with $f(U_x) \subset V$. Then $x \in U_x \subset f^{-1}(V)$. Taking the union over all $x \in f^{-1}(V)$ we find that $\bigcup_{x \in f^{-1}(V)} U_x = f^{-1}(V)$ is a union of open sets, hence is open. □

2.6.3 Continuity in terms of closed sets and the closure

Theorem 2.6.9. *Let X and Y be topological spaces, and $f\colon X \to Y$ a function. The following are equivalent:*

(1) f is continuous.

(2) For every subset $A \subset X$ we have $f(\bar{A}) \subset \overline{f(A)}$.

(3) For every closed B in Y the preimage $f^{-1}(B)$ is closed.

Proof. (1) $\implies$ (2): Assume that f is continuous and $A \subset X$. Each point in $f(\bar{A})$ has the form $f(x)$ for some $x \in \bar{A}$. We must show that $f(x) \in \overline{f(A)}$. Let V be a neighborhood of $f(x)$. By continuity, $f^{-1}(V)$ is a neighborhood of x. Since $x \in \bar{A}$, the intersection $A \cap f^{-1}(V)$ is nonempty. Choose a $y \in A \cap f^{-1}(V)$. Then $f(y) \in f(A) \cap V$, since $y \in A$ implies $f(y) \in f(A)$ and $y \in f^{-1}(V)$ implies $f(y) \in V$. In particular, $f(A)$ meets V. Since V was an arbitrary neighborhood of $f(x)$ we have $f(x) \in \overline{f(A)}$.

(2) $\implies$ (3): Let $B \subset Y$ be closed, and let $A = f^{-1}(B)$. We will show that $A = \bar{A}$, so that A is closed. Now $f(A) \subset B$, so $\overline{f(A)} \subset B$, since B is closed. By hypothesis $f(\bar{A}) \subset \overline{f(A)}$, so $f(\bar{A}) \subset B$, hence $\bar{A} \subset f^{-1}(B) = A$. This implies $A = \bar{A}$.

(3) $\implies$ (1): Let $V \subset Y$ be open, then $B = Y - V$ is closed. By hypothesis

$$f^{-1}(B) = f^{-1}(Y - V) = X - f^{-1}(V)$$

is closed, so $f^{-1}(V)$ is open. Hence f is continuous. □

2.6.4 Homeomorphism = topological equivalence

Definition 2.6.10. A bijective function $f\colon X \to Y$ between topological spaces with the property that both f and $f^{-1}\colon Y \to X$ are continuous, is called a *homeomorphism*. If there exists a homeomorphism $f\colon X \to Y$ we say that X and Y are *homeomorphic spaces*, or that they are *topologically equivalent*, and write $X \cong Y$.

Lemma 2.6.11. *Being homeomorphic is an equivalence relation on any set of topological spaces:*

(1) For each space X the identity function $id\colon X \to X$, with $id(x) = x$ for all $x \in X$, is a homeomorphism.

(2) If $f\colon X \to Y$ is a homeomorphism, then so is the inverse map $f^{-1}\colon Y \to X$.

(3) If $f\colon X \to Y$ and $g\colon Y \to Z$ are homeomorphisms, then so is the composite map $gf\colon X \to Z$.

Lemma 2.6.12. *Let $f\colon X \to Y$ be a bijective function between topological spaces. The following are equivalent:*

(1) f is a homeomorphism.

(2) A set $U \subset X$ is open in X if and only if the image $f(U) \subset Y$ is open in Y.

(3) A set $V \subset Y$ is open in Y if and only if the preimage $f^{-1}(V) \subset X$ is open in X.

Proof. To say that f is continuous means that $V \subset Y$ open implies $f^{-1}(V) \subset X$ open. To say that f^{-1} is continuous means that $U \subset X$ open implies $f(U) \subset Y$ open. Now each $U \subset X$ has the form $U = f^{-1}(V)$ for a unique $V \subset Y$, with $f(U) = f(f^{-1}(V)) = V$. Hence to say that f^{-1} is continuous also means that $f^{-1}(V) \subset X$ open implies $V \subset Y$ open.

This proves that (1) and (3) are equivalent. Replacing f by f^{-1} proves that (1) and (2) are equivalent. □

In other words, f is a homeomorphism if and only the image function

$$f\colon \mathscr{P}(X) \to \mathscr{P}(Y)$$

induces a bijection from the topology on X to the topology on Y, or equivalently, if the preimage function

$$f^{-1}\colon \mathscr{P}(Y) \to \mathscr{P}(X)$$

induces a bijection from the topology on Y to the topology on X.

Remark 2.6.13. Any property of X that can be expressed in terms of its elements and its open subsets is called a *topological property* of X. If $f\colon X \to Y$ is a homeomorphism, than any topological property of X is logically equivalent to the corresponding topological property of Y obtained by replacing each element $x \in X$ by its image $f(x) \in Y$, and each open subset $U \subset X$ by its image $f(U) \subset Y$. Such topological properties are thus preserved by homeomorphisms.

Example 2.6.14. For instance, being a finite topological space, having the discrete, trivial or cofinite topology, or being a Hausdorff space, are all examples of topological properties. So if X is a Hausdorff space and $X \cong Y$ then Y is a Hausdorff space. We shall study other topological properties, like compactness and connectedness, in later sections.

2.6.5 Examples and nonexamples

Example 2.6.15. The two open intervals $X = (0, 1)$ and $Y = (a, b)$ with $a < b$, each with the subspace topology from $\mathbb{R}$, are homeomorphic. One example of a homeomorphism $f\colon (0, 1) \to (a, b)$ is given by the linear map $f(x) = a + (b - a)x = (1 - x)a + xb$. The inverse $f^{-1}\colon (a, b) \to (0, 1)$ is given by the linear map $f^{-1}(y) = (y - a)/(b - a)$. Hence any two open intervals are homeomorphic.

Example 2.6.16. The function $f\colon (-1, 1) \to \mathbb{R}$ given by $f(x) = x/(1-x^2)$ is a homeomorphism. To find the inverse we rewrite the equation $x/(1 - x^2) = y$ as $yx^2 + x - y = 0$ and solve for $x \in (-1, 1)$ as a function of $y \in \mathbb{R}$, namely

$$f^{-1}(y) = (-1 + \sqrt{1 + 4y^2})/2y = 2y/(1 + \sqrt{1 + 4y^2})\,.$$

It is well known that both f and f^{-1} are continuous, hence $\mathbb{R}$ is homeomorphic to any open interval (a, b).

Example 2.6.17. The identity function $id\colon \mathbb{R}_\ell \to \mathbb{R}$ is a bijection that is continuous but not a homeomorphism, since the inverse function ($id\colon \mathbb{R} \to \mathbb{R}_\ell$) is not continuous.

Example 2.6.18. Let $X = [0, 1)$ and $Y = S^1$, where

$$S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$$

is the unit circle in the xy-plane, with the subspace topology. Let $f\colon X \to Y$ be given by $f(t) = (\cos(2\pi t), \sin(2\pi t))$ for $t \in [0, 1)$. It is a continuous bijection, but the inverse function $g = f^{-1}\colon S^1 \to [0, 1)$ is not continuous at $f(0) = (1, 0)$. For $U = [0, 1/2)$ is open in $[0, 1)$, while the preimage $g^{-1}(U)$ is not open in S^1. This preimage equals the image $f(U)$, which is the part of S^1 that lies strictly in the upper half-plane (where $y > 0$), together with the point $(1, 0)$. No neighborhood of $(1, 0)$ in S^1 is contained in $f(U)$, so $f(U)$ is not open.

2.6.6 Constructing maps

Theorem 2.6.19. *Let A be a subset of a topological space X. The subspace topology on A is the coarsest topology for which the inclusion $i\colon A \to X$ is continuous, where $i(a) = a$ for all $a \in A$.*

Proof. For i to be continuous in a topology $\mathscr{T}'$ on A, the inverse image $i^{-1}(U) = A \cap U$ must be open in A for each open $U \subset X$, and conversely. This just means that $\mathscr{T}'$ must contain the subspace topology on A. □

Corollary 2.6.20. *The restriction $f|A\colon A \to Y$ of any continuous function $f\colon X \to Y$ to a subspace $A \subset X$ is continuous.*

Lemma 2.6.21. *The corestriction $g\colon X \to B$ of any continuous function $f\colon X \to Y$ is continuous, where $B \subset Y$ is a subspace containing $f(X)$.*

Proof. Each open subset $V \subset B$ has the form $B \cap U$, where $U \subset Y$ is open, hence $g^{-1}(V) = \{x \in X \mid g(x) \in V\} = \{x \in X \mid f(x) \in U\} = f^{-1}(U)$ is open. □

Definition 2.6.22. A map $f\colon X \to Y$ is called an *embedding* (also called an *imbedding*) if the corestriction $g\colon X \to f(X)$ is a homeomorphism, where the image $f(X) \subset Y$ has the subspace topology.

Lemma 2.6.23. *A map $f\colon X \to Y$ is an embedding if and only if it factors as the composite of a homeomorphism $h\colon X \to B$ and the inclusion $j\colon B \to Y$ of a subspace. In particular, any embedding is an injective map.*

Proof. It is clear that an embedding f factors in this way, with $B = f(X)$. Conversely, if $f = j \circ h$ with $j\colon B \to Y$ the inclusion, then $f(X) = B$ and $h\colon X \to B$ equals the corestriction of f. □

Example 2.6.24. The map $f\colon [0, 1) \to \mathbb{R}^2$ given by $f(t) = (\cos(2\pi t), \sin(2\pi t))$ is an example of an injective continuous function that is not an embedding, since the corestriction $g\colon [0, 1) \to S^1$ to its image is not a homeomorphism.

Lemma 2.6.25. *Let $P = \{p\}$ be a singleton set, with the unique topology. For each topological space X the unique map $f\colon X \to P$ is continuous.*

Proof. The only open subsets of P are $\varnothing$ and P, with preimages $\varnothing$ and X, respectively, and these are open. □

Corollary 2.6.26. *Each constant function $c\colon X \to Y$ to a point $p \in Y$ is continuous, where $c(x) = p$ for all $x \in X$.*

Theorem 2.6.27. *Let $f\colon X \to Y$ be a function, and suppose that $X = \bigcup_{\alpha \in J} U_\alpha$ is covered by a collection of open subsets $U_\alpha \subset X$. Then f is continuous if (and only if) each restriction $f|U_\alpha\colon U_\alpha \to Y$ is continuous.*

Proof. Let $V \subset Y$ be open. For each $\alpha \in J$, the preimage $(f|U_\alpha)^{-1}(V)$ is open in U_α, since $f|U_\alpha$ is continuous, hence is open in X, since U_α is open. Then

$$f^{-1}(V) = \bigcup_{\alpha \in J} (f|U_\alpha)^{-1}(V)$$

is a union of open subsets of X, hence is open. □

Theorem 2.6.28. *Let $f\colon X \to Y$ be a function, and suppose that $X = A_1 \cup \dots \cup A_n$ is covered by a finite collection of closed subsets $A_i \subset X$. Then f is continuous if (and only if) each restriction $f|A_i\colon A_i \to Y$ is continuous.*

Proof. Let $B \subset Y$ be closed. For each $1 \le i \le n$, the preimage $(f|A_i)^{-1}(B)$ is closed in A_i, since $f|A_i$ is continuous, hence is closed in X, since A_i is closed. Then

$$f^{-1}(B) = (f|A_1)^{-1}(B) \cup \dots \cup (f|A_n)^{-1}(B)$$

is a finite union of closed subsets of X, hence is closed. □

[[See Munkres p. 109 for examples.]]

2.6.7 Maps into products

Theorem 2.6.29. *Let A, X and Y be topological spaces. A function $f\colon A \to X \times Y$ is continuous if and only if both of its components $f_1 = \pi_1 \circ f\colon A \to X$ and $f_2 = \pi_2 \circ f\colon A \to Y$ are continuous.*

Proof. We use the subbasis criterion for continuity. Recall that the product topology on $X \times Y$ is generated by the subbasis $\mathscr{S}$ consisting of the open subsets $U \times Y$ for U open in X and $X \times Y$ for V open in Y. The function f is continuous if and only if $f^{-1}(S)$ is open in A for each S in this subbasis, i.e., if and only if $f^{-1}(U \times Y) = f_1^{-1}(U)$ is open for each U open in X, and $f^{-1}(X \times V) = f_2^{-1}(V)$ is open for each V open in Y. This is equivalent to the continuity of f_1 and f_2. □

Corollary 2.6.30. *The product topology on $X \times Y$ is the coarsest topology for which both of the projection maps $\pi_1\colon X \times Y \to X$ and $\pi_2\colon X \times Y \to Y$ are continuous.*

Proof. Let $A = X \times Y$ as a set, with some topology $\mathscr{T}$, and consider the identity function $f\colon A \to X \times Y$. Then $f_1 = \pi_1$ and $f_2 = \pi_2$ are both continuous if and only if f is continuous, which holds if and only if $\mathscr{T}$ is finer than the product topology. □

Example 2.6.31. Let $A = (a, b) \subset \mathbb{R}$ and $X = Y = \mathbb{R}$. A function $f\colon (a, b) \to \mathbb{R}^2$ can be written $f(t) = (f_1(t), f_2(t))$. Then f is continuous if and only if both of the component functions f_1 and f_2 are continuous.

2.7 (§19) The Product Topology

Let $\{X_\alpha\}_{\alpha\in J}$ be an indexed collection of sets. The *cartesian product*

$$\prod_{\alpha\in J} X_\alpha$$

is the set of J-indexed sequences $(x_\alpha)_{\alpha\in J}$ with $x_\alpha \in X_\alpha$, for each $\alpha \in J$.

For each $\beta \in J$ there is a projection function

$$\pi_\beta\colon \prod_{\alpha\in J} X_\alpha \to X_\beta$$

taking $(x_\alpha)_{\alpha\in J}$ to x_β.

Now suppose that each X_α is a topological space.

Definition 2.7.1. The *product topology* on $\prod_{\alpha\in J} X_\alpha$ is the topology generated by the subbasis

$$\mathscr{S} = \{\pi_\beta^{-1}(U_\beta) \mid \beta \in J,\, U_\beta \subset X_\beta \text{ open}\}$$

consisting of all preimages

$$\pi_\beta^{-1}(U_\beta)$$

where β ranges over the indexing set J and U_β ranges over the open subsets of X_β.

Lemma 2.7.2. *The elements of the subbasis $\mathscr{S}$ generating the product topology on $\prod_{\alpha\in J} X_\alpha$ are precisely the products*

$$S = \prod_{\alpha\in J} U_\alpha$$

where each $U_\alpha \subset X_\alpha$ is open, and $U_\alpha \neq X_\alpha$ for at most one $\alpha \in J$.

The elements of the associated basis $\mathscr{B}$ for the product topology on $\prod_{\alpha \in J} X_\alpha$ are precisely the products

$$B = \prod_{\alpha \in J} U_\alpha$$

where each $U_\alpha \subset X_\alpha$ is open, and $U_\alpha \neq X_\alpha$ for only finitely many $\alpha \in J$.

Proof. For each $\beta \in J$ and $U_\beta \subset X_\beta$ open we have

$$\pi_\beta^{-1}(U_\beta) = \prod_{\alpha \in J} U_\alpha$$

where $U_\alpha = X_\alpha$ for all $\alpha \neq \beta$.

The intersection of two subbasis elements $\pi_\beta^{-1}(U)$ and $\pi_\beta^{-1}(V)$, with $U, V \subset X_\beta$ for the same $\beta \in J$, is equal to the subbasis element $\pi_\beta^{-1}(U \cap V)$. Hence the basis elements all have the form

$$\pi_{\beta_1}^{-1}(U_{\beta_1}) \cap \cdots \cap \pi_{\beta_n}^{-1}(U_{\beta_n})$$

for some finite subset $\{\beta_1, \dots, \beta_n\} \subset J$ and open subsets $U_{\beta_i} \subset X_{\beta_i}$ for $1 \leq i \leq n$, where all of the β_i are distinct. This finite intersection equals

$$\prod_{\alpha \in J} U_\alpha$$

where $U_\alpha = X_\alpha$ for all $\alpha \notin \{\beta_1, \dots, \beta_n\}$. □

As always, each open subset of $\prod_{\alpha \in J} X_\alpha$ is the union of a collection of basis elements, and each such union is open.

2.7.1 Properties of general product spaces

Theorem 2.7.3. *Let $\{X_\alpha\}_{\alpha \in J}$ be a collection of topological spaces. A function $f \colon A \to \prod_{\alpha \in J} X_\alpha$ is continuous if and only if all of its components $f_\beta = \pi_\beta \circ f \colon A \to X_\alpha$ are continuous.*

Corollary 2.7.4. *The product topology on $\prod_{\alpha \in J} X_\alpha$ is the coarsest topology for which all of the projection maps $\pi_\beta \colon \prod_{\alpha \in J} X_\alpha \to X_\beta$ are continuous.*

Theorem 2.7.5. *Let A_α be a subspace of X_α for each $\alpha \in J$. The product topology on $\prod_{\alpha \in J} A_\alpha$ equals the subspace topology from $\prod_{\alpha \in J} X_\alpha$.*

Theorem 2.7.6. *If X_α is a Hausdorff space, for each $\alpha \in J$, then $\prod_{\alpha \in J} X_\alpha$ is Hausdorff.*

Theorem 2.7.7. *Let A_α be a subset of X_α for each $\alpha \in J$. The closure of the product $\prod_{\alpha \in J} A_\alpha$ equals the product of the closures:*

$$\overline{\prod_{\alpha \in J} A_\alpha} = \prod_{\alpha \in J} \bar{A}_\alpha \,.$$

[[See Munkres page 116.]]

2.8 (§20) The Metric Topology

2.8.1 Bounded metrics

Let (X, d) be a metric space. Recall the associated metric topology $\mathscr{T}_d$ on X.

Definition 2.8.1. A topological space $(X, \mathscr{T})$ is *metrizable* if there exists a metric d on X so that $\mathscr{T}$ is the topology associated to d.

Definition 2.8.2. A metric space (X, d) is *bounded* if there exists a number M such that $d(x, y) \le M$ for all $x, y \in X$. If (X, d) is bounded the *diameter* of X is the least upper bound

$$\operatorname{diam}(X) = \sup\{d(x, y) \mid x, y \in X\}\,.$$

Being bounded is a metric, not a topological property:

Theorem 2.8.3. *Let (X, d) be a metric space. Define the* standard bounded metric $\bar{d}\colon X \times X \to \mathbb{R}$ *by*

$$\bar{d}(x, y) = \min\{d(x, y), 1\}\,.$$

Then $\bar{d}$ is a bounded metric on X that defines the same topology as d.

Proof. Checking that $\bar{d}(x, y) = 0$ if and only if $x = y$, and $\bar{d}(y, x) = \bar{d}(x, y)$, is trivial. To prove the triangle inequality

$$\bar{d}(x, z) \le \bar{d}(x, y) + \bar{d}(y, z)$$

we divide into two cases. If $d(x, y) \ge 1$ or $d(y, z) \ge 1$, then $\bar{d}(x, y) = 1$ or $\bar{d}(y, z) = 1$, so

$$\bar{d}(x, z) \le 1 \le \bar{d}(x, y) + \bar{d}(y, z)\,.$$

Otherwise, $\bar{d}(x, y) = d(x, y)$ and $\bar{d}(y, z) = d(y, z)$, so

$$\bar{d}(x, z) \le d(x, z) \le d(x, y) + d(y, z) = \bar{d}(x, y) + \bar{d}(y, z)\,.$$

It is clear that $\bar{d}(x, y) \le 1$ for all $x, y \in X$, so $(X, \bar{d})$ is bounded.

In any metric space, the collection of ϵ-balls with $\epsilon < 1$ is a basis for the associated topology. These collections are the same for d and $\bar{d}$. □

2.8.2 Euclidean n-space

Definition 2.8.4. Let $X = \mathbb{R}^n$ be the set of real n-tuples $x = (x_1, \dots, x_n)$. The *Euclidean norm* on $\mathbb{R}^n$ is given by

$$\|x\| = \sqrt{x_1^2 + \cdots + x_n^2}$$

and the *Euclidean metric* is defined by

$$d(x, y) = \|y - x\|\,.$$

The *sup norm* = *max norm* on $\mathbb{R}^n$ is given by

$$\|x\|_\infty = \max\{|x_1|, \dots, |x_n|\}$$

and the *square metric* is defined by

$$\rho(x, y) = \|y - x\|_\infty\,.$$

Theorem 2.8.5. *For each $n \ge 0$, the Euclidean metric d and the square metric ρ define the same topology on $\mathbb{R}^n$.*

2.8.3 Infinite dimensional Euclidean space

For any indexing set J, consider the set $\mathbb{R}^J$ of real J-tuples $x = (x_\alpha)_{\alpha \in J}$, or equivalently, of functions

$$x \colon J \to \mathbb{R}\,.$$

For example, when $J = \{1, 2, \dots, n\}$ we can identify $\mathbb{R}^{\{1,2,\dots,n\}}$ with $\mathbb{R}^n$.

When $J = \{1, 2, \dots\} = \mathbb{N}$ we write $\mathbb{R}^\omega$ for the set of real sequences $x = (x_n)_{n=1}^\infty$.

The formulas

$$\|x\| = \sqrt{x_1^2 + x_2^2 + \dots}$$

and

$$\|x\|_\infty = \sup\{|x_1|, |x_2|, \dots\}$$

are not well-defined for all $x \in \mathbb{R}^\omega$.

However, $\|x\|_\infty$ does make sense for all bounded sequences in $\mathbb{R}$. Replacing the usual metric on $\mathbb{R}$ with the standard bounded metric will therefore allow us to generalize the square metric to infinite dimensions.

Definition 2.8.6. Let (X, d) be any metric space, with associated standard bounded metric $\bar{d}$. Let J be any set, and let X^J be the set of J-tuples $x \colon J \to X$ in X. The function

$$\bar{\rho}(x, y) = \sup\{\bar{d}(x_\alpha, y_\alpha) \mid \alpha \in J\}$$

defines a metric on X^J, called the *uniform metric* on X. The associated topology is called the *uniform topology.*

Theorem 2.8.7. *The uniform topology on $\mathbb{R}^J$ is finer than the product topology.*

Proof. Consider a point $x = (x_\alpha)_{\alpha \in J}$ in $\mathbb{R}^J$ and a product topology basis neighborhood

$$B = \prod_{\alpha \in J} U_\alpha$$

of x, where each U_α is open and $U_\alpha \neq \mathbb{R}$ only for $\alpha \in \{\alpha_1, \dots, \alpha_n\} \subset J$. For each i choose an $\epsilon_i > 0$ so that $B_{\bar{d}}(x_i, \epsilon_i) \subset U_i$. Let $\epsilon = \min\{\epsilon_1, \dots, \epsilon_n\}$. Then $B_{\bar{\rho}}(x, \epsilon) \subset B$, since if $y \in B_{\bar{\rho}}(x, \epsilon)$ then

$$\bar{\rho}(x, y) = \sup\{\bar{d}(x_\alpha, y_\alpha) \mid \alpha \in J\} < \epsilon$$

so that $\bar{d}(x_\alpha, y_\alpha) < \epsilon$ for all $\alpha \in J$. In particular, $y_\alpha \in U_\alpha$ for all $\alpha \in J$. □

[[Exercise: The uniform topology is strictly finer than the product topology for J infinite.]]

The product topology is metrizable when J is countable. It suffices to consider the case $J = \mathbb{N}$.

Theorem 2.8.8. *Let $\bar{d}(x, y) = \min\{|y - x|, 1\}$ be the standard bounded metric on $\mathbb{R}$. For points $x = (x_n)_{n=1}^\infty$, $y = (y_n)_{n=1}^\infty \in \mathbb{R}^\omega$ define*

$$D(x, y) = \sup\Big\{\frac{\bar{d}(x_n, y_n)}{n} \mid n \in \mathbb{N}\Big\}\,.$$

Then D is a metric that induces the product topology on $\mathbb{R}^\omega$.

Proof. ((See Munkres p. 125 for the proof that D is a metric.))

Let us show that the product topology on $\mathbb{R}^\omega$ is finer than the metric topology from D. Suppose that $x \in V$ where $V \subset \mathbb{R}^\omega$ is open in the metric topology. Then there is an $\epsilon > 0$ with $B_D(x, \epsilon) \subset V$. We seek an open set U in the product topology with $x \in U \subset B_D(x, \epsilon)$.

Let

$$U_n = B_{\bar{d}}(x_n, n\epsilon/2)$$

for all $n \in \mathbb{N}$. More explicitly, $U_n = (x_n - n\epsilon/2, x_n + n\epsilon/2)$ for all $1/n \geq \epsilon/2$, and $U_n = \mathbb{R}$ for all $1/n < \epsilon/2$. Then each U_n is open, and $U_n \neq \mathbb{R}$ only for finitely many n, so

$$U = \prod_{n=1}^{\infty} U_n$$

is open in the product topology, with $x \in U$. If $y \in U$ then $y_n \in U_n$, so $\bar{d}(x_n, y_n) < n\epsilon/2$ and $\bar{d}(x_n, y_n)/n < \epsilon/2$, for each $n \in \mathbb{N}$. Hence $D(x, y) \leq \epsilon/2 < \epsilon$ and $U \subset B_D(x, \epsilon)$, as required.

Conversely, we show that the metric topology from D is finer than the product topology on $\mathbb{R}^\omega$. Let $x \in U$ where $U \subset \mathbb{R}^\omega$ is open in the product topology. Then there is a basis element B for the product topology, with $x \in B \subset U$. We seek an $\epsilon > 0$ such that $B_D(x, \epsilon) \subset B$.

The basis element has the form

$$B = \prod_{n=1}^{\infty} U_n$$

where $x_n \in U_n$ is open for each n, and $U_n \neq \mathbb{R}$ only for finitely many n. If $U_n = \mathbb{R}$ let $\delta_n = n$, otherwise choose $0 < \delta_n < 1$ such that $(x_n - \delta_n, x_n + \delta_n) \subset U_n$. Let

$$\epsilon = \min\left\{\frac{\delta_n}{n} \mid n \in \mathbb{N}\right\}.$$

The minimum is well-defined since only finitely many of these numbers are different from 1. If $y \in B_D(x, \epsilon)$ then

$$\bar{d}(x_n, y_n)/n < \epsilon \leq \delta_n/n$$

for all n. It follows that $y_n \in U_n$, since there is only something to check if $U_n \neq \mathbb{R}$, in which case $\delta_n < 1$, and $\bar{d}(x_n, y_n) < \delta_n$ implies $y_n \in (x_n - \delta_n, x_n + \delta_n) \subset U_n$. Hence $B_D(x, \epsilon) \subset B$, as required. □

2.9 (§21) The Metric Topology (continued)

Definition 2.9.1. A topological space X has a *countable basis at a point* $x \in X$ if there is a countable collection $\{B_n\}_{n=1}^{\infty}$ of neighborhoods of x in X, such that each neighborhood U of x contains (at least) one of the B_n. A space with a countable basis at each of its points is said to satisfy the *first countability axiom.* Replacing each B_n by $B_1 \cap \cdots \cap B_n$ we may assume that the neighborhoods are nested:

$$B_1 \supset B_2 \supset \cdots \supset B_n \supset \dots .$$

Lemma 2.9.2. *Each metric space (X, d) satisfies the first countability axiom.*

Proof. A countable basis at $x \in X$ is given by the neighborhoods $B_d(x, 1/n)$ for $n \in \mathbb{N}$. □

Lemma 2.9.3 (The sequence lemma). *Let A be a subspace of a topological space X. If there is a sequence of points in A that converges to x, then $x \in \bar{A}$. If X is metrizable, then the converse holds.*

Proof. Converse: Let $x \in \bar{A}$. Then for each $n \in \mathbb{N}$ the neighborhood $B_d(x, 1/n)$ of x meets A, so we can choose an $x_n \in A \cap B_d(x, 1/n)$. Then $x_n \to x$ as $n \to \infty$, as each neighborhood U of x contains some $B_d(x, 1/N)$, hence also each $B_d(x, 1/n)$ for $n \geq N$, so $x_n \in U$ for all $n \geq N$. □

Theorem 2.9.4. *Let $f\colon X \to Y$ be a function. If f is continuous, then for every convergent sequence $x_n \to x$ in X the image sequence $f(x_n)$ converges to $f(x)$ in Y. If X is metrizable, then the converse holds.*

Proof. Converse: Let $A \subset X$ be a subset. We prove that $f(\bar{A}) \subset \overline{f(A)}$. Any point in $f(\bar{A})$ has the form $f(x)$ with $x \in \bar{A}$. By the lemma above, there exists a sequence $(x_n)_{n=1}^\infty$ in A with $x_n \to x$ as $n \to \infty$. By hypothesis, $f(x_n) \to f(x)$ as $n \to \infty$, where $(f(x_n))_{n=1}^\infty$ is a sequence of points in $f(A) \subset Y$. Hence $f(x)$ is in the closure of $f(A)$. □

Lemma 2.9.5. *An uncountable product $\mathbb{R}^J$ of copies of $\mathbb{R}$ is not metrizable.*

Proof. Let J be uncountable. We show that $X = \mathbb{R}^J$ does not satisfy the converse claim in the sequence lemma. Let

$$A = \{(x_\alpha)_{\alpha \in J} \mid x_\alpha \neq 1 \text{ for only finitely many } \alpha \in J\}$$

and $x = 0 = (0)_{\alpha \in J}$. We shall prove (1) that $0 \in \bar{A}$, but (2) there is no sequence $(x_n)_{n=1}^\infty$ of points in A that converges to 0. Here each x_n is a J-tuple $(x_{n,\alpha})_{\alpha \in J}$.

Claim (1): Consider any basis element $B = \prod_{\alpha \in J} U_\alpha$ containing 0. Here $0 \in U_\alpha \subset \mathbb{R}$ is open, and $U_\alpha \neq \mathbb{R}$ only for finitely many $\alpha \in J$. Let

$$y_\alpha = \begin{cases} 0 & \text{if } U_\alpha \neq \mathbb{R}, \\ 1 & \text{if } U_\alpha = \mathbb{R}. \end{cases}$$

Then $y = (y_\alpha)_{\alpha \in J}$ is in A and in B, hence $A \cap B \neq \varnothing$. So $0 \in \bar{A}$.

Claim (2): Suppose that $(x_n)_{n=1}^\infty$ is a sequence of points in A. For each $n \in \mathbb{N}$, let J_n be the finite set of $\alpha \in J$ such that $x_{n,\alpha} \neq 1$. Then $\bigcup_{n=1}^\infty J_n$ is a countable union of finite sets, hence is countable. Since J is uncountable there exists a $\beta \in J - \bigcup_{n=1}^\infty J_n$. Then $x_{n,\beta} = 1$ for all n, and

$$U = \pi_\beta^{-1}(-1/2, 1/2)$$

is a neighborhood of 0 that does not contain any x_n. Hence $(x_n)_{n=1}^\infty$ cannot converge to 0. □

2.10 (§22) The Quotient Topology

Each injective function $f\colon X \to Y$ factors as a bijection $g\colon X \to B$ followed by an inclusion $i\colon B \subset Y$, where $B = f(X)$ is the image of f. When Y is a topological space, we defined the subspace topology on B to be the coarsest topology making $i\colon B \to Y$ continuous. We may also give X the unique topology making $g\colon X \to B$ a homeomorphism, and then $f\colon X \to Y$ is an embedding.

In this section we consider the more-or-less dual situation of a surjective function $f\colon X \to Y$. When X is a topological space we shall explain how to give Y the finest topology making f continuous, called the *quotient topology*.

2.10.1 Equivalence relations

Definition 2.10.1. A *relation* $\sim$ (read: "tilde") on a set X is a subset $R \subset X \times X$, where we write $x \sim y$ if and only of $(x, y) \in R$. An *equivalence relation* on X is a relation $\sim$ satisfying the three properties:

(1) $x \sim x$ for each $x \in X$.

(2) $x \sim y$ implies $y \sim x$ for $x, y \in X$.

(3) $x \sim y$ and $y \sim z$ implies $x \sim z$ for $x, y, z \in X$.

Definition 2.10.2. If $\sim$ is an equivalence relation on X, let

$$[x] = \{y \in X \mid x \sim y\}$$

be the *equivalence class* of $x \in X$. (This is a subset of X.) Note that $x \in [x]$ for each $x \in X$, and $[x] = [y]$ if and only if $x \sim y$. Hence the equivalence classes are nonempty subsets of X that cover X, and which are mutually disjoint. Let

$$X/{\sim} = \{[x] \mid x \in X\}$$

(read: "X mod tilde") be the set of equivalence classes. (This is a set of subsets of X.) The *canonical surjection*

$$\pi \colon X \to X/{\sim}$$

is given by $\pi(x) = [x]$.

Lemma 2.10.3. *Let $f \colon X \to Y$ be a surjective function. Define an equivalence relation $\sim$ on X by $x \sim y$ if and only if $f(x) = f(y)$. There is an induced bijection*

$$h \colon X/{\sim} \to Y$$

given by $h([x]) = f(x)$. Its inverse h^{-1} takes $y \in Y$ to the preimage $f^{-1}(y)$, which equals $[x]$ for any choice of $x \in f^{-1}(y)$. The surjection $f \colon X \to Y$ thus factors as the composite of the canonical surjection $\pi \colon X \to X/{\sim}$ and the bijection $h \colon X/{\sim} \to Y$.

Proof. The function h is well-defined since $[x] = [y]$ only if $x \sim y$, in which case $f(x) = f(y)$ by assumption. It is surjective since each element of Y has the form $f(x) = h([x])$ for some $x \in X$. It is injective since $h([x]) = h([y])$ implies $f(x) = f(y)$ so $x \sim y$ and $[x] = [y]$. $\square$

In this way we can go back and forth between equivalence relations on X and surjective functions $X \to Y$, at least up to a bijection.

2.10.2 Quotient maps

Definition 2.10.4. Let $f \colon X \to Y$ be a surjective function, where X is a topological space. The *quotient topology* on Y (from X) is the collection of subsets $U \subset Y$ such that $f^{-1}(U)$ is open in X.

Lemma 2.10.5. *The quotient topology is a topology on Y.*

Proof. (1): $f^{-1}(\varnothing) = \varnothing$ and $f^{-1}(Y) = X$ are both open in X, so $\varnothing$ and Y are open in the quotient topology on Y.

(2): If $\{U_\alpha\}_{\alpha\in J}$ is a collection of open subsets of Y then each $f^{-1}(U_\alpha)$ is open in X, so

$$f^{-1}(\bigcup_{\alpha\in J} U_\alpha) = \bigcup_{\alpha\in J} f^{-1}(U_\alpha)$$

is a union of open sets in X hence is open in X, so $\bigcup_{\alpha\in J} U_\alpha$ is open in the quotient topology on Y.

(2): If $\{U_1, \dots, U_n\}$ is a finite collection of open subsets of Y then each $f^{-1}(U_i)$ is open in X, so

$$f^{-1}(U_1 \cap \cdots \cap U_n) = f^{-1}(U_1) \cap \cdots \cap f^{-1}(U_n)$$

is a finite intersection of open sets in X hence is open in X, so $U_1 \cap \cdots \cap U_n$ is open in the quotient topology on Y. □

Definition 2.10.6. A surjective function $f: X \to Y$ between topological spaces is called a *quotient map* if Y has the quotient topology from X. i.e., if $U \subset Y$ is open if and only if $f^{-1}(U)$ is open in X.

Lemma 2.10.7. *A quotient map $f: X \to Y$ is continuous.*

Proof. When Y has the quotient topology from X, $U \subset Y$ is open if and only if $f^{-1}(U)$ is open in X. In particular f is continuous. □

Lemma 2.10.8. *Let $f: X \to Y$ be a surjective function, where X is a topological space. The quotient topology is the finest topology on Y such that $f: X \to Y$ is continuous.*

Proof. Let $\mathscr{T}$ be a topology on Y such that $f: X \to Y$ is continuous. Then for each $U \in \mathscr{T}$ we have that $f^{-1}(U)$ is open in X, so U is in the quotient topology. Hence $\mathscr{T}$ is coarser than the quotient topology. □

Lemma 2.10.9. *A surjective function $f: X \to Y$ is a quotient map if and only if the following condition holds: a subset $A \subset Y$ is closed if and only if $f^{-1}(A)$ is closed in X.*

Proof. Let $U = Y - A$. Then A is closed if and only if U is open, and $f^{-1}(A)$ is closed if and only if

$$X - f^{-1}(A) = f^{-1}(Y - A) = f^{-1}(U)$$

is open. □

Lemma 2.10.10. *A bijective quotient map $f: X \to Y$ is a homeomorphism, and conversely.*

2.10.3 Open and closed maps

Definition 2.10.11. A continuous function $f: X \to Y$ is called an *open map* if $f(U)$ is open in Y for each open $U \subset X$. It is called a *closed map* if $f(A)$ is closed in Y for each closed $A \subset X$.

Lemma 2.10.12. *Let $\mathscr{B}$ be a basis for a topology on X. A map $f: X \to Y$ is open if and only if $f(B)$ is open in Y for each basis element $B \in \mathscr{B}$.*

Proof. Each basis element is open in X, so if f is an open map then $f(B)$ will be open in Y.

Conversely, any open subset $U \subset X$ is a union of basis elements $U = \bigcup_{\alpha\in J} B_\alpha$, so if each $f(B_\alpha)$ is open in Y then

$$f(U) = f(\bigcup_{\alpha\in J} B_\alpha) = \bigcup_{\alpha\in J} f(B_\alpha)$$

is open. □

Lemma 2.10.13. *(1) Each surjective, open map $f\colon X \to Y$ is a quotient map.*

(2) Each surjective, closed map $f\colon X \to Y$ is a quotient map.

Proof. (1): Let $U \subset Y$. If U is open then $f^{-1}(U)$ is open since f is continuous. Conversely, if $f^{-1}(U)$ is open in X then

$$U = f(f^{-1}(U))$$

because f is surjective, and this is an open subset of Y since f is an open map.

(2): Let $A \subset Y$. If A is closed then $f^{-1}(A)$ is closed since f is continuous. Conversely, if $f^{-1}(A)$ is closed in X then

$$A = f(f^{-1}(A))$$

because f is surjective, and this is a closed subset of Y since f is a closed map. □

Example 2.10.14. Let $X = [0,1]$ in the subspace topology from $\mathbb{R}$, and let $Y = S^1$ be the *circle* in the subspace topology from $\mathbb{R}^2$. Let

$$f\colon [0,1] \to S^1$$

be given by $f(t) = (\cos(2\pi t), \sin(2\pi t))$. It is clearly continuous and surjective. It can also be shown to be closed (using compactness?), hence is a quotient map. It is not open, since $U = [0, 1/2)$ is open in $[0,1]$, but the image $f(U)$ is not open in S^1.

Define an equivalence relation $\sim$ on $[0,1]$ by $s \sim t$ if and only if $f(s) = f(t)$. The equivalence classes for this relation are $[0] = [1] = \{0,1\}$ and $[t] = \{t\}$ for $0 < t < 1$. We get an induced bijection

$$h\colon [0,1]/{\sim} \to S^1$$

and $f = h \circ \pi$. (We might also write $0 \sim 1$ for this equivalence relation.) Then h is a homeomorphism from $[0,1]/{\sim}$ with the quotient topology from $[0,1]$ to S^1 with the quotient topology from $[0,1]$, which equals the subspace topology from $\mathbb{R}^2$.

Example 2.10.15. Let $X = [0,1] \times [0,1]$ be a square in the subspace topology from $\mathbb{R} \times \mathbb{R} = \mathbb{R}^2$, and let $Y = S^1 \times S^1$ in the product topology, which equals the subspace topology from $\mathbb{R}^2 \times \mathbb{R}^2 = \mathbb{R}^4$. We call Y a *torus*. Let

$$g\colon [0,1] \times [0,1] \to S^1 \times S^1$$

be given by $g(s,t) = (\cos(2\pi s), \sin(2\pi s), \cos(2\pi t), \sin(2\pi t))$. This is the product $f \times f$ of two copies of the quotient map $f\colon [0,1] \to S^1$ discussed above.

The function g is continuous and surjective. It is also closed, hence a quotient map, and induces a homeomorphism

$$h\colon ([0,1] \times [0,1])/{\sim} \to S^1 \times S^1$$

where $\sim$ is the equivalence relation given by $(s,t) \sim (s',t')$ precisely if $g(s,t) = g(s',t')$. The equivalence classes of this relation are the 4-element set

$$\{(0,0), (0,1), (1,0), (1,1)\},$$

the 2-element sets

$$\{(s,0), (s,1)\} \quad \text{and} \quad \{(0,t), (1,t)\},$$

and the singleton sets (s,t), for $0 < s, t < 1$

In this way the torus $S^1 \times S^1$ is realized, up to homeomorphism, as a quotient space of the square $[0,1] \times [0,1]$, with respect to the equivalence relation $\sim$ generated by the relations $(s,0) \sim (s,1)$ and $(0,t) \sim (1,t)$ for all $s,t \in [0,1]$.

Example 2.10.16. Let $X = \mathbb{R}$ and $Y = \{n, z, p\}$. Define $f\colon X \to Y$ by

$$f(x) = \begin{cases} n & \text{if } x < 0, \\ z & \text{if } x = 0, \\ p & \text{if } x > 0. \end{cases}$$

The quotient topology on Y from $\mathbb{R}$ is the collection

$$\{\varnothing, \{n\}, \{p\}, \{n, p\}, Y\} .$$

It is a non-Hausdorff topology, where z is the only closed point.

Example 2.10.17. Let $X = \mathbb{R} \times \mathbb{R}$ and $Y = \mathbb{R}$. Let $\pi_1\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be the projection $\pi_1(x, y) = x$. Then π_1 is continuous and surjective. It is an open map, since for each basis element $B = U \times V \subset \mathbb{R} \times \mathbb{R}$ for the product topology, with $U, V \subset \mathbb{R}$ open, the image $\pi_1(B) = U$ is open in $\mathbb{R}$. Hence π_1 is a quotient map.

It is not a closed map, since the hyperbola

$$C = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid xy = 1\}$$

is a closed subset of $\mathbb{R} \times \mathbb{R}$, but $\pi_1(C) = \mathbb{R} - \{0\}$ is not closed in $\mathbb{R}$. To see that C is closed, use the continuous function $m\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ taking (x, y) to xy, and note that $C = m^{-1}(1)$ is the preimage of the closed point $\{1\}$.

Example 2.10.18. Let $A = C \cup \{(0, 0)\}$ be a subspace of $X = \mathbb{R}^2$, and consider the restricted map $f = \pi_1|A\colon A \to \mathbb{R}$. It is continuous and surjective, but not a quotient map. For $\{0\}$ is not open in $\mathbb{R}$, but its preimage $f^{-1}(0) = \{(0, 0)\}$ is open in the subspace topology on A.

Theorem 2.10.19. *Let $f\colon X \to Y$ be a quotient map, let $A \subset X$ and $B = f(A) \subset Y$ be subspaces, and let $g\colon A \to B$ be the restricted map. Assume that $A = f^{-1}(B)$.*

(1) If A is open (or closed) in X, the g is a quotient map.

(2) If f is an open map (or a closed map), then g is a quotient map.

Proof. Consider the diagram

$$\begin{array}{ccccccc} g^{-1}(V) & \rightarrowtail & A & \rightarrowtail & X & \leftarrowtail & U \\ & & \downarrow g & & \downarrow f & & \\ V & \rightarrowtail & B & \rightarrowtail & Y & \leftarrowtail & f(U) \end{array}$$

We first prove the identities

$$g^{-1}(V) = f^{-1}(V)$$

for $V \subset B$, and

$$g(A \cap U) = B \cap f(U)$$

for $U \subset X$. It is clear that $g^{-1} \subset f^{-1}(V)$. Conversely, if $x \in f^{-1}(V)$ then $f(x) \in V \subset B$, so $x \in f^{-1}(B) = A$, hence $x \in g^{-1}(V)$. It is also clear that $g(A \cap U) = f(A \cap U) \subset B \cap f(U)$. Conversely, if $y \in B \cap f(U)$ then there is an $x \in U$ with $f(x) = y$. Since $f(x) = y \in B$ we get $x \in f^{-1}(B) = A$, so $x \in A \cap U$. Hence $y \in f(A \cap U)$.

Next, suppose that A is open. Given $V \subset B$ with $g^{-1}(V)$ open in A we want to prove that V is open in B. Since A is assumed to be open in X we know that $g^{-1}(V) = f^{-1}(V)$ is open in

X. Since f is a quotient map, V is open in Y. Hence $V = B \cap V$ is open in B. (Same argument for A closed.)

Finally, suppose that f is an open map. Given $V \subset B$ with $g^{-1}(V)$ open in A we want to prove that V is open in B. There is an open subset $U \subset X$ with $g^{-1}(V) = A \cap U$. Then

$$g(g^{-1}(V)) = g(A \cap U) = B \cap f(U) .$$

Here $V = g(g^{-1}(V))$ since g is surjective, and $f(U)$ is open in Y since f is assumed to be an open map. Hence $V = B \cap f(U)$ is open in B. (Same argument for f a closed map.) □

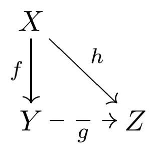
Remark 2.10.20. The composite of two quotient maps $f\colon X \to Y$ and $g\colon Y \to Z$ is a quotient map $gf\colon X \to Z$.

The product of two quotient maps is in general not a quotient map. Some condition like local compactness is usually needed.

The image of a Hausdorff space X under a quotient map $f\colon X \to Y$ needs not be a Hausdorff space.

The quotient topology has a universal mapping property, somewhat dual to that of the subspace and product topologies.

Theorem 2.10.21. *Let $f\colon X \to Y$ be a quotient map, and Z any topological space. Let $h\colon X \to Z$ be a function such that $h(x) = h(y)$ whenever $f(x) = f(y)$. Then h induces a unique function $g\colon Y \to Z$ with $h = g \circ f$.*



The induced function g is continuous if and only if h is continuous. Furthermore, g is a quotient map if and only if h is a quotient map.

Proof. Each element $y \in Y$ has the form $y = f(x)$ for $x \in X$, since f is surjective, so we can (and must) define $g(y) = h(x)$.

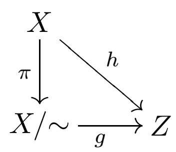
If g is continuous, then so is the composite $h = g \circ f$. Conversely, suppose that h is continuous. To prove that g is continuous, let $V \subset Z$ be open. To prove that $g^{-1}(V)$ is open in Y it suffices to show that $f^{-1}(g^{-1}(V))$ is open in X, since f is a quotient map. But $f^{-1}(g^{-1}(V)) = h^{-1}(V)$, and $h^{-1}(V)$ is open in X by the assumption that h is continuous.

If g is a quotient map, then so is the composite $h = g \circ f$. Conversely, suppose that h is a quotient map. Then g is surjective, since any $z \in Z$ has the form $z = h(x) = g(f(x))$ for some $x \in X$. Lastly, let $V \subset Z$ and suppose that $g^{-1}(V)$ is open. We must show that V is open. Now $f^{-1}(g^{-1}(V)) = h^{-1}(V)$ is open, since f is continuous. Hence V is open, by the assumption that h is a quotient map. □

Corollary 2.10.22. *Let $h\colon X \to Z$ be a surjective, continuous map. Let $\sim$ be the equivalence relation on X given by $x \sim y$ if and only if $h(x) = h(y)$, and let $X/\sim$ be the set of equivalence classes:*

$$X/\sim\, = \{h^{-1}(z) \mid z \in Z\}$$

Give $X/\sim$ the quotient topology from the canonical surjection $\pi\colon X \to X/\sim$.



(1) The map h induces a bijective, continuous map $g\colon X/{\sim} \to Z$.

(2) The map $g\colon X/{\sim} \to Z$ is a homeomorphism if and only if h is a quotient map.

(3) If Z is Hausdorff, then so is $X/{\sim}$.

Proof. □

Chapter 3

Connectedness and Compactness

3.1 (§23) Connected Spaces

3.1.1 Disjoint unions

Definition 3.1.1. Let C and D be topological spaces. Assume that $C \cap D = \varnothing$. We then write $C \sqcup D$ for the *disjoint union* $C \cup D$. There are canonical inclusions

$$i_C \colon C \to C \sqcup D \qquad \text{and} \qquad i_D \colon D \to C \sqcup D\,.$$

The *disjoint union topology* on $C \sqcup D$ is the collection of subsets $W \subset C \sqcup D$ such that $C \cap W$ is open in C and $D \cap W$ is open in D. It is the finest topology on $C \sqcup D$ for which both i_C and i_D are continuous.

Remark 3.1.2. The disjoint union $C \sqcup D$ is also known as the sum, or coproduct, of the two spaces C and D. It has a universal property dual to that of the product $C \times D$.

Each space X is homeomorphic to a disjoint union $C \sqcup D$ in some trivial ways, if $C = \varnothing$ or $D = \varnothing$. If $X \cong C \sqcup D$ in a non-trivial way, with both C and D nonempty, then we say that X is *disconnected*. Otherwise, X is a *connected* space.

3.1.2 Separations

Definition 3.1.3. Let X be a topological space. A *separation* of X is a pair U, V of disjoint, nonempty open subsets of X whose union is X. The space X is said to be *connected* (norsk: sammenhengende) if there does not exist a separation of X. Otherwise it is *disconnected*.

Remark 3.1.4. Being connected is a topological property. The empty space $X = \varnothing$ may require special care. Some authors make an exception, and say that it is not connected. The situation is similar to that of prime factorization, where the unit 1 has no proper factors in natural numbers, but is still not counted as a prime.

Lemma 3.1.5. *A space X is connected if and only if the only subsets of X that are both open and closed are $\varnothing$ and X itself.*

Proof. In a separation U, V of X, since $U \cap V = \varnothing$ and $U \cup V = X$ we must have $V = X - U$. Asking that U and V are open is equivalent to asking that U is both open and closed. Asking that both U and V are nonempty is equivalent to asking that U is different from $\varnothing$ and X. □

Lemma 3.1.6. *If U, V is a separation of X, then $X \cong U \sqcup V$ is homeomorphic to the disjoint union of the subspaces U and V.*

Proof. If $W \subset X$ is open then $U \cap W$ and $V \cap W$ are open in the subspace topologies on U and V, respectively, so W is open in $U \sqcup V$. Conversely, if W is open in $U \sqcup V$, then $U \cap W$ and $V \cap W$ are open in U and V, respectively, hence are open in X, since U and V are open in X. Thus

$$W = (U \cap W) \cup (V \cap W)$$

is a union of open sets, hence is open in X. □

Lemma 3.1.7. *If $X = C \sqcup D$ with C and D nonempty, then C, D is a separation of X.*

Proof. It is clear that C and D are disjoint, nonempty subsets of X whose union equals X. To see that C is open in the disjoint union topology, not that $C \cap C = C$ is open in C and $D \cap C = \varnothing$ is open in D. Similarly, D is open in the disjoint union topology. □

Lemma 3.1.8. *A separation of a topological space X is a pair of disjoint, nonempty subsets A and B whose union is X, neither of which contains any limit points of the other. (In symbols, $A \cap B' = \varnothing$ and $A' \cap B = \varnothing$.)*

Proof. If A and B form a separation of X, then A is closed, so $A' \subset \bar{A} = A$ does not meet B. Similarly, B' does not meet A.

Conversely, suppose that A and B are disjoint, nonempty sets with union X, $A' \cap B = \varnothing$ and $A \cap B' = \varnothing$. Then $\bar{A} \cap B = \varnothing$, since $\bar{A} = A \cup A'$, so $\bar{A} \subset X - B = A$, hence $A = \bar{A}$ is closed and B is open. Likewise, B is closed and A is open. Hence A and B form a separation of X. □

Example 3.1.9. Each 1-point space $X = \{a\}$ is connected, since there are no proper, nonempty subsets.

Example 3.1.10. Let $X = \{a, b\}$ with the Sierpinski topology $\mathscr{T}_a = \{\varnothing, \{a\}, X\}$. The proper, nonempty subsets of X are $\{a\}$ and $\{b\}$, where the first is open and the second is closed, but neither is both open and closed. Hence X is connected.

Example 3.1.11. Let $X = [-1, 0) \cup (0, 1]$ be a subspace of $\mathbb{R}$. Then $U = [-1, 0)$ and $V = (0, 1]$ is a separation of X, so X is disconnected.

Remark 3.1.12. We shall prove in the next section that $\mathbb{R}$ is connected, as is each interval $[a, b]$, $[a, b)$, $(a, b]$ and (a, b) for $-\infty \le a \le b \le \infty$.

Example 3.1.13. Each subspace $X \subset \mathbb{Q}$ with at least 2 elements is disconnected: If $p < q \in X$ choose an irrational $a \in (p, q)$. Then $U = X \cap (-\infty, a)$ and $V = X \cap (a, \infty)$ is a separation of X.

3.1.3 Constructions with connected spaces

Lemma 3.1.14. *If U and V form a separation of X, and A is a connected subspace, then $A \subset U$ or $A \subset V$.*

Proof. The intersection $A \cap U$ is open and closed in A. Since A is connected, $A \cap U$ is empty or all of A. In the first case, $A \subset V$. In the second case, $A \subset U$. □

Theorem 3.1.15. *The union of a collection of connected subspaces of X, that all have a point in common, is connected.*

Proof. Let $\{A_\alpha\}_{\alpha\in J}$ be a collection of connected subspaces of X, let $p \in \bigcap_{\alpha\in J} A_\alpha$, and let $Y = \bigcup_{\alpha\in J} A_\alpha$. To show that Y is connected, suppose that $Y = U \cup V$ is a separation of Y. Then $p \in U$ or $p \in V$. Suppose, without loss of generality, that $p \in U$. For each $\alpha \in J$ the connected space A_α is contained in U or in V. Since $p \in A_\alpha$ and $p \notin V$ we must have $A_\alpha \subset U$. This holds for each α, hence $Y \subset U$. This contradicts the assumption that V is nonempty. □

Theorem 3.1.16. *Let $A \subset B \subset \bar{A}$ be subspaces of X. If A is connected then B is connected.*

Proof. Suppose that $B = U \cup V$ is a separation. Since A is connected, we have $A \subset U$ or $A \subset V$. Without loss of generality assume that $A \subset U$. Then $B \subset \bar{A} \subset \bar{U}$. Since U is closed in B, it equals its closure $B \cap \bar{U}$ in B. Combining $B \subset \bar{U}$ and $U = B \cap \bar{U}$ we deduce that $B \subset U$. This contradicts the assumption that V is nonempty. □

Theorem 3.1.17. *The continuous image of a connected space is connected.*

Proof. Let $f\colon X \to Y$ be continuous, with X connected. We prove that the image space $Z = f(X)$ is connected. Suppose that $Z = U \cup V$ is a separation. Then $f^{-1}(U)$ and $f^{-1}(V)$ are disjoint, nonempty open subsets of X whose union equals X. This contradicts the assumption that X is connected. □

3.2 (§24) Connected Subspaces of the Real Line

We now use the existence of least upper bounds for nonempty, bounded subsets of $\mathbb{R}$ to prove that $\mathbb{R}$ is connected.

Definition 3.2.1. A subset $C \subset \mathbb{R}$ is *convex* if for any two points $a < b$ in C the closed interval $[a, b]$ is a subset of Y.

Example 3.2.2. The convex subsets of $\mathbb{R}$ are $\varnothing$, the intervals (a, b), $[a, b)$, $(a, b]$ and $[a, b]$, the rays $(-\infty, b)$, $(-\infty, b]$, (a, ∞) and $[a, \infty)$, and $\mathbb{R}$ itself.

Theorem 3.2.3. *Each convex subset $C \subset \mathbb{R}$ is connected.*

Proof. We first reduce to the special case $C = [a, b]$.

Suppose that C is the union of two disjoint, nonempty subsets A and B, each of which is open in C. Choose $a \in A$ and $b \in B$. We may assume, without loss of generality, that $a < b$. The interval $[a, b]$ is contained in C. Let $U = [a, b] \cap A$ and $V = [a, b] \cap B$. Then $[a, b]$ is the union of the disjoint, nonempty subsets U and V, each of which is open in $[a, b]$. Note that $a \in A$ and $b \in B$.

Let $c = \sup U$ be the least upper bound of U.

Clearly $c \in [a, b]$, since $a \in U$ and b is an upper bound for U. For each n the interval $(c - 1/n, c]$ contains a point $x_n \in U$, since otherwise $c - 1/n$ would have been a smaller upper bound for U. The sequence x_n converges to c, so c is in the closure of U. But U is closed in $[a, b]$, since its complement V is open, so $c \in U$. In particular, $c \neq b$, since $b \in V$.

On the other hand, c is open in U, so U contains a neighborhood $[a, b] \cap (c - \epsilon, c + \epsilon)$ of c in $[a, b]$, for some $\epsilon > 0$. We know that $c < b$, so this neighborhood contains elements greater than c. This contradicts the fact that c is an upper bound for U.

This contradiction tells us that C is connected. □

Theorem 3.2.4 (Intermediate value theorem). *Let $f\colon X \to \mathbb{R}$ be a continuous map, where X is a connected space. If $a, b \in X$ are points, and $r \in \mathbb{R}$ lies between $f(a)$ and $f(b)$, then there exists a point $c \in X$ with $f(c) = r$.*

Proof. Suppose that $f(X) \subset \mathbb{R} - \{r\} = (-\infty, r) \cup (r, \infty)$. Then X is the union of the disjoint, nonempty subsets $U = f^{-1}((-\infty, r))$ and $V = f^{-1}((r, \infty))$, each of which is open in X. This contradicts the assumption that X is connected. □

3.2.1 Path connected spaces

Definition 3.2.5. Given points $x, y \in X$ a *path* in X from x to y is a map $f\colon [a, b] \to X$ with $f(a) = x$ and $f(b) = y$, where $[a, b] \subset \mathbb{R}$.

A space X is *path connected* (norsk: veisammenhengende) if for any two points x and y of X there exists a path in X from x to y.

Lemma 3.2.6. *A path connected space is connected.*

Proof. Let X be path connected, and suppose that U and V separate X. Choose points $x \in U$ and $y \in V$, and a path $f\colon [a, b] \to X$ in X from x to y. Then $f^{-1}(U)$ and $f^{-1}(V)$ form a separation of the connected space $[a, b]$, which is impossible. □

Lemma 3.2.7. *The continuous image of a path connected space is path connected.*

Proof. If $g\colon X \to Y$ is a map, any two points in $f(X)$ can be written as $g(x)$ and $g(y)$ for $x, y \in X$. Since X is path connected, there is a path $f\colon [a, b] \to X$ in X from x to y. Then $g \circ f\colon [a, b] \to Y$ is a path in $f(X)$ from $f(x)$ to $f(y)$. Hence $f(X)$ is path connected. □

Definition 3.2.8. A subset C of a real vector space V is *convex* if for each pair of points $x, y \in C$ the straight-line path $f\colon [0, 1] \to V$ defined by

$$f(t) = (1 - t)x + ty$$

takes all of its values in C.

Example 3.2.9. Any convex subset of $\mathbb{R}^n$ is connected, since f is continuous. For example, the n-dimensional unit ball

$$B^n = \{x \in \mathbb{R}^n \mid \|x\| \le 1\}$$

is convex for $n \ge 0$, hence path connected.

Example 3.2.10. The punctured Euclidean space $\mathbb{R}^n - \{0\}$ is path connected for $n \ge 2$. For $n = 1$, the space $\mathbb{R}^1 - \{0\}$ is not (path-)connected. For $n = 0$ the space $\mathbb{R}^0 - \{0\}$ is empty, hence is path connected by convention.

Example 3.2.11. The $(n - 1)$-dimensional unit sphere

$$S^{n-1} = \{x \in \mathbb{R}^n \mid \|x\| = 1\}$$

is the continuous image of $g\colon \mathbb{R}^n - \{0\} \to S^{n-1}$ given by $g(x) = x/\|x\|$, hence is path connected for $n \ge 2$. For $n = 1$, the 0-sphere $S^0 = \{+1, -1\}$ is not path connected. For $n = 0$, the (-1)-sphere S^{-1} is empty, hence is also path connected, by convention.

Example 3.2.12. Let

$$S = \{(x, \sin(1/x)) \mid 0 < x \le 1\}$$

be a subset of $\mathbb{R}^2$. It is the image of the connected space $(0, 1]$ under the continuous map $g(t) = (t, \sin(1/t))$, hence is connected. Therefore its closure $\bar{S}$ in $\mathbb{R}^2$ is connected. This closure

$$\bar{S} = S \cup V$$

is the union of S with the vertical interval $V = \{0\} \times [-1, 1]$. The space $\bar{S}$ is called the *topologist's sine curve.*

We show that $\bar{S}$ is not path connected. Suppose that $f\colon [a, b] \to \bar{S}$ is a map with $f(a) \in V$ and $f(b) \in S$. The set of $t \in [a, b]$ with $f(t) \in V$ is closed, since V is closed in $\bar{S}$, hence has a greatest element c. The restricted function $f|[c, b]\colon [c, b] \to \bar{S}$ is then a path in $\bar{S}$ starting in V and ending in S. By reparametrizing, we may replace $[c, b]$ by $[0, 1]$. We then have a map $f\colon [0, 1] \to \bar{S}$ with $f(0) \in V$ and $f(t) \in S$ for all $t \in (0, 1]$.

Write $f(t) = (x(t), y(t))$. Then $x(0) = 0$ and $x(t) > 0$ for all $t \in (0, 1]$. We show that there is a sequence of points $t_n \in (0, 1]$ with $0 < t_n < 1/n$ and $y(t_n) = (-1)^n$. Then $t_n \to 0$ as $n \to \infty$, but the sequence $(y(t_n))_{n=1}^{\infty} = ((-1)^n)_{n=1}^{\infty}$ does not converge. This contradicts the continuity of y.

For each n we choose a $v > 1/x(1/n)$ such that $\sin(v) = (-1)^n$. Let $u = 1/v$, then $0 = x(0) < u < x(1/n)$ and $\sin(1/u) = (-1)^n$. By the intermediate value theorem for x, there is a $t_n \in (0, 1/n)$ with $x(t_n) = u$. Then $y(t_n) = (-1)^n$, as desired.

3.3 (§25) Components and Local Connectedness

Definition 3.3.1. Define an equivalence relation $\sim$ for points in a topological space X by $x \sim y$ if there is a connected subset $C \subset X$ with $x, y \in C$. The equivalence classes for $\sim$ are called the *(connected) components* of X.

Lemma 3.3.2. *$\sim$ is an equivalence relation on X.*

Theorem 3.3.3. *The components of X are connected, disjoint subspaces whose union is X. Each nonempty, connected subset of X is contained in precisely one component.*

Proof. It is clear that the components are disjoint, nonempty subspaces whose union is X, since the components are the equivalence classes for an equivalence relation. If $A \subset X$ is connected and meets two components C_1 and C_2, in points x_1 and x_2, say, then $x_1 \sim x_2$, so $C_1 = C_2$.

To show that each component C is connected, let $x_0 \in C$. For each $x \in C$ we have $x_0 \sim x$, so there exists a connected subset A_x with $x_0, x \in A_x$. Then $A_x \subset C$, so $\bigcup_{x \in C} A_x = C$. Since all A_x are connected, and all contain x_0, it follows that the union is connected. □

Theorem 3.3.4. *A finite cartesian product of connected spaces is connected.*

Proof. Let X and Y be connected spaces. We show that any two points (x, y) and (x', y') in $X \times Y$ are in the same component. First, $X \times \{y\}$ is homeomorphic to X, hence is connected. So (x, y) and (x', y) are in the same component. Next, $\{x'\} \times Y$ is homeomorphic to Y, hence is connected. So (x', y) and (x', y') are in the same component. The claim follows by associativity of $\sim$.

Given n connected spaces $X_1, \dots, X_n$, the homeomorphism

$$X_1 \times \cdots \times X_n \cong (X_1 \times \cdots \times X_{n-1}) \times X_n$$

and induction on n shows that $X_1 \times \cdots \times X_n$ is connected. □

Remark 3.3.5. In fact, an arbitrary product of connected spaces is connected, in the product topology.

3.3.1 Path components

Definition 3.3.6. Define another equivalence relation $\simeq$ for points in a topological space X by $x \simeq y$ if there is a path in X from x to y. The equivalence classes for $\simeq$ are called the *path components* of X.

Lemma 3.3.7. *$\simeq$ is an equivalence relation on X.*

Theorem 3.3.8. *The path components of X are path connected, disjoint subspaces whose union is X. Each nonempty, path connected subset of X is contained in precisely one path component.*

Example 3.3.9. The topologist's sine curve $\bar{S} = S \cup V$ is connected, hence consist of only one component. It has two path components, S and V. Note that S is open in $\bar{S}$, but not closed, and V is closed in $\bar{S}$, but not open.

3.3.2 Locally connected spaces

Definition 3.3.10. A space X is *locally connected at a point* $x \in X$ if for each neighborhood U of x there is a connected neighborhood V of x contained in U:

$$x \in V \subset U \subset X$$

We say that X is *locally connected* if it is locally connected at each of its points.

Definition 3.3.11. A space X is *locally path connected at a point* $x \in X$ if for each neighborhood U of x there is a path connected neighborhood V of x contained in U:

$$x \in V \subset U \subset X$$

We say that X is *locally path connected* if it is locally path connected at each of its points.

Example 3.3.12. The real line is locally (path) connected, since each neighborhood U of any point x contains a (path) connected basis neighborhood $(x - \epsilon, x + \epsilon)$ for some $\epsilon > 0$.

Example 3.3.13. The topologist's sine curve is not locally (path) connected, since small neighborhoods of points in V are not connected.

Theorem 3.3.14. *If X is locally connected, then each component C of X is open in X.*

Proof. Let $x \in C$. Then $U = X$ is a neighborhood of x, so by local connectivity there is a connected neighborhood V of x with $V \subset U = X$. Since V is connected, $V \subset C$. Hence C contains a neighborhood around each of its points, and must be open. □

Theorem 3.3.15. *If X is locally path connected, then each path component P of X is open in X.*

Theorem 3.3.16. *If X is a topological space, each path component of X lies in a unique component of X. If X is locally path connected, then the components and the path components of X are the same.*

Proof. Each path component P is nonempty and connected, hence lies in a unique component C. If X is locally path connected we show that $P = C$.

Let U be the union of the path components Q of X that are different from P and meet C. Since each such path component Q is connected, it lies in C, so that

$$C = P \cup U$$

is a disjoint union. Because X is locally path connected, each path component P or Q is open in X, hence so is the union U. Hence P is a nonempty, open and closed subset of C. Since C is connected, it follows that $P = C$. □

3.4 (§26) Compact Spaces

3.4.1 Open covers and finite subcovers

Definition 3.4.1. A collection $\mathscr{A} = \{U_\alpha\}_{\alpha \in J}$ of subsets of X is said to *cover* X, or to be a *covering* of X, if the union of its elements is equal to X, so $X = \bigcup_{\alpha \in J} U_\alpha$. If each element in the collection is an open subset of X, then we say that $\mathscr{A}$ is an *open cover*.

A subcollection $\mathscr{B} \subset \mathscr{A}$ that also covers X is called a *subcover* of $\mathscr{A}$. If, furthermore, $\mathscr{B}$ has finitely many elements, then we call $\mathscr{B}$ a *finite subcover* of $\mathscr{A}$.

Definition 3.4.2. A space X is said to be *compact* if for each open cover $\mathscr{A} = \{U_\alpha\}_{\alpha \in J}$ of X there exists a finite subcollection $\{U_{\alpha_1}, \dots, U_{\alpha_n}\}$ that also covers X. In other words, X is compact if each open cover of X contains a finite subcover.

Example 3.4.3. A finite topological space X is compact, since there are only finitely many different open subsets $U \subset X$, so any collection covering X is finite.

Example 3.4.4. The real line $\mathbb{R}$ is not compact, since the open cover $\mathscr{A} = \{(n-1, n+1) \mid n \in \mathbb{N}\}$ does not admit a finite subcover.

Remark 3.4.5. From S. G. Krantz' "Mathematical apocrypha".

> There is a story about Sir Michael Atiyah (1929–) and Graeme Segal (1941–) giving an oral exam to a student at Cambridge. Evidently the poor student was a nervous wreck, and it got to a point where he could hardly answer any questions as all.
>
> At one point, Atiyah (endeavoring to be kind) asked the student to give an example of a compact set. The student said: "The real line." Trying to play along, Segal said: "In what topology?"

Example 3.4.6. The real line $\mathbb{R}$ in the trivial topology is compact, since the only open covers are the collections $\{\mathbb{R}\}$ and $\{\varnothing, \mathbb{R}\}$, which are finite.

Example 3.4.7. The subspace

$$X = \{0\} \cup \{1/n \mid n \in \mathbb{N}\}$$

of $\mathbb{R}$ is compact. Given an open covering $\mathscr{A}$ of X, choose $U \in \mathscr{A}$ with $0 \in U$. Since U is open in the subspace topology, there is an $N \in \mathbb{N}$ such that $1/n \in U$ for all $n > N$. For each $1 \le n \le N$ choose $U_n \in \mathscr{A}$ with $1/n \in U_n$. Then $\mathscr{B} = \{U, U_1, \dots, U_N\}$ is a finite subcover of $\mathscr{A}$.

Example 3.4.8. Any space X in the cofinite topology is compact. If X is empty, this is trivially true. Otherwise, if $\mathscr{A} = \{U_\alpha\}_{\alpha \in J}$ is an open cover, not all U_α can be empty. Choose a $\beta \in J$ so that $U_\beta \subset X$ is nonempty. Then $X - U_\beta = \{x_1, \dots, x_n\}$ is a finite set. For each i choose an $\alpha_i \in J$ so that $x_i \in U_{\alpha_i}$. Then $\{U_\beta, U_{\alpha_1}, \dots, U_{\alpha_n}\}$ is a finite subcover of $\mathscr{A}$. Since $\mathscr{A}$ was an arbitrary open cover of X, it follows that X is compact.

Definition 3.4.9. If Y is a subspace of X, a collection $\mathscr{A}$ of subsets of X *covers* Y if the union of the elements of $\mathscr{A}$ contains Y.

Lemma 3.4.10. *Let Y be a subspace of X. Then Y is compact if and only if each covering of Y by open subsets of X contains a finite subcollection covering Y.*

Proof. Suppose that Y is compact, and that $\mathscr{A} = \{U_\alpha\}_{\alpha \in J}$ is a covering of Y by open subsets of X. Then $\{Y \cap U_\alpha\}_{\alpha \in J}$ is an open cover of Y, hence contains a finite subcover $\{Y \cap U_{\alpha_1}, \dots, Y \cap U_{\alpha_n}\}$. Then $U_{\alpha_1}, \dots, U_{\alpha_n}\}$ is a finite subcollection of $\mathscr{A}$ that covers Y.

Conversely, suppose that each covering of Y by open subsets of X contains a finite subcollection covering Y, and let $\mathscr{A} = \{V_\alpha\}_{\alpha \in J}$ be an open covering of Y. For each $\alpha \in J$ we can write $V_\alpha = Y \cap U_\alpha$ for some open subset $U_\alpha \subset X$. Then the collection $\{U_\alpha\}_{\alpha \in J}$ is a covering of Y by open subsets of X, which we have assumed contains a finite subcollection $\{U_{\alpha_1}, \dots, U_{\alpha_n}\}$ covering Y. Then $\{V_{\alpha_1}, \dots, V_{\alpha_n}\}$ is a finite subcollection of $\mathscr{A}$ that covers Y. Since $\mathscr{A}$ was an arbitrary open cover, it follows that Y is compact. □

3.4.2 Compact subspaces of Hausdorff spaces

Theorem 3.4.11. *Every closed subspace of a compact space is compact.*

Proof. ((See Munkres, page 165.)) □

Recall that finite subsets of Hausdorff spaces are closed. Compact subspaces generalize finite sets in this respect.

Theorem 3.4.12. *Every compact subspace of a Hausdorff space is closed.*

Proof. Let X be a Hausdorff space and let $K \subset X$ be a compact subspace. Let $p \in X - K$ be any point. We prove that $p \notin \bar{K}$, so that $K = \bar{K}$ is closed.

For each point $q \in K$ we have $p \neq q$, so there exist neighborhoods U_q and V_q of p and q, respectively, with $U_q \cap V_q = \varnothing$. The collection $\{V_q \mid q \in K\}$ of open subsets in X covers K, since

$$K \subset \bigcup_{q \in K} V_q \,.$$

By compactness of K, there is a finite subcollection $\{V_{q_1}, \dots, V_{q_n}\}$ that also covers K:

$$K \subset V = V_{q_1} \cup \dots \cup V_{q_n} \,.$$

Let $U = U_{q_1} \cap \dots \cap U_{q_n}$. Then U is neighborhood of p. We claim that $U \cap K = \varnothing$, so p is not in $\bar{K}$. In fact $U \cap V = \varnothing$, for if $x \in V$ then $x \in V_{q_i}$ for some i, but then $x \notin U_{q_i}$ so $x \notin U$. □

We have proved:

Lemma 3.4.13. *If X is a Hausdorff space, $K \subset X$ a compact subspace, and $p \in X - K$, then there exist disjoint open subsets U and V of X with $p \in U$ and $K \subset V$.*

Example 3.4.14. The intervals $(a, b]$, $[a, b)$ and (a, b) are not closed in $\mathbb{R}$, hence cannot be compact. We shall prove in the next section that each closed interval $[a, b]$ in $\mathbb{R}$ is compact.

Example 3.4.15. If X is an infinite set with the cofinite topology, then any subspace has the cofinite topology, hence is compact, but not every subspace is closed.

Theorem 3.4.16. *The continuous image of a compact space is compact.*

Proof. Let $f \colon X \to Y$ be continuous, and assume that X is compact. We prove that $f(X)$ is a compact subspace of Y. Let $\mathscr{A} = \{U_\alpha\}_{\alpha \in J}$ be a covering of $f(X)$ by open subsets of Y. Then $\{f^{-1}(U_\alpha)\}_{\alpha \in J}$ is an open cover of X. By compactness there exists a finite subcover $\{f^{-1}(U_{\alpha_1}), \dots, f^{-1}(U_{\alpha_n})\}$. Then the sets $\{U_{\alpha_1}, \dots, U_{\alpha_n}\}$ cover $f(X)$. □

Theorem 3.4.17. *Let $f \colon X \to Y$ be a map from a compact space X to a Hausdorff space Y.*

(1) f is a closed map.

(2) If f is surjective, then f is a quotient map.

(3) If f is bijective, then f is a homeomorphism.

(4) If f is injective, then f is an embedding.

Proof. (1): Let $A \subset X$ be a closed subset. Since X is compact, A is compact. Since f is continuous, $f(A)$ is compact. Since Y is Hausdorff, $f(A) \subset Y$ is closed.

(2): Any closed, surjective map is a quotient map.

(3): If f is bijective, the inverse function $h = f^{-1}\colon Y \to X$ is continuous, since for each closed subset $A \subset X$ the preimage $h^{-1}(A) = f(A)$ is closed in Y.

(4): If f is injective, the corestriction $g\colon X \to f(X)$ is bijective, and $f(X)$ is Hausdorff, so g is a homeomorphism. □

Example 3.4.18. The map $\mathbb{R} \to \{0\}$ shows that the continuous preimage of a compact space needs not be compact.

Definition 3.4.19. A map $f\colon X \to Y$ is said to be *proper* if for each compact subspace $K \subset Y$ the preimage $f^{-1}(K)$ is compact.

3.4.3 Finite products of compact spaces

Theorem 3.4.20. *Let X and Y be compact spaces. Then $X \times Y$ is compact.*

Corollary 3.4.21. *Any finite product of compact spaces is compact.*

Lemma 3.4.22 (The tube lemma). *Consider the product $X \times Y$, let $p \in X$, and assume that Y is compact. If $N \subset X \times Y$ is open, with $\{p\} \times Y \subset N$, then there exists a neighborhood $U \subset X$ of p with $U \times Y \subset N$.*

Proof. For each $q \in Y$ we have $(p, q) \in \{p\} \times Y \subset N$. Since N is open there is a basis element $U_q \times V_q \subset N$ for the product topology on $X \times Y$, with $p \in U_q$ open in X and $q \in V_q$ open in Y. The collection $\{V_q\}_{q\in Y}$ is an open cover of Y. By compactness of Y, there exists a finite subcover $\{V_{q_1}, \dots, V_{q_n}\}$. Let $U = U_{q_1} \cap \cdots \cap U_{q_n}$. Then $p \in U$ is open in X. We claim that $U \times Y \subset N$. For any $(x, y) \in U \times Y$ there is an $1 \le i \le n$ with $y \in V_{q_i}$. Then $x \in U \subset U_{q_i}$, so $(x, y) \in U_{q_i} \times V_{q_i} \subset N$. □

Example 3.4.23. The tube lemma fails if Y is not compact. Consider the neighborhood

$$N = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid |xy| \le 1\}$$

of $\{0\} \times \mathbb{R}$.

Proof of theorem. Let $\mathscr{A} = \{W_\alpha\}_{\alpha\in J}$ be an open cover of $X \times Y$. For each point $p \in X$, the subspace $\{p\} \times Y$ is compact, and is therefore covered by a finite subcollection $\mathscr{B}_p = \{W_{\alpha_1}, \dots, W_{\alpha_n}\}$ of $\mathscr{A}$. Let $N = W_{\alpha_1} \cup \cdots \cup W_{\alpha_n}$. Then $N \subset X \times Y$ is an open subset containing $\{p\} \times Y$. By the tube lemma, there is a neighborhood $U_p \subset X$ of p with $U_p \times Y \subset N$. Note that $U_p \times Y$ is covered by the finite subcollection $\mathscr{B}_p$ of $\mathscr{A}$.

Now let $p \in X$ vary. The collection $\{U_p\}_{p\in X}$ is an open cover of X, hence admits a finite subcover $\{U_{p_1}, \dots, U_{p_m}\}$. For each $1 \le j \le m$ the subspace $U_{p_j} \times Y$ is covered by the finite subcollection $\mathscr{B}_{p_j}$ of $\mathscr{A}$. Hence the union

$$X \times Y = \bigcup_{j=1}^{m} U_{p_j} \times Y$$

is covered by the subcollection

$$\mathscr{B}_{p_1} \cup \cdots \cup \mathscr{B}_{p_m}$$

of $\mathscr{A}$. This is a finite union of finite collections, hence is a finite subcollection of $\mathscr{A}$. Since $\mathscr{A}$ was an arbitrary open cover, it follows that $X \times Y$ is compact. □

3.4.4 The finite intersection property

Let $\mathscr{A}$ be a collection of open subsets of a space X. Let $\mathscr{C} = \{X - U \mid U \in \mathscr{A}\}$ be the collection of closed complements. To say that $\mathscr{A}$ is a cover of X is equivalent to saying that $\mathscr{C}$ has empty intersection:

$$\bigcap_{C \in \mathscr{C}} C = \bigcap_{U \in \mathscr{A}} (X - U) = X - \bigcup_{U \in \mathscr{A}} U$$

is empty if and only if $X = \bigcup_{U \in \mathscr{A}} U$.

Definition 3.4.24. A collection $\mathscr{C}$ of subsets of X has the *finite intersection property* if for each finite subcollection $\{C_1, \dots, C_n\} \subset \mathscr{C}$ the intersection

$$C_1 \cap \cdots \cap C_n$$

is nonempty.

Theorem 3.4.25. *A topological space X is compact if and only if for each collection $\mathscr{C}$ of closed subsets of X, having the finite intersection property, the intersection $\bigcap_{C \in \mathscr{C}} C$ is nonempty.*

Proof. "X is compact" is equivalent to the assertion:

> For any collection $\mathscr{A}$ of open subsets of X, if $\mathscr{A}$ covers X then some finite subcollection of $\mathscr{A}$ covers X.

This is logically equivalent to the contrapositive statement:

> For any collection $\mathscr{A}$ of open subsets of X, if no finite subcollection of $\mathscr{A}$ covers X, then $\mathscr{A}$ does not cover X.

Translated to a statement about the complementary collection of closed subsets $\mathscr{C}$, this is equivalent to:

> For any collection $\mathscr{C}$ of closed subsets of X, if no finite subcollection of $\mathscr{C}$ has empty intersection, then $\mathscr{C}$ does not have empty intersection.

In other words:

> For any collection $\mathscr{C}$ of closed subsets of X, if each finite subcollection of $\mathscr{C}$ has nonempty intersection, then $\mathscr{C}$ has nonempty intersection.

□

3.5 (§27) Compact Subspaces of the Real Line

Theorem 3.5.1. *Each closed interval $[a, b] \subset \mathbb{R}$ of the real line is compact.*

(Here $a, b \in \mathbb{R}$, we are not considering infinite intervals.)

Proof. Let $\mathscr{A} = \{U_\alpha\}_{\alpha \in J}$ be a covering of $[a, b]$ by open subsets of $\mathbb{R}$. Consider the set S of all $x \in [a, b]$ such that $[a, x]$ can be covered by a finite subcollection of $\mathscr{A}$. Then $a \in S$, since $a \in U_\alpha$ for some $\alpha \in J$, and then $\{U_\alpha\}$ is a finite subcollection of $\mathscr{A}$ that covers $[a, a] = \{a\}$. Hence S is nonempty and bounded above. Let $c = \sup S$ be the least upper bound of S. Clearly $c \in [a, b]$.

We claim that $c \in S$. Choose $\beta \in J$ with $c \in U_\beta$. Since U_β is open there exists an $\epsilon > 0$ with $(c - \epsilon, c + \epsilon) \subset U_\beta$. Since the supremum c is in the closure of S, there is some point $x \in S \cap (c - \epsilon, c + \epsilon)$. Since c is an upper bound for S, $x \le c$. Then $[a, x]$ can be covered by a finite subcollection $\{U_{\alpha_1}, \dots, U_{\alpha_n}\}$ of $\mathscr{A}$, and $[x, c]$ is contained in U_β. This implies that $[a, c] = [a, x] \cup [x, c]$ is covered by the finite subcollection $\{U_{\alpha_1}, \dots, U_{\alpha_n}, U_\beta\}$ of $\mathscr{A}$, so that $x \in S$.

We also claim that $c = b$. Suppose that $c < b$, to achieve a contradiction. Then there is a $y \in [a, b] \cap (c - \epsilon, c + \epsilon)$ with $c < y$ such that $[a, y]$ is covered by the same finite subcollection $\{U_{\alpha_1}, \dots, U_{\alpha_n}, U_\beta\}$. Hence $y \in S$, contradicting the assumption that c is an upper bound. □

Example 3.5.2. The surjective map $f: [0, 1] \to S^1$ given by $f(t) = (\cos(2\pi t), \sin(2\pi t))$ is a quotient map, since $[0, 1]$ is compact and $S^1 \subset \mathbb{R}$ is Hausdorff. Similarly for $f \times f: [0, 1] \times [0, 1] \to S^1 \times S^1$.

Theorem 3.5.3. *A subspace Y of $\mathbb{R}^n$ is compact if and only if it is closed and bounded (in any of the equivalent metrics coming from a norm).*

Proof. Since $[a, b] \subset \mathbb{R}$ is compact, any finite product

$$[a_1, b_1] \times \cdots \times [a_n, b_n] \subset \mathbb{R}^n$$

is compact, as is any closed subset Y of such a finite product. (These are the bounded subsets in the square metric.)

Conversely, the collection of open subsets

$$U_M = (-M, M) \times \cdots \times (-M, M) \subset \mathbb{R}^n$$

for $M \in \mathbb{N}$ has union $\mathbb{R}^n$, so if Y is compact then there is a finite subcollection $\{U_{M_1}, \dots, U_{M_k}\}$ of these that covers Y. Let $M = \max\{M_1, \dots, M_k\}$. Then $Y \subset U_M$ is bounded. Since $\mathbb{R}^n$ is Hausdorff we must also have that Y is closed. □

Theorem 3.5.4 (Extreme value theorem). *Let $f: X \to \mathbb{R}$ be continuous, with X compact. Then there exist points $c, d \in X$ with*

$$f(c) \le f(x) \le f(d)$$

for all $x \in X$.

Proof. The continuous image $f(X) \subset \mathbb{R}$ is compact, hence closed, so contains both its infimum and its supremum. Writing these as $f(c)$ and $f(d)$, we get the conclusion. □

3.5.1 The Lebesgue number

Definition 3.5.5. Let (X, d) be a metric space and let $A \subset X$ be a nonempty subset. For each $x \in X$ the *distance from x to A* is

$$d(x, A) = \inf\{d(x, a) \mid a \in A\}\,.$$

The *diameter of A* is

$$\operatorname{diam}(A) = \sup\{d(a, b) \mid a, b \in A\}\,.$$

Lemma 3.5.6. *The function $x \mapsto d(x, A)$ is continuous.*

Proof. Let $x, y \in X$. By the triangle inequality,

$$d(x, A) \leq d(x, a) \leq d(x, y) + d(y, a)$$

for all $a \in A$, so

$$d(x, A) - d(x, y) \leq \inf\{d(y, a) \mid a \in A\} = d(y, A)$$

and

$$|d(x, A) - d(y, A)| \leq d(x, y)$$

by symmetry in x and y. Hence $x \mapsto d(x, A)$ is continuous. □

Lemma 3.5.7. *Let $\mathscr{A}$ be an open cover of a compact metric space (X, d). There exists a $\delta > 0$ such that for each subset $B \subset X$ of diameter $< \delta$ there exists an element $U \in \mathscr{A}$ with $B \subset U$. The number δ is called a* Lebesgue number *of $\mathscr{A}$.*

Proof. If $X \in \mathscr{A}$ then any positive number is a Lebesgue number for $\mathscr{A}$. Otherwise, by compactness there is a finite subcollection $\{U_1, \ldots, U_n\}$ of $\mathscr{A}$ that covers X. Let $C_i = X - U_i$ be the closed complement; each C_i is nonempty. To say that $x \in U_i$ is equivalent to saying $d(x, C_i) > 0$, since $d(x, C_i) = 0$ if and only if $x \in C_i$.

Define $f \colon X \to \mathbb{R}$ as the average

$$f(x) = \frac{1}{n} \sum_{i=1}^{n} d(x, C_i) .$$

Claim: $f(x) > 0$ for all $x \in X$. Proof: For any given $x \in X$ there is an i with $x \in U_i$. Since U_i is open, there exists an $\epsilon > 0$ with $B_d(x, \epsilon) \subset U_i$. Then $d(x, C_i) \geq \epsilon$. Hence $f(x) \geq \epsilon/n > 0$.

Since f is continuous, it has a positive minimum value δ. Claim: δ is a Lebesgue number for $\{U_1, \ldots, U_n\}$, hence for $\mathscr{A}$. Proof: Let $B \subset X$ have diameter $< \delta$. There is only something to prove for B nonempty; choose a point $p \in B$. Then $B \subset B_d(p, \delta)$. Consider the numbers $d(p, C_i)$ for $1 \leq i \leq n$. Choose m so that $d(p, C_m)$ is the largest of these numbers. Then

$$\delta \leq f(p) \leq d(p, C_m)$$

so $B_d(p, \delta) \cap C_m = \varnothing$. Hence $B \subset B_d(p, \delta) \subset U_m$. □

3.5.2 Uniform continuity

Definition 3.5.8. Let $f \colon (X, d_X) \to (Y, d_Y)$ be a function between metric spaces. We say that f is *uniformly continuous* if given $\epsilon > 0$ there exists a $\delta > 0$ such that for any two points $x, x' \in X$ with $d_X(x, x') < \delta$ we have $d_Y(f(x), f(x')) < \epsilon$, or equivalently, if for any $x \in X$ we have $f(B_{d_X}(x, \delta)) \subset B_{d_Y}(f(x), \epsilon)$.

Theorem 3.5.9. *Let $f \colon (X, d_X) \to (Y, d_Y)$ be a continuous map om metric spaces. If X is compact then f is uniformly continuous.*

Proof. Given $\epsilon > 0$ cover Y by the balls $B(y, \epsilon/2)$ and let

$$\mathscr{A} = \{f^{-1}(B(y, \epsilon/2))\}_{y \in Y}$$

be the open covering of X by the preimages of these balls. Choose a Lebesgue number $\delta > 0$ for this open covering. If $x, x' \in X$ with $d(x, x') < \delta$ then $\{x, x'\} \subset f^{-1}(B(y, \epsilon/2))$ for some $y \in Y$, hence $\{f(x), f(x')\} \subset B(y, \epsilon/2)$. Thus $d_Y(f(x), f(x')) \leq d_Y(f(x), y) + d_Y(y, f(x')) < \epsilon$. □

3.6 (§28) Limit Point Compactness

Recall that a space X is compact if for each open cover $\mathscr{A} = \{U_\alpha\}_{\alpha \in J}$ there is a finite subcover $\{U_{\alpha_1}, \dots, U_{\alpha_n}\}$.

Recall also that a limit point of a subset $A \subset X$ is a point $p \in X$ such that each neighborhood of p meets A in a point different from p.

Definition 3.6.1. A space X is *limit point compact* if every infinite subset $A \subset X$ has a limit point.

Theorem 3.6.2. *Any compact space is limit point compact.*

Proof. Let X be compact and suppose that a subset $A \subset X$ has no limit points. We prove that A is finite.

Since A has no limit points, $\bar{A} = A \cup A' = A$ is closed. For each $p \in A$ we can choose a neighborhood U_p of p such that $A \cap U_p = \{p\}$. The collection consisting of $X - A$ and the U_p for $p \in A$ cover X. By compactness, there are finitely many points $p_1, \dots, p_n$ such that $X - A$ and $U_{p_1}, \dots, U_{p_n}$ cover X. Since A does not meet $X - A$, and meets each U_{p_i} in only one point, $A = \{p_1, \dots, p_n\}$ must be finite. □

Let $(x_n)_{n=1}^\infty$ be a sequence of points in X. If

$$n_1 < n_2 < \dots < n_k < \dots$$

is a strictly increasing sequence of natural numbers, the sequence

$$x_{n_1}, x_{n_2}, \dots, x_{n_k}, \dots$$

is called a subsequence of $(x_n)_{n=1}^\infty$. It is a convergent subsequence if $x_{n_k} \to y$ as $k \to \infty$, for some $y \in X$.

Definition 3.6.3. A space X is *sequentially compact* if every sequence $(x_n)_{n=1}^\infty$ in X has a convergent subsequence $(x_{n_k})_{k=1}^\infty$.

Theorem 3.6.4. *Let X be a metrizable space. Then the following are equivalent:*

(1) X is compact.

(2) X is limit point compact.

(3) X is sequentially compact.

Proof. The implication compact $\implies$ limit point compact was proved above.

To prove limit point compact $\implies$ sequentially compact, assume that X is limit point compact and consider any sequence $(x_n)_{n=1}^\infty$ in X. let $A = \{x_n \mid n \in \mathbb{N}\}$ be the set of points in the sequence.

If A is finite then at least one $p \in A$ appears as $p = x_n$ for infinitely many n, in which case the constant sequence at p is a convergent subsequence of $(x_n)_{n=1}^\infty$.

Otherwise A is infinite. By assumption, it has a limit point p. For each $k \geq 1$ the intersection $A \cap B(p, 1/k)$ contains other points than p. In fact, since X is Hausdorff it contains infinitely many other points than p. (See Theorem 17.9 on page 99 in Munkres.)

Choose a convergent subsequence $(x_{n_k})_{k=1}^\infty$ of $(x_n)_{n=1}^\infty$ as follows. First choose n_1 so that $x_{n_1} \in A \cap B(p, 1)$. Inductively suppose (for an $m \geq 2$) that we have chosen an increasing sequence of natural numbers

$$n_1 < n_2 < \dots < n_{m-1}$$

and points $x_{n_k} \in A \cap B(p, 1/k)$ for $1 \leq k < m$. Then $A \cap B(p, 1/k)$ contains infinitely many points. In particular, it contains points x_n with $n > n_{m-1}$. Choose such a point x_{n_m} with $n_m > n_{m-1}$. This proves the inductive step. Since $d(x_{n_k}, p) < 1/k$ for all $k \geq 1$, it follows that $x_{n_k} \to p$ as $k \to \infty$.

The proof of the implication sequentially compact $\implies$ compact remains. First we show that if X is sequentially compact then the Lebesgue number lemma holds for X:

Lemma 3.6.5. *Let $\mathscr{A}$ be an open cover of a sequentially compact metric space (X, d). Then there exists a $\delta > 0$ such that for each subset $B \subset X$ of diameter $< \delta$ there is an element $U \in \mathscr{A}$ with $B \subset U$.*

We assume that no such δ exists, and achieve a contradiction. For each $n \in \mathbb{N}$ there is a set C_n of diameter $< 1/n$ that is not contained in any element of $\mathscr{A}$. Choose $x_n \in C_n$. By the assumed sequential compactness, the sequence $(x_n)_{n=1}^\infty$ has a convergent subsequence $(x_{n_k})_{k=1}^\infty$, with $(n_k)_{k=1}^\infty$ a strictly increasing sequence. Let $p \in X$ be its limit: $x_{n_k} \to p$ as $k \to \infty$. There is an $U \in \mathscr{A}$ with $p \in U$. Since U is open, there is an $\epsilon > 0$ with $B_d(p, \epsilon) \subset U$. For k sufficiently large we have $1/n_k < \epsilon/2$ and $d(x_{n_k}, p) < \epsilon/2$. Then C_{n_k} lies in the $\epsilon/2$-neighborhood of x_{n_k}, hence in the ϵ-neighborhood of p:

$$C_{n_k} \subset B_d(x_{n_k}, \epsilon/2) \subset B_d(p, \epsilon) \subset U$$

This contradicts the choice of C_{n_k}, not being contained in any element of $\mathscr{A}$.

Next we show that if X is sequentially compact then it is totally bounded:

Lemma 3.6.6. *Let (X, d) be sequentially compact. For each $\epsilon > 0$ there exists a finite covering of X by ϵ-balls.*

Assume that for some $\epsilon > 0$ there is no finite covering of X by ϵ-balls, to reach a contradiction. Construct a sequence $(x_n)_{n=1}^\infty$ as follows. Choose any point $x_1 \in X$. Having chosen $x_1, \dots, x_n$, note that the finite union

$$B_d(x_1, \epsilon) \cup \cdots \cup B_d(x_n, \epsilon)$$

is not all of X, so we can choose x_{n+1} in its complement, and continue. By construction, $d(x_m, x_n) \geq \epsilon$ for all $m \neq n$, so $(x_n)_{n=1}^\infty$ does not contain any convergent subsequence. This contradicts sequential compactness of X.

We can now finish the proof. Let (X, d) be sequentially compact. To prove that X is compact, consider any open cover $\mathscr{A}$ of X. It has a Lebesgue number $\delta > 0$. Let $\epsilon = \delta/3$. Choose a finite covering of X by ϵ-balls. Each ϵ-ball has diameter $\leq 2\epsilon < \delta$, hence is contained in an element of $\mathscr{A}$. Hence X is covered by finitely many of the elements of $\mathscr{A}$, so $\mathscr{A}$ has a finite subcover. □

3.7 (§29) Local Compactness

We have seen that the closed subspaces of a compact Hausdorff space are the same as the compact subspaces. We shall now consider a condition satisfied by the open subspaces of compact Hausdorff spaces.

Definition 3.7.1. A space X is *locally compact at x* if there is a compact subspace C of X that contains a neighborhood V of x:

$$x \in V \subset C \subset X$$

It is *locally compact* if it is locally compact at each of its points.

Example 3.7.2. Any compact space is locally compact.

Example 3.7.3. The real line $\mathbb{R}$ is locally compact. Each point $x \in \mathbb{R}$ is contained in the compact subspace $C = [x-1, x+1]$, which contains the neighborhood $V = (x-1, x+1)$.

Example 3.7.4. The set of rational numbers $\mathbb{Q}$, in the subspace topology from $\mathbb{R}$, is not locally compact. Any subset $C \subset \mathbb{Q}$ containing a basis neighborhood $\mathbb{Q} \cap (x-\epsilon, x+\epsilon)$ cannot be compact. [[More details?]]

Example 3.7.5. Euclidean n-space $\mathbb{R}^n$ is locally compact. Each point $(x_1, \dots, x_n) \in \mathbb{R}^n$ is contained in the compact subspace

$$C = [x_1 - 1, x_1 + 1] \times \cdots \times [x_n - 1, x_n + 1],$$

which contains the neighborhood

$$V = (x_1 - 1, x_1 + 1) \times \cdots \times (x_n - 1, x_n + 1).$$

Example 3.7.6. The infinite product $\mathbb{R}^\omega$ is not locally compact. Any neighborhood V of $0 = (0)_{n=1}^\infty$ contains a basis neighborhood

$$(-\epsilon, \epsilon) \times \cdots \times (-\epsilon, \epsilon) \times \mathbb{R} \times \cdots \times \mathbb{R} \times \dots$$

for some $\epsilon > 0$. If V were contained in a compact subspace C, then the closure

$$[-\epsilon, \epsilon] \times \cdots \times [-\epsilon, \epsilon] \times \mathbb{R} \times \cdots \times \mathbb{R} \times \dots$$

of the basis neighborhood would be compact, which it is not.

3.7.1 The one-point compactification

Definition 3.7.7. Let X be a locally compact Hausdorff space. Let $Y = X \cup \{\infty\}$ where $\infty \notin X$. Give Y the topology $\mathscr{T}_\infty$ consisting of

(1) the open subsets $U \subset X$, and

(2) the complements $Y - C$ of compact subsets $C \subset X$.

We call Y the *one-point compactification* of X.

Theorem 3.7.8. *Let X be a locally compact Hausdorff space. The one-point compactification $Y = X \cup \{\infty\}$ is a compact Hausdorff space, $X \subset Y$ is a subspace, and $Y - X$ consists of a single point.*

Proof. We first prove that the given collection $\mathscr{T}_\infty$ is a topology on Y. The empty set is of type (1) and Y is of type (2). To check that the intersection of two open sets is open, there are three cases:

$$\begin{aligned} U_1 \cap U_2 &\subset X \\ (Y - C_1) \cap (Y - C_2) &= Y - (C_1 \cup C_2) \\ U_1 \cap (Y - C_2) &= U_1 \cap (X - C_2) \end{aligned}$$

These are of type (1), (2) and (1), respectively, since $C_1 \cup C_2$ is compact and $X - C_2$ is open, since X is assumed to be Hausdorff.

To check that the union of a (nonempty) collection of open sets is open, there are three cases:

$$\bigcup_{\alpha\in J} U_\alpha = U \subset X$$
$$\bigcup_{\beta\in K} (Y - C_\beta) = Y - \bigcap_{\beta\in K} C_\beta = Y - C$$
$$U \cup (Y - C) = Y - (C - U)$$

These are of type (1), (2) and (2), respectively, since $C = \bigcap_{\beta\in K} C_\beta \subset X$ is a closed subspace of some compact space C_β, hence is compact, and $C - U$ is a closed subspace of C, hence is compact.

Next we show that $X \subset Y$ is a subspace: The open sets in the subspace topology are of the form $X \cap V$ where V is open in Y. If $V = U \subset X$ is of type (1), then $X \cap V = U$ is open in X. If $V = Y - C$ is of type (2), then $X \cap V = X - C$ is open in X since $C \subset X$ is compact, hence closed, in the Hausdorff space X. Conversely, if $U \subset X$ is open, then U is open of type (1) in Y.

To show that Y is compact, let $\mathscr{A}$ be an open cover of Y. Some element $V \in \mathscr{A}$ must contain $\infty \notin X$, hence be of the form $V = Y - C$. The collection $\mathscr{A}$ of open subsets of Y covers the compact space C, so there is a finite subcollection $\{U_1, \dots, U_n\} \subset \mathscr{A}$ that covers C. Then $\{V, U_1, \dots, U_n\}$ is a finite subcover of $\mathscr{A}$.

To show that Y is Hausdorff, let $x, y \in Y$. If both lie in X, then there are open subsets $U, V \subset X$ with $x \in U$, $y \in V$, $U \cap V = \varnothing$. Then U and V are also open and disjoint in Y. Otherwise, we may assume that $x \in X$ and $y = \infty$. Since X is locally compact at x there exists a compact $C \subset X$ containing a neighborhood U of x. Let $V = Y - C$. Then $x \in U$, $\infty \in V$, U and V are open in Y and $U \cap V = \varnothing$. □

Here is a converse.

Proposition 3.7.9. *Let $X \subset Y$ be a subspace of a compact Hausdorff space, such that $Y - X$ consists of a single point. Then X is locally compact and Hausdorff.*

Proof. As a subspace of a Hausdorff space, it is clear that X is Hausdorff. We prove that it is locally compact. Let $x \in X$ and let y be the single point of $Y - X$. Since Y is Hausdorff, there are open sets $U, V \subset Y$ with $x \in U$, $y \in V$, $U \cap V = \varnothing$. Let $C = Y - V$. It is a closed subset of a compact space, hence compact. Thus $x \in U \subset C \subset X$, as required for local compactness at x. □

There is also the following uniqueness statement, which justifies why we say "the one-point compactification", not just "a one-point compactification". The notation Y' just refers to a variant of Y, not its set of limit points.

Proposition 3.7.10. *Let X be locally compact Hausdorff, with one-point compactification $Y = X \cup \{\infty\}$, and suppose that Y' is a compact Hausdorff space such that $X \subset Y'$ is a subspace and $Y' - X$ is a single point. Then the unique bijection $Y' \to Y$ that is the identity on X is a homeomorphism.*

Proof. It suffices that the bijection $f\colon Y' \to Y$ is continuous, since Y' is compact and Y is Hausdorff. An open subset of Y is of the form U or $Y - C$, with $U \subset X$ open and $C \subset X$ compact. The preimage $f^{-1}(U) = U$ is then open in X, hence also in Y', since X must be open in the Hausdorff space Y' because its complement is a single point. The preimage $f^{-1}(Y - C) = Y' - C$ will also be open in Y', because C is compact and Y' is Hausdorff, so $C \subset Y'$ is closed. □

Example 3.7.11. The one-point compactification of the open interval $(0, 1)$ is homeomorphic to the circle S^1. This follows from the uniqueness statement above, and the homeomorphism

$$f : (0, 1) \to S^1 - \{(1, 0)\}$$

given by $f(t) = (\cos(2\pi t), \sin(2\pi t))$. The closed interval $[0, 1]$ is a different compactification, with $[0, 1] - (0, 1) = \{0, 1\}$ consisting of two points.

Since $(0, 1) \cong \mathbb{R}$, the one-point compactification of $\mathbb{R}$ is also homeomorphic to the circle:

$$\mathbb{R} \cup \{\infty\} \cong S^1$$

Example 3.7.12. The one-point compactification of the open unit n-ball

$$B(0, 1) = \{x \in \mathbb{R}^n \mid \|x\| < 1\}$$

is homeomorphic to the n-sphere S^n. The closed n-ball

$$\bar{B}(0, 1) = \{x \in \mathbb{R}^n \mid \|x\| \le 1\}$$

is a different compactification, with $\bar{B}(0, 1) - B(0, 1) = S^{n-1}$ consisting of the $(n-1)$-sphere.

Since $B(0, 1) \cong \mathbb{R}^n$, the one-point compactification of $\mathbb{R}^n$ is also homeomorphic to the n-sphere:

$$\mathbb{R}^n \cup \{\infty\} \cong S^n$$

3.7.2 The local nature of local compactness

For Hausdorff spaces, the property of being locally compact is a local property in the following sense.

Theorem 3.7.13. *Let X be a Hausdorff space. Then X is locally compact if and only if for each point $x \in X$ and each neighborhood U of x there is a neighborhood V of x with compact closure $\bar{V}$ contained in U:*

$$x \in V \subset \bar{V} \subset U$$

Proof. The stated property implies local compactness at x, by taking $U = X$ and $C = \bar{V}$.

For the converse, suppose that X is locally compact (and Hausdorff), and let $x \in U \subset X$ be a neighborhood. Let $Y = X \cup \{\infty\}$ be the one-point compactification, and let $K = Y - U$. Then $K \subset Y$ is closed, hence compact. Since Y is Hausdorff and $x \notin K$ we can find open subsets $V, W \subset Y$ with $x \in V$, $K \subset W$ and $V \cap W = \varnothing$. Then $\infty \in K \subset W$, so $W = Y - C$ for some compact $C \subset X$. Hence $x \in V \subset C \subset U$ with V open and C compact. Hence $\bar{V}$ is also compact. □

Corollary 3.7.14. *Let X be locally compact Hausdorff. If $A \subset X$ is an open or closed subspace, then A is locally compact.*

Proof. Suppose that A is open in X. Let $x \in A$. By the previous theorem there is a neighborhood V of x with $\bar{V}$ compact and $\bar{V} \subset A$. This shows that A is locally compact at x.

Suppose instead that A is closed in X. Let $x \in A$. Since X is locally compact there is a compact subspace $C \subset X$ that contains a neighborhood V of x. Then $A \cap C$ is closed in C, hence compact, and contains the neighborhood $A \cap V$ of x in the subspace topology on A. □

Corollary 3.7.15. *A space X is homeomorphic to an open subspace of a compact Hausdorff space if and only if X is locally compact and Hausdorff.*

Proof. If X is an open subspace of a compact Hausdorff space, then X is locally compact by the corollary above, and obviously Hausdorff. The same applies if X is homeomorphic to such an open subspace.

Conversely, if X is locally compact and Hausdorff then X is an open subspace of its one-point compactification $Y = X \cup \{\infty\}$, which is compact Hausdorff. $\square$

Chapter 4

Countability and Separation Axioms

We have seen the first countability axiom (each point has a countable neighborhood basis) and the Hausdorff separation axiom (two points can be separated by disjoint neighborhoods).

Our aim is to prove the Urysohn metrization theorem, saying that if a topological space X satisfies a countability axiom (it is second countable) and a separation axiom (is it regular), then we can construct enough continuous functions $X \to \mathbb{R}$ to embed X into a metric space, so that X is metrizable.

4.1 (§30) The Countability Axioms

It may be a good idea to look through §7 on Countable and Uncountable Sets, if this material is unfamiliar. A set C is *countable* if can be put in bijective correspondence with a subset of $\mathbb{N}$. Finite sets are countable. A set is *countably infinite* if it is countable but not finite, or equivalently, if it can be put in bijective correspondence with all of $\mathbb{N}$.

A subset of a countable set is countable, a finite or countable union of countable sets is countable, and a finite product of countable sets is countable.

The set of rational numbers is countable. The set of real numbers is uncountable, i.e., not countable, and an uncountable product of sets with 2 or more elements each is uncountable.

4.1.1 First- and second-countable

Definition 4.1.1. A space X has a *countable basis at x* if there is a countable collection $\mathscr{B}$ of neighborhoods of x such that each neighborhood of x contains at least one of the elements of $\mathscr{B}$. A space having a countable basis at each of its points is said to satisfy the *first countability axiom*, or to be *first-countable*.

Every metric space is first-countable.

Definition 4.1.2. If a space has a countable basis for its topology, then it is said to satisfy the *second countability axiom*, or to be *second-countable*.

Lemma 4.1.3. *Second-countability implies first-countability.*

Proof. Let $\mathscr{B} = \{U_n\}_{n=1}^{\infty}$ be a countable basis for the topology of a space X. For each point $x \in X$ the subcollection $\mathscr{B}_x = \{U \in \mathscr{B} \mid x \in U\}$ is a countable neighborhood basis at x. □

Example 4.1.4. The real line is second-countable. A countable basis for the topology is given by the open intervals (a, b) with $a < b \in \mathbb{Q}$ both rational.

Euclidean n-space $\mathbb{R}^n$ is second-countable. A countable basis is given by the products

$$(a_1, b_1) \times \cdots \times (a_n, b_n)$$

where all $a_i, b_i \in \mathbb{Q}$ are rational.

Even the infinite product $\mathbb{R}^\omega$ is second-countable. A countable basis is given by the products

$$\prod_{n=1}^{\infty} U_n$$

where $U_n = (a_n, b_n)$ with rational endpoints, for finitely many n, and $U_n = \mathbb{R}$ for all other n.

Not every metric space is second-countable. A counterexample is $\mathbb{R}^\omega$ in the uniform topology.

Theorem 4.1.5. *A subspace of a first-countable (resp. second-countable) space is first-countable (resp. second-countable).*

A (finite or) countable product of first-countable (resp. second-countable) spaces is first-countable (resp. second-countable)

Proof. We do the second-countable cases. If $\mathscr{B} = \{U_n\}_{n=1}^{\infty}$ is a countable basis for X, and $A \subset X$, then $\{A \cap U_n\}_{n=1}^{\infty}$ is a countable basis for A.

If $\mathscr{B}_n$ is a countable basis for X_n, for $n \in \mathbb{N}$, then the collection of products

$$\prod_{n=1}^{\infty} U_n$$

where $U_n \in \mathscr{B}_n$ for finitely many n, and $U_n = X_n$ for the remaining n, is a countable basis for the product topology on $\prod_{n=1}^{\infty} X_n$.

The first-countable cases are similar. □

4.1.2 Dense, Lindelöf, separable

Definition 4.1.6. A subset $A \subset X$ is said to be *dense* of $\bar{A} = X$, i.e., if each nonempty open subset of X meets A.

Example 4.1.7. The rational numbers $\mathbb{Q} \subset \mathbb{R}$ are dense in the real line.

Theorem 4.1.8. *Suppose that X is second-countable. Then*

(1) Every open covering of X admits a countable subcovering.

(2) There exists a countable dense subset of X.

A space with property (1) is called a *Lindelöf space.* A space with property (2) is called a *separable space.*

Proof. Let $\{B_n\}_{n=1}^{\infty}$ be a countable basis for the topology on X.

(1) Let $\mathscr{A}$ be any open covering of X. For each $n \in \mathbb{N}$ choose an $A_n \in \mathscr{A}$ such that $B_n \subset A_n$, if such an A_n exists. Let $\mathscr{C}$ be the collection of these A_n's. It is a countable subcollection of $\mathscr{A}$.

Claim: $\mathscr{C}$ covers X. Proof: Let $x \in X$. Choose $A \in \mathscr{A}$ with $x \in A$. Since A is open, and $\mathscr{B}$ is a basis for the topology, there is a $B_n \in \mathscr{B}$ with $x \in B_n \subset A$. Then A_n exists in $\mathscr{C}$, and $x \in B_n \subset A_n$.

(2) Choose a point $x_n \in B_n$ for each non-empty $B_n \in \mathscr{B}$. Let D be the set of these x_n's. It is a countable subset of X.

Claim: D is dense in X. Proof: Let $x \in X$. Each neighborhood U of x contains a basis element B_n with $x \in B_n$, so $x_n \in D$ exists and $x_n \in D \cap U$. Hence $x \in \bar{D}$. □

4.2 (§31) The Separation Axioms

((See Munkres.))

4.3 (§32) Normal Spaces

((See Munkres.))

4.4 (§33) The Urysohn Lemma

Theorem 4.4.1 (Urysohn's lemma). *Let A and B be disjoint, closed subsets of a normal space X. There exists a map*

$$f\colon X \to [0,1]$$

such that $f(x) = 0$ for all $x \in A$ and $f(x) = 1$ for all $x \in B$.

Proof. A *dyadic number* is a rational number of the form $r = a/2^n$, where a and n integers with $n \ge 0$. The dyadic numbers are dense in $\mathbb{R}$.

For each dyadic number $0 \le r \le 1$ we shall construct an open subset $U_r \subset X$, with $A \subset U_r \subset X - B$, so that for each pair of dyadic numbers $0 \le p < q \le 1$ we have $\bar{U}_p \subset U_q$.

Let $U_1 = X - B$. Then $A \subset U_1$, so by normality there exists an open U_0 with $A \subset U_0 \subset \bar{U}_0 \subset U_1$. By normality again, there exists an open $U_{1/2}$ with $\bar{U}_0 \subset U_{1/2} \subset \bar{U}_{1/2} \subset U_1$.

Let $n \ge 2$ and assume inductively that we have constructed the U_r for all $0 \le r \le 1$ of the form $b/2^{n-1} = 2b/2^n$. We must construct the U_r for r of the form $a/2^n$ with $a = 2b+1$ odd. By induction we have constructed $U_{2b/2^n}$ and $U_{(2b+2)/2^n}$ with $\bar{U}_{2b/2^n} \subset U_{(2b+2)/2^n}$. Using normality we can choose an open $U_{(2b+1)/2^n}$ with

$$\bar{U}_{2b/2^n} \subset U_{(2b+1)/2^n} \subset \bar{U}_{(2b+1)/2^n} \subset U_{(2b+2)/2^n} \,.$$

Continuing for all natural numbers n, we are done.

Extend the definition of the U_r to all dyadic numbers r, by letting $U_r = \varnothing$ for $r < 0$, and $U_r = X$ for $r > 1$. We still have the key property that $\bar{U}_p \subset U_q$ for all dyadic numbers $p < q$.

Let $x \in X$ and consider the set

$$D(x) = \{r \text{ dyadic} \mid x \in U_r\} \,.$$

Since $x \notin U_r$ for $r < 0$, the displayed set is bounded below by 0. Since $x \in U_r$ for all $r > 1$, the displayed set contains all dyadic $r > 1$, and is nonempty. Hence the greatest lower bound

$$f(x) = \inf D(x)$$

exists as a real number, and lies in the interval $[0,1]$.

Claim (1): If $x \in \bar{U}_r$ then $f(x) \le r$.

If $x \in \bar{U}_r$ then $x \in U_q$ for all $r < q$, so $D(x)$ contains all dyadic numbers greater than r. The dyadic numbers are dense in the reals, so $f(x) \le r$.

Claim (2): If $x \notin U_r$ then $r \le f(x)$.

If $x \notin U_r$ then $x \notin U_p$ for all $p < r$, so $D(x)$ contains no dyadic numbers less than r. Hence r is a lower bound for $D(x)$, and $r \le f(x)$.

Claim (3): f is continuous.

Let $x \in X$ and consider any neighborhood (c,d) in $\mathbb{R}$ of $f(x)$. We shall find a neighborhood U of x with $f(U) \subset (c,d)$.

Choose dyadic numbers p and q with $c < p < f(x) < q < d$. Then $x \notin \bar{U}_p$ by (1), and $x \in U_q$ by (2), so $U = U_q - \bar{U}_p$ is a neighborhood of x.

If $y \in U$ then $y \notin U_p \subset \bar{U}_p$, so $c < p \leq f(y)$. Also $y \in U_q \subset \bar{U}_q$, so $f(y) \leq q < d$. Hence $f(U) \subset (c, d)$. □

4.4.1 Completely regular spaces

Definition 4.4.2. Let A and B be subsets of a topological space X. We say that A and B *can be separated by a continuous function* if there is a continuous function $f: X \to \mathbb{R}$ such that $f(x) = 0$ for all $x \in A$ and $f(x) = 1$ for all $x \in B$.

Clearly A and B must be disjoint for this to be possible.

Definition 4.4.3. A space X is *completely regular* if one-point sets in X are closed, and if each pair of disjoint, closed subsets $\{x\}$ and B can be separated by a continuous function.

Completely regular spaces are sometimes known as $T_{3\frac{1}{2}}$-spaces:

Lemma 4.4.4. *Any normal space is completely regular, and any completely regular space is regular.*

Proof. The first claim follows from Urysohn's lemma for $A = \{x\}$. The second claim follows since $U = f^{-1}([0, 1/2))$ and $V = f^{-1}((1/2, 1])$ are disjoint open subsets of X that separate $\{x\} \subset f^{-1}(0)$ and $B \subset f^{-1}(1)$. □

Theorem 4.4.5. *A subspace of a completely regular space is completely regular. A product of completely regular spaces is completely regular.*

((Proof omitted.))

4.5 (§34) The Urysohn Metrization Theorem

Theorem 4.5.1 (Urysohn's metrization theorem). *Every second-countable regular space is metrizable.*

Proof. Let X be a second-countable regular space. By Theorem 32.1 this is the same as a second-countable normal space. We shall prove that X is metrizable by embedding it into the metrizable space $\mathbb{R}^\omega = \prod_{k=1}^\infty \mathbb{R}$ (in the product topology).

Claim 1: There is a countable collection of maps $f_k: X \to [0, 1]$, such that for any

$$p \in U \subset X$$

with U open there is an f_k in the collection with $f_k(p) = 1$ and $f_k(X - U) \subset \{0\}$.

Let $\mathscr{B} = \{B_n\}_{n=1}^\infty$ be a countable basis for the topology on X. For each pair (m, n) of indices with $\bar{B}_m \subset B_n$ use Urysohn's lemma to choose a map $g_{m,n}: X \to [0, 1]$ with $g_{m,n}(\bar{B}_m) \subset \{1\}$ and $g_{m,n}(X - \bar{B}_n) \subset \{0\}$. Then the collection $\{g_{m,n}\}$ satisfies the claim.

To see this, consider $p \in U$ open in X. Since $\mathscr{B}$ is a basis, there is a basis element B_n with $p \in B_n \subset U$. By regularity, there is an open V with $p \in V \subset \bar{V} \subset B_n$, and by the basis property there is a basis element B_m with $p \in B_m \subset V$. Then $\bar{B}_m \subset \bar{V} \subset B_n$, so $p \in \bar{B}_m \subset B_n \subset U$. Then $g_{m,n}$ is defined, and satisfies $g_{m,n}(p) = 1$ and $g_{m,n}(X - U) \subset \{0\}$.

We reindex the countable collection $\{g_{m,n}\}$ as $\{f_k\}_{k=1}^\infty$. Define a map

$$F: X \to \prod_{k=1}^\infty \mathbb{R} = \mathbb{R}^\omega$$

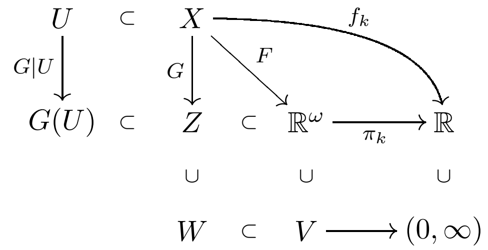
by the rule

$$F(x) = (f_1(x), f_2(x), \dots) .$$

Claim 2: F is an embedding of X into $\mathbb{R}^\omega$.

It is clear that F is continuous, since each component f_k is continuous and $\mathbb{R}^\omega$ has the product topology. It is also clear that F is injective, since for $x \neq y$ in X the complement $U = X - \{y\}$ is a neighborhood of x, so there is an index k with $f_k(x) = 1$ and $f_k(X - U) \subset \{0\}$, so $f_k(y) = 0$. Hence the k-th coordinates of $F(x)$ and $F(y)$ are different, so $F(x) \neq F(y)$.

Let $Z = F(X) \subset \mathbb{R}^\omega$ be the image of F in the subspace topology. We have proved that the corestriction $G\colon X \to Z$ of F is a continuous bijection. In order to show that G is a homeomorphism, so that F is an embedding, it remains to show that G is an open map. Let $U \subset X$ be open. We must show that $G(U)$ is open in Z.



Let $q \in G(U) \subset Z$ be any point. We shall find an open $W \subset Z$ with $q \in W \subset G(U)$. Let $p \in U$ be the (unique) point with $G(p) = q$. Choose an index k such that $f_k(p) = 1$ and $f_k(X - U) \subset \{0\}$. Let

$$V = \pi_k^{-1}((0, \infty))$$

be the open set of real sequences $(x_1, x_2, \dots,)$ with $x_k > 0$, and let $W = Z \cap V$. Then W is open in the subspace topology on Z.

Claim 3: $q \in W \subset G(U)$.

We have $\pi_k(q) = \pi_k(F(p)) = f_k(p) = 1$, so $q \in V$. Since $q \in Z$ we get $q \in Z \cap V = W$.

Let $y \in W$ be any point. Then $y = G(x)$ for some (unique) $x \in X$, and $\pi_k(y) = \pi_k(F(x)) = f_k(x) \in (0, \infty)$, which implies $x \in U$, since $f_k(X - U) \subset \{0\}$. Hence $y = G(x) \in G(U)$, so $W \subset G(U)$. □

4.6 (§35) The Tietze Extension Theorem

Theorem 4.6.1. *Let A be a closed subspace of a normal space X. Any map $f\colon A \to [0, 1]$ (resp. $f\colon A \to \mathbb{R}$) may be extended to a map $g\colon X \to [0, 1]$ (resp. $g\colon X \to \mathbb{R}$) with $g|A = f$.*

4.7 (§36) Embeddings of Manifolds

Definition 4.7.1. An *m-dimensional manifold* is a second-countable Hausdorff space X such that each point $p \in X$ has a neighborhood U that is homeomorphic to an open subset V of $\mathbb{R}^m$.

Any neighborhood V of a point q in $\mathbb{R}^m$ contains a neighborhood that is homeomorphic to $\mathbb{R}^m$. Hence we may just as well ask that each point $p \in X$ has a neighborhood that is homeomorphic to $\mathbb{R}^m$. We say that X is *locally homeomorphic to* $\mathbb{R}^m$.

A 0-dimensional manifold is a discrete, countable set. A 1-dimensional manifold is called a *curve*. A 2-dimensional manifold is called a *surface*.

Theorem 4.7.2. *If X is a compact m-dimensional manifold, then X can be embedded in $\mathbb{R}^n$ for some $n \in \mathbb{N}$.*

Chapter 5

The Tychonoff Theorem

5.1 (§37) The Tychonoff Theorem

Theorem 5.1.1 (Tychonoff theorem). *Any product of compact spaces is compact.*

In other words, for any set J and any collection $\{X_\alpha\}_{\alpha\in J}$ of compact topological spaces X_α, the product space

$$\prod_{\alpha\in J} X_\alpha$$

is compact in the product topology.

We proved this for finite J, but the result is also true for infinite J. We omit the proof, and consider an application instead.

5.1.1 The profinite integers

Let $n \in \mathbb{N}$ be a natural number. We say that two integers a and b are *congruent modulo* n, and write $a \equiv b \mod n$, if $b - a$ is a multiple of n, i.e., if $n \mid b - a$. Congruence modulo n is an equivalence relation on $\mathbb{Z}$, and the equivalence class of an integer a is

$$[a]_n = a + n\mathbb{Z} = \{a + kn \mid k \in \mathbb{Z}\}\,.$$

The n integers

$$\{0, 1, 2, \dots, n-1\}$$

are commonly chosen representatives for the n different equivalence classes for this equivalence relation. The set of equivalence classes for congruence modulo n is denoted $\mathbb{Z}/n\mathbb{Z}$, $\mathbb{Z}/(n)$ or $\mathbb{Z}/n$. It is a commutative ring, called the *ring of integers modulo* n, with sum

$$[a]_n + [b]_n = [a+b]_n$$

and product

$$[a]_n \cdot [b]_n = [ab]_n$$

defined by choosing representatives. There is a surjective ring homomorphism

$$\phi_n \colon \mathbb{Z} \to \mathbb{Z}/n$$

taking a to $[a]_n$.

Lemma 5.1.2. *View $\mathbb{Z}$ and $\mathbb{Z}/n$ as discrete topological spaces. The product space*

$$\prod_{n=1}^{\infty} \mathbb{Z}/n = \mathbb{Z}/1 \times \mathbb{Z}/2 \times \mathbb{Z}/3 \times \dots$$

is a compact Hausdorff space. The function

$$\Phi \colon \mathbb{Z} \to \prod_{n=1}^{\infty} \mathbb{Z}/n$$

with components $(\phi_1, \phi_2, \phi_3, \dots)$, taking a to $([a]_1, [a]_2, [a]_3, \dots)$, is an injective, continuous function

Proof. Each $\mathbb{Z}/n$ has only finitely many open subsets, hence any open cover is finite, so $\mathbb{Z}/n$ is compact. By Tychonoff's theorem, the product space $\prod_{n=1}^{\infty} \mathbb{Z}/n$ is also compact. (This is not too hard to prove directly)

Each $\mathbb{Z}/n$ is discrete, hence Hausdorff, so also the product space $\prod_{n=1}^{\infty} \mathbb{Z}/n$ is Hausdorff.

The function Φ is injective, since if $\Phi(a) = \Phi(b)$ then $\phi_n(a) = \phi_n(b)$ for all $n \in \mathbb{N}$, so $b - a$ is divisible by each natural number n. Taking $n > |b - a|$ it follows that $b - a = 0$, so $a = b$.

Each function ϕ_n is continuous, since $\mathbb{Z}$ has the discrete topology. Hence Φ is continuous, since each of its components is continuous and $\prod_{n=1}^{\infty} \mathbb{Z}/n$ has the product topology. □

Let $m, d \in \mathbb{N}$ be natural numbers, and suppose that m is a multiple of d, so that $d \mid m$. The function

$$\rho_{m,d} \colon \mathbb{Z}/m \to \mathbb{Z}/d$$

taking $[a]_m$ to $[a]_n$ is then a well-defined ring homomorphism. We call $\rho_{m,d}(x)$ the *reduction modulo d* of $x \in \mathbb{Z}/m$. Notice that $\rho_{m,d} \circ \phi_m = \phi_d$

$$\begin{array}{ccc} \mathbb{Z} & \xrightarrow{\phi_m} & \mathbb{Z}/m \\ & \searrow^{\phi_d} & \downarrow{\scriptstyle \rho_{m,d}} \\ & & \mathbb{Z}/d \end{array}$$

as functions $\mathbb{Z} \to \mathbb{Z}/d$. Hence the image $\Phi(a) \in \prod_{n=1}^{\infty} \mathbb{Z}/n$ of an integer $a \in \mathbb{Z}$ is a sequence $(x_n)_{n=1}^{\infty}$ with the property that

$$\rho_{m,d}(x_m) = x_d$$

for all $d \mid m$. Let

$$\hat{\mathbb{Z}} = \{(x_n)_{n=1}^{\infty} \in \prod_{n=1}^{\infty} \mathbb{Z}/n \mid \rho_{m,d}(x_m) = x_d \text{ for all } d \mid m\}$$

be the subspace defined by this property. In other words, an element of $\hat{\mathbb{Z}}$ is a sequence $(x_n)_{n=1}^{\infty}$ with x_n an integer modulo n, such that x_d is the reduction modulo d of x_m, for each $d \mid m$.

The set $\hat{\mathbb{Z}}$ is a commutative ring, called the *ring of profinite integers*, with sum

$$(x_n)_{n=1}^{\infty} + (y_n)_{n=1}^{\infty} = (x_n + y_n)_{n=1}^{\infty}$$

and product

$$(x_n)_{n=1}^{\infty} \cdot (y_n)_{n=1}^{\infty} = (x_n \cdot y_n)_{n=1}^{\infty}$$

defined termwise. It is a topological ring, in the sense that the ring operations are continuous.

Lemma 5.1.3. *The ring of profinite integers $\hat{\mathbb{Z}}$ is a closed subspace of the product space $\prod_{n=1}^{\infty} \mathbb{Z}/n$, hence is a compact Hausdorff space.*

Proof. For each $d \mid m$ consider the continuous function

$$f_{m,d} \colon \prod_{n=1}^{\infty} \mathbb{Z}/n \to \mathbb{Z}/d$$

taking $(x_n)_{n=1}^{\infty}$ to $\rho_{m,d}(x_m) - x_d$. The preimage

$$C_{m,d} = f_{m,d}^{-1}([0]_d)$$

of $[0]_d$ is closed, since $\mathbb{Z}/d$ is discrete. Hence the intersection

$$\hat{\mathbb{Z}} = \bigcap_{d \mid m} C_{m,d}$$

is also closed in $\prod_{n=1}^{\infty} \mathbb{Z}/n$. □

Lemma 5.1.4. *The function Φ corestricts to an injective, continuous ring homomorphism*

$$\Psi \colon \mathbb{Z} \to \hat{\mathbb{Z}}$$

taking a to $([a]_n)_{n=1}^{\infty}$, from the discrete space $\mathbb{Z}$ to the compact Hausdorff space $\hat{\mathbb{Z}}$. The image of Ψ is dense in $\hat{\mathbb{Z}}$.

Proof. Let $p = (x_n)_{n=1}^{\infty}$ be any point in $\hat{\mathbb{Z}}$. Any neighborhood V of p contains a basis element $\hat{\mathbb{Z}} \cap U$ for the subspace topology, where

$$U = \prod_{n=1}^{\infty} U_n \subset \prod_{n=1}^{\infty} \mathbb{Z}/n$$

is a basis element for the product topology. Here $x_n \in U_n$ for all n, and $U_n = \mathbb{Z}/n$ for all but finitely many n. Let $N \in \mathbb{N}$ be a common multiple of all the n with $U_n \neq \mathbb{Z}/n$. Choose any integer a with $[a]_N = x_N$. Claim:

$$\Psi(a) \in \Psi(\mathbb{Z}) \cap U\,,$$

so that $\Psi(\mathbb{Z}) \cap V \neq \varnothing$, and $\Psi(\mathbb{Z})$ is dense in $\hat{\mathbb{Z}}$.

To prove the claim, it is enough to prove that $\Phi(a) \in U$, or equivalently, that $[a]_n \in U_n$ for all n. This is clear when $U_n = \mathbb{Z}/n$. When $U_n \neq \mathbb{Z}/n$ we have $n \mid N$, and then

$$[a]_n = \rho_{N,n}([a]_N) = \rho_{N,n}(x_N) = x_n \in U_n\,.$$

□

The map $\Psi \colon \mathbb{Z} \to \hat{\mathbb{Z}}$ is not an embedding of the discrete space $\mathbb{Z}$ in the topological sense. The topology on $\mathbb{Z}$ that makes Ψ an embedding, i.e., the subspace topology from $\hat{\mathbb{Z}}$, may be called the *Fürstenberg topology.*

Theorem 5.1.5 (Euclid, ca. 300 BC). *There are infinitely many prime numbers.*

Proof. Here is Fürstenberg's proof from 1955. The open sets of the Fürstenberg topology on $\mathbb{Z}$ are of the form $\Psi^{-1}(V)$ with V open in $\hat{\mathbb{Z}}$, or equivalently, of the form $\Phi^{-1}(U)$ with U open in $\prod_{n=1}^{\infty} \mathbb{Z}/n$. It follows that each open set in the Fürstenberg topology on $\mathbb{Z}$ is either empty or infinite.

For each prime number p, the subset

$$p\mathbb{Z} = \{kp \mid k \in \mathbb{Z}\}$$

of $\mathbb{Z}$ is closed in the Fürstenberg topology. Consider the subset

$$A = \bigcup_{p \text{ prime}} p\mathbb{Z}$$

of $\mathbb{Z}$. Its complement is

$$X - A = \{\pm 1\},$$

since the only integers not divisible by any primes as the units 1 and -1.

If there is only a finite set of primes, then A is a finite union of closed subsets, hence is closed in $\mathbb{Z}$, so that $X - A = \{\pm 1\}$ is open. This contradicts the fact that the open subsets of $\mathbb{Z}$ are either empty or infinite. □

Chapter 6

Complete Metric Spaces and Function Spaces

6.1 (§43) Complete Metric Spaces

Definition 6.1.1. Let (X, d) be a metric space. A sequence $(x_n)_{n=1}^{\infty}$ of points in X is a *Cauchy sequence* if for each $\epsilon > 0$ there is an $N \in \mathbb{N}$ such that

$$d(x_m, x_n) < \epsilon$$

for all $m, n \geq N$.

Each convergent sequence is a Cauchy sequence.

Definition 6.1.2. A metric space (X, d) is *complete* if each Cauchy sequence in X is convergent.

Lemma 6.1.3. *A metric space (X, d) is complete if every Cauchy sequence in X has a convergent subsequence.*

Proof. If $(x_n)_n$ is Cauchy and $x_{n_k} \to y$ as $k \to \infty$ then $x_n \to y$ as $n \to \infty$. □

Proposition 6.1.4. *Every compact metric space is complete.*

Proof. Every Cauchy sequence in a compact metric space contains a convergent subsequence, by (sequential) compactness, so by the lemma above it is convergent. □

Theorem 6.1.5. *Euclidean space $\mathbb{R}^n$ is complete (in any of the equivalent metrics coming from a norm).*

6.2 (§45) Compactness in Metric Spaces

Definition 6.2.1. A metric space (X, d) is *totally bounded* if for every $\epsilon > 0$ there is a finite covering of X by ϵ-balls.

Proposition 6.2.2. *Every compact metric space is totally bounded.*

Proof. Let (X, d) be a compact metric space and consider any $\epsilon > 0$. The collection of all ϵ-balls is an open covering of X. By compactness there exists a finite subcover, which is a finite covering of X by ϵ-balls. □

Theorem 6.2.3. *A metric space (X, d) is compact if and only if it is complete and totally bounded.*

Proof. We have proved that a compact metric space is complete and totally bounded. Conversely, we will prove that a complete and totally bounded is sequentially compact. This implies that it is compact, as we proved in §28.

Let $(x_n)_{n=1}^{\infty}$ be any sequence of points in X. We shall construct a Cauchy subsequence $(x_{n_k})_{k=1}^{\infty}$. By the assumed completeness of X, this will be a convergent subsequence.

First cover X by finitely many balls of radius 1. At least one of these, call it B_1, will contain x_n for infinitely many n. Let

$$J_1 = \{n \in \mathbb{N} \mid x_n \in B_1\}$$

be this infinite set of indices.

Inductively suppose, for some $k \geq 1$, that we have chosen a ball B_k of radius $1/k$ that contains x_n for all n in an infinite set $J_k \subset \mathbb{N}$. Cover X by finitely many balls of radius $1/(k+1)$. At least one of these, call it B_{k+1}, will contain x_n for infinitely many $n \in J_k$. Let

$$J_{k+1} = \{n \in J_k \mid x_n \in B_{k+1}\}$$

be this infinite set. Continue for all k, to get an infinite descending sequence of infinite sets:

$$J_1 \supset \cdots \supset J_k \supset J_{k+1} \supset \ldots .$$

Choose $n_1 \in J_1$. Inductively suppose, for some $k \geq 1$, that we have chosen $n_k \in J_k$. Since J_{k+1} is infinite, we can choose an $n_{k+1} \in J_{k+1}$ with $n_{k+1} > n_k$. Continue for all k. The sequence

$$n_1 < \cdots < n_k < n_{k+1} < \ldots$$

is strictly increasing, so $(x_{n_k})_{k=1}^{\infty}$ is a subsequence of $(x_n)_{n=1}^{\infty}$.

We claim that it is a Cauchy sequence. Let $\epsilon > 0$ and choose k with $1/k < \epsilon/2$. For all $i, j \geq k$ we have $n_i \in J_i \subset J_k$ and $n_j \in J_j \subset J_k$, so $x_{n_i}, x_{n_j} \in B_k$. Since B_k has diameter $\leq 2/k < \epsilon$, we get that $d(x_{n_i}, x_{n_j}) < \epsilon$. Hence $(x_{n_k})_{k=1}^{\infty}$ is Cauchy. □

6.3 (§46) Pointwise and Compact Convergence

6.3.1 Function spaces

Let X and Y be topological spaces, and consider the set

$$Y^X = \{f : X \to Y\} = \prod_X Y$$

of functions from X to Y.

Definition 6.3.1. For $x \in X$ and $U \subset Y$ open let

$$S(x, U) = \{f : X \to Y \mid f(x) \in U\} .$$

This equals $\pi_x^{-1}(U)$, where $\pi_x : \prod_X Y \to Y$ projects to the x-th factor. The collection of these subsets $S(x, U) \subset Y^X$ is a subbasis for the *topology of pointwise convergence*, equal to the product topology on $\prod_X Y$.

Lemma 6.3.2. *A sequence $(f_n)_{n=1}^{\infty}$ of functions $f_n : X \to Y$ converges to $f : X \to Y$ in the topology of pointwise convergence if and only if $f_n(x) \to f(x)$ as $n \to \infty$ for each $x \in X$.*

Now suppose that (Y, d) is a metric space.

Definition 6.3.3. For $f: X \to Y$ and $\epsilon > 0$ let

$$B(f, \epsilon) = \{g: X \to Y \mid \sup_{x \in X} d(f(x), g(x)) < \epsilon\}$$

be the set of functions $g: X \to Y$ such that the distances $d(f(x), g(x))$ for $x \in X$ are bounded above by a number less than ϵ. If $g \in B(f, \epsilon)$ let

$$\delta = \epsilon - \sup_{x \in X} d(f(x), g(x)) .$$

Then $B(g, \delta) \subset B(f, \epsilon)$. If $g \in B(f_1, \epsilon_1) \cap B(f_2, \epsilon_2)$ we define similar numbers $\delta_1, \delta_2 > 0$, let $\delta = \min\{\delta_1, \delta_2\}$, and note that $B(g, \delta) \subset B(f_1, \epsilon_1) \cap B(f_2, \epsilon_2)$. The collection of these subsets $B(f, \epsilon) \subset Y^X$ is a basis for the *topology of uniform convergence*.

Lemma 6.3.4. *A sequence $(f_n)_{n=1}^{\infty}$ of functions $f_n: X \to Y$ converges to $f: X \to Y$ in the topology of uniform convergence if and only if the sequence f_n converges uniformly to f as $n \to \infty$.*

Theorem 6.3.5. *The uniform limit of a sequence of continuous functions is continuous. In other words, if the sequence $(f_n)_{n=1}^{\infty}$ of functions $f_n: X \to Y$ converges to $f: X \to Y$ in the topology of uniform convergence, and each f_n is continuous, then f is continuous.*

The topology of uniform convergence is finer than the topology of pointwise convergence. There is an intermediate topology, of compact convergence, or uniform convergence on compact sets, which also has the property that the limit of a sequence of continuous functions is continuous, at least for reasonable X. If X is compact, then the topology of compact convergence equals the topology of uniform convergence.

Definition 6.3.6. For $f: X \to Y$, $C \subset X$ compact, and $\epsilon > 0$ let

$$B_C(f, \epsilon) = \{g: X \to Y \mid \sup_{x \in C} d(f(x), g(x)) < \epsilon\}$$

be the set of functions $g: X \to Y$ such that the distances $d(f(x), g(x))$ for $x \in C$ are bounded above by a number less than ϵ. If $g \in B_C(f, \epsilon)$ let

$$\delta = \epsilon - \sup_{x \in C} d(f(x), g(x)) .$$

Then $B_C(g, \delta) \subset B_C(f, \epsilon)$. The collection of these subsets $B_C(f, \epsilon) \subset Y^X$ is a basis for the *topology of compact convergence*, also known as the topology of *uniform convergence on compact subsets*.

Lemma 6.3.7. *A sequence $(f_n)_{n=1}^{\infty}$ of functions $f_n: X \to Y$ converges to $f: X \to Y$ in the topology of compact convergence if and only if for each compact subset $C \subset X$ the sequence $f_n|C: C \to Y$ converges uniformly to $f|C: C \to Y$ as $n \to \infty$.*

Example 6.3.8. Consider functions $\mathbb{R} \to \mathbb{R}$. The Taylor polynomials

$$f_n(x) = \sum_{k=0}^{n} \frac{x^k}{k!}$$

converge pointwise, and uniformly on compact subsets, but not uniformly, to the exponential function $f(x) = e^x$.

If $(f_n)_{n=1}^\infty$ is a sequence of continuous functions $f_n \colon X \to Y$ that converges uniformly on compact subsets to $f \colon X \to Y$, we get for each compact $C \subset X$ that $(f_n|C)_{n=1}^\infty$ is a sequence of continuous functions $f_n|C \colon C \to Y$ that converges uniformly to $f|C \colon C \to Y$. Hence $f|C \colon C \to Y$ is continuous, for each such C.

In order to deduced that f itself is continuous, we make the following mild assumption about the space X. For any open subset $A \subset X$ we have that $A \cap C$ is open in the subspace topology on C, for each compact subset $C \subset X$ We are interested in spaces where the converse holds:

Definition 6.3.9. A topological space X is *compactly generated* if the following condition holds: a subset $A \subset X$ is open if (and only if) $A \cap C$ is open in C for each compact subspace $C \subset X$.

Lemma 6.3.10. *X is compactly generated if and only if the following condition holds: a subset $B \subset X$ is closed if (and only if) $B \cap C$ is closed in C for each compact subspace $C \subset X$.*

Proof. Suppose that the condition of the lemma holds. Let $A \subset X$ be such that $A \cap C$ is open in C for each compact subspace $C \subset X$. Let $B = X - A$. Then $B \cap C = C - (A \cap C)$ is the complement of an open set in C, hence is closed in C, for each compact subspace $C \subset X$. By the lemma B is closed in X, hence its complement A is open in X. □

Lemma 6.3.11. *Let X be a compactly generated space, and let Y be any topological space. A function $f \colon X \to Y$ is continuous if (and only if) the restriction $f|C \colon C \to Y$ is continuous for each compact subspace $C \subset X$.*

Proof. Let $U \subset Y$ be open. We prove that $f^{-1}(U)$ is open in X, by checking that $f^{-1}(U) \cap C$ is open in C for each compact $C \subset X$. In fact,

$$f^{-1}(U) \cap C = (f|C)^{-1}(U)$$

is open in C since $f|C \colon C \to Y$ is continuous. □

Theorem 6.3.12. *Let X be compactly generated and let (Y, d) be a metric space. If a sequence of continuous functions $f_n \colon X \to Y$ converges uniformly on compact subspaces to a function $f \colon X \to Y$, then f is continuous.*

Proposition 6.3.13. *If X is locally compact, or first-countable (as is every metrizable space), then X is compactly generated.*

((Proof: Munkres page 284.))

6.3.2 Mapping spaces

The uniform, compact and pointwise topologies on Y^X restrict to give topologies on the subspace

$$\mathscr{C}(X, Y) = \{f \colon X \to Y \mid f \text{ is continuous}\} \subset Y^X$$

of continuous maps $f \colon X \to Y$. Other notations for this *mapping space* are $F(X, Y)$ and $\mathrm{Map}(X, Y)$. It turns out that the topology of compact convergence on $\mathscr{C}(X, Y)$ will not depend on the choice of metric on Y, and can be extended to the case of arbitrary topological spaces Y.

Let X and Y be topological spaces.

Definition 6.3.14. For $C \subset X$ compact and $U \subset Y$ open, let

$$S(C, U) = \{f \in \mathscr{C}(X, Y) \mid f(C) \subset U\}.$$

The collection of these subsets $S(C, U) \subset \mathscr{C}(X, Y)$ is a subbasis for the *compact-open topology.*

The compact-open topology on $\mathscr{C}(X, Y)$ is finer than the topology of pointwise convergence, since one-point sets are compact.

Theorem 6.3.15. *Let X be a topological space and (Y, d) a metric space. On $\mathscr{C}(X, Y)$ the compact-open topology equals the topology of compact convergence.*

Proof. If $A \subset Y$ and $\epsilon > 0$ let
$$B(A, \epsilon) = \bigcup_{p \in A} B(p, \epsilon)$$
be the ϵ-neighborhood of A in Y. If $A \subset U \subset Y$ with V open then the distance
$$x \mapsto d(x, Y - U)$$
to the complement of U is a continuous, positive function on A. If A is compact, then its minimum is attained, so there is an $\epsilon > 0$ with $B(A, \epsilon) \subset U$.

Claim: The topology of compact convergence is finer than the compact-open topology.

Let $S(C, U) \subset \mathscr{C}(X, Y)$ be a subbasis element for the compact-open topology, and let $f \in S(C, U)$. We will find a neighborhood $B_C(f, \epsilon)$ of f in the topology of compact convergence that is contained in $S(C, U)$.

Because f is continuous, $f(C)$ is a compact subset of the open set U. Hence there is an $\epsilon > 0$ so that $B(f(C), \epsilon) \subset U$. For any $g \in B_C(f, \epsilon)$ we have $g(C) \subset B(f(C), \epsilon)$, hence $f \in B_C(f, \epsilon) \subset S(C, U)$.

Claim: The compact-open topology is finer than the topology of compact convergence.

Let $f \colon \mathscr{C}(X, Y)$. Any open neighborhood of f in the topology of compact convergence contains a basis element of the form $B_C(f, \epsilon)$ for some compact $C \subset X$ and $\epsilon > 0$. We shall find a neighborhood
$$S(C_1, U_1) \cap \cdots \cap S(C_n, U_n)$$
of f in the compact-open topology that is contained in $B_C(f, \epsilon)$.

Each point $x \in X$ has a neighborhood V_x such that $f(\bar{V}_x)$ lies in an open subset $U_x \subset Y$ of diameter $< \epsilon$. For example, choose $x \in V_x \subset f^{-1}(B(f(x), \epsilon/4))$. Then $f(V_x) \subset B(f(x), \epsilon/4)$ so
$$f(\bar{V}_x) \subset \overline{f(V_x)} \subset U_x = B(f(x), \epsilon/3)$$
has diameter $\leq 2\epsilon/3 < \epsilon$.

The collection $\{V_x \mid x \in X\}$ is an open covering of C, and admits a finite subcover $\{V_{x_1}, \dots, V_{x_n}\}$. Let $C_i = \bar{V}_{x_i} \cap C$ and $U_i = U_{x_i}$, for $1 \leq i \leq n$. Then C_i is a closed subset of the compact space C, hence is compact.

Since $f(C_i) \subset f(\bar{V}_{x_i}) \subset U_i$ for each i we have
$$f \in S(C_1, U_1) \cap \cdots \cap S(C_n, U_n) .$$
On the other hand, if $g \in S(C_1, U_1) \cap \cdots \cap S(C_n, U_n)$ then we claim that $g \in B_C(f, \epsilon)$.

For each $x \in C$ there is an i such that $x \in C_i \subset \bar{V}_{x_i}$. Since $g \in S(C_i, U_i)$ we have $g(C_i) \subset U_i$. Hence $g(x) \in U_i$. We also have $f(x) \in f(C_i) \subset U_i$. Since U_i has diameter $\leq 2\epsilon/3$, we get that $d(f(x), g(x)) \leq 2\epsilon/3$. Thus
$$\sup_{x \in C} d(f(x), g(x)) \leq 2\epsilon/3 < \epsilon$$
so that $g \in B_C(f, \epsilon)$. ∎

Corollary 6.3.16. *Let Y be a metric space. The compact convergence topology on $\mathscr{C}(X, Y)$ only depends on the underlying topology on Y. If X is compact, the uniform topology on $\mathscr{C}(X, Y)$ only depends on the underlying topology on Y.*

6.3.3 Joint continuity

A function

$$F\colon X \times Y \to Z$$

corresponds to a function

$$f\colon X \to Z^Y$$

defined by the equation $f(x)(y) = F(x,y)$, and conversely. In this situation, we may call F the *left adjoint* and f the *right adjoint*

If F is continuous, each function $f(x)\colon Y \to Z$ is continuous, so that f is a function

$$f\colon X \to \mathscr{C}(Y,Z)\,.$$

In fact, f is then continuous if we give $\mathscr{C}(Y,Z)$ the compact-open topology. The converse holds e.g. if X is locally compact Hausdorff.

Theorem 6.3.17. *Let X, Y and Z be spaces, and give $\mathscr{C}(Y,Z)$ the compact-open topology. If $F\colon X \times Y \to Z$ is continuous, then so is the right adjoint function $f\colon X \to \mathscr{C}(Y,Z)$.*

Proof. Consider a point $p \in X$ and a subbasis element $S(C,U) \subset \mathscr{C}(Y,Z)$ that contains $f(p)$, so that $f(p)(C) \subset U$, or equivalently, $F(\{p\} \times C) \subset U$. We wish to find a neighborhood V of p such that $f(V) \subset S(C,U)$.

By continuity of F, $F^{-1}(U)$ is an open subset of $X \times Y$ that contains $\{p\} \times C$. Then $F^{-1}(U) \cap X \times C$ is an open subset in $X \times C$ that contains $\{p\} \times C$. By compactness of C, using the "tube lemma" of §26, there is a neighborhood V of $\{p\}$ such that $V \times C \subset F^{-1}(U)$. Then for $q \in V$ we have $F(\{q\} \times C) \subset U$, so $f(q)(C) \subset U$ and $f(q) \in S(C,U)$. Hence $f(p) \in f(V) \subset S(C,U)$. □

Proposition 6.3.18. *Let Y be a locally compact Hausdorff space, and give $\mathscr{C}(Y,Z)$ the compact-open topology. The* evaluation map

$$e\colon \mathscr{C}(Y,Z) \times Y \to Z$$

given by $e(g,y) = g(y)$ is continuous.

Proof. Let $(g,p) \in \mathscr{C}(Y,Z) \times Y$ and consider any neighborhood U of $e(g,p) = g(p) \in Z$. We wish to find a neighborhood W of (g,p) such that $e(W) \subset U$.

By the continuity of g, the preimage $g^{-1}(U)$ is a neighborhood of $p \in Y$. By the assumption that Y locally compact and Hausdorff (recall the local nature of local compactness, in §29), there is a neighborhood V of p with compact closure $\bar{V} \subset g^{-1}(U)$. Then

$$W = S(\bar{V},U) \times V \subset \mathscr{C}(Y,Z) \times Y$$

is an open subset containing (g,p), since $g(\bar{V}) \subset U$ and $p \in V$. Furthermore, if $(h,q) \in S(\bar{V},U) \times V$ then $e(h,q) = h(q) \in U$, so $e(W) \subset U$. □

Theorem 6.3.19. *Let X, Y and Z be spaces, with Y locally compact and Hausdorff, and give $\mathscr{C}(Y,Z)$ the compact-open topology. If $f\colon X \to \mathscr{C}(Y,Z)$ is continuous, then so is the left adjoint function $F\colon X \times Y \to Z$.*

Proof. This follows from the proposition, since F is the composite

$$X \times Y \xrightarrow{f \times id} \mathscr{C}(Y,Z) \times Y \xrightarrow{e} Z\,.$$

□

The unit interval $I = [0,1]$ in the subspace topology from $\mathbb{R}$ is compact and Hausdorff, and in particular is locally compact Hausdorff.

Corollary 6.3.20. *There is a one-to-one correspondence between continuous maps*

$$\begin{aligned} F &\colon X \times I \to Y \\ f &\colon X \to \mathscr{C}(I, Y) \\ G &\colon I \times X \to Y \end{aligned}$$

(and

$$g \colon I \to \mathscr{C}(X, Y)$$

if X is locally compact and Hausdorff) subject to the relations

$$F(x,t) = f(x)(t) = G(t,x) = g(t)(x) \,.$$

Each of these maps constitutes a homotopy *between the maps $g(0)\colon X \to Y$ and $g(1)\colon X \to Y$.*

Chapter 7

The Fundamental Group

7.1 (§51) Homotopy of Paths

7.1.1 Path homotopy

Let X and Y be topological spaces. Recall that $I = [0, 1]$.

Definition 7.1.1. Two maps $f, g\colon X \to Y$ are *homotopic* if there exists a map

$$F\colon X \times I \to Y$$

with $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$ for all $x \in X$. In this case we write $f \simeq g$, and call F a *homotopy* from f to g.

Remark 7.1.2. We often use $t \in I$ to indicate the homotopy parameter. For each $t \in I$ let $f_t\colon X \to Y$ be given by $f_t(x) = F(x, t)$. Then each f_t is continuous, and the rule $t \mapsto f_t$ defines a map $I \to \mathscr{C}(X, Y)$, connecting $f = f_0$ to $f_1 = g$. If X is locally compact and Hausdorff we can conversely recover the homotopy F from this map.

We are particularly interested in the case when $X = I$, so that $f, g\colon I \to Y$ are paths in Y. We often use $s \in I$ as the path parameter.

Definition 7.1.3. Let $y_0, y_1 \in Y$. A *path in Y from y_0 to y_1* is a map $f\colon I \to Y$ with $f(0) = y_0$ and $f(1) = y_1$.

Definition 7.1.4. Two paths $f, g\colon I \to Y$ in Y from y_0 to y_1, are *path homotopic* if there is a map

$$F\colon I \times I \to Y$$

with $F(s, 0) = f(s)$ and $F(s, 1) = g(s)$ for all $s \in I$, and $F(0, t) = y_0$ and $F(1, t) = y_1$ for all $t \in I$ In this case we write $f \simeq_p g$, and call F a *path homotopy* from f to g.

Lemma 7.1.5. *The relations $\simeq$ and $\simeq_p$ are equivalence relations.*

Proof. $f \simeq f$ by the constant homotopy $F(x, t) = f(x)$.

If F is a homotopy from f to g, then $\bar{F}\colon X \times I \to Y$ given by $\bar{F}(x, t) = F(x, 1 - t)$ is a homotopy from g to f.

If F is a homotopy from f to g and G is a homotopy from g to h then $H = F \star G\colon X \times I \to Y$ given by

$$(F \star G)(x, t) = \begin{cases} F(x, 2t) & \text{for } 0 \le t \le 1/2 \\ G(x, 2t - 1) & \text{for } 1/2 \le t \le 1 \end{cases}$$

is a homotopy from f to h. Note that $F(x,1) = g(x) = G(x,0)$, so the two maps agree along $t = 1/2$, and $F \star G$ is continuous by the pasting lemma.

In the case $X = I$, the constant homotopy is a path homotopy, if F is a path homotopy then so is $\bar{F}$, and if F and G are path homotopies then so is $F \star G$. □

Definition 7.1.6. We write $[f]$ for the homotopy class of a map $f\colon X \to Y$. If $X = I$, so that f is a path, we shall instead let $[f]$ denote its path homotopy class.

Example 7.1.7. Let $f, g\colon X \to \mathbb{R}^2$ be any two maps to the plane. Then $f \simeq g$ via the *straight-line homotopy*

$$F(x,t) = (1-t)f(x) + tg(x)\,.$$

For each x, the rule $t \mapsto F(x,t)$ parametrizes the straight line in $\mathbb{R}^2$ from $F(x,0) = f(x)$ to $F(x,1) = g(x)$.

If $X = I$ and f, g are both paths from y_0 to y_1, then the straight-line homotopy is a path homotopy, since $F(0,t) = (1-t)f(0) + tg(0) = (1-t)y_0 + ty_0 = y_0$ and $F(1,t) = (1-t)f(1) + tg(1) = (1-t)y_1 + ty_1 = y_1$ for all $t \in I$.

More generally, if $Y \subset \mathbb{R}^n$ is any convex subspace of $\mathbb{R}^n$, so that $(1-t)a + tb \in Y$ for all $a, b \in Y$ and $t \in I$, then any two maps $f, g\colon X \to Y$ are homotopic.

Example 7.1.8. Let $Y = \mathbb{R}^2 - \{(0,0)\}$ be the *punctured plane*. The paths

$$f(s) = (\cos(\pi s), \sin(\pi s))$$
$$g(s) = (\cos(\pi s), 2\sin(\pi s))$$

are path homotopic; the straight-line homotopy in $\mathbb{R}^2$ factors through Y. However, the straight-line homotopy from f to the path

$$h(s) = (\cos(\pi s), -\sin(\pi s))$$

passes through the origin at $s = t = 1/2$, hence does not give a (path) homotopy from f to h as paths in Y. We shall prove later that f and h are not path homotopic in Y. (They are homotopic as maps, ignoring the endpoint conditions, which is one reason why it is path homotopy that is the interesting relation.)

7.1.2 Composition of paths

We now only consider maps from I, and change the name of the target space from Y to X.

Definition 7.1.9. If $f\colon I \to X$ is a path from x_0 to x_1, and $g\colon I \to X$ is a path from x_1 to x_2, the *product* $f * g\colon I \to X$ is defined to be the path

$$(f*g)(s) = \begin{cases} f(2s) & \text{for } 0 \le s \le 1/2 \\ g(2s-1) & \text{for } 1/2 \le s \le 1 \end{cases}$$

from x_0 to x_2. Note that $f(1) = x_1 = g(0)$, so $f * g$ is continuous by the pasting lemma.

Lemma 7.1.10. *If $f_0 \simeq_p f_1$ are path homotopic paths from x_0 to x_1, and $g_0 \simeq_p g_1$ are path homotopic paths from x_1 to x_2, then the products $f_0 * g_0 \simeq_p f_1 * g_1$ are path homotopic paths from x_0 to x_2. Hence there is a well-defined product of path homotopy classes*

$$[f] * [g] = [f * g]$$

when f is a path to x_1 and g is a path from x_1.

Proof. Let $F\colon I \times I \to I$ be a path homotopy from f_0 to f_1, and let $G\colon I \times I \to I$ be a path homotopy from g_0 to g_1. Then $F * G\colon I \times I \to I$ defined by

$$(F * G)(s,t) = \begin{cases} F(2s,t) & \text{for } 0 \le s \le 1/2 \\ G(2s-1,t) & \text{for } 1/2 \le s \le 1 \end{cases}$$

is a path homotopy from $f_0 * g_0$ to $f_1 * g_1$. □

Theorem 7.1.11. *The operation $*$ on path homotopy classes in a space X has the following properties:*

1. *(Associativity) If f is a path to x_1, g a path from x_1 to x_2, and h a path from x_2, then*

$$([f] * [g]) * [h] = [f] * ([g] * [h]) .$$

2. *(Left and right units) For $x \in X$ let $e_x\colon I \to X$ denote the* constant path *at y, with $e_y(s) = y$ for all $s \in I$. If f is a path from x_0 to x_1 then*

$$[e_{x_0}] * [f] = [f] = [f] * [e_{x_1}] .$$

3. *(Inverse) For $f\colon I \to X$ a path from x_0 to x_1 let $\bar{f}$ be the* reverse path *from x_1 to x_0, with $\bar{f}(s) = f(1-s)$ for all $s \in I$. Then*

$$[f] * [\bar{f}] = [e_{x_0}] \qquad \text{and} \qquad [\bar{f}] * [f] = [e_{x_1}] .$$

Proof. (1) The product $([f] * [g]) * [h]$ is the path homotopy class of the path $(f * g) * h$ given by

$$((f * g) * h)(s) = \begin{cases} f(4s) & \text{for } 0 \le s \le 1/4 \\ g(4s-1) & \text{for } 1/4 \le s \le 1/2 \\ h(2s-1) & \text{for } 1/2 \le s \le 1 \end{cases}$$

while $[f] * ([g] * [h])$ is the path homotopy class of the path $f * (g * h)$ given by

$$(f * (g * h))(s) = \begin{cases} f(2s) & \text{for } 0 \le s \le 1/2 \\ g(4s-2) & \text{for } 1/2 \le s \le 3/4 \\ h(4s-3) & \text{for } 3/4 \le s \le 1 . \end{cases}$$

Give a homotopy F from $(f * g) * h$ to $f * (g * h)$ that at time t traverses f for

$$0 \le s \le (1-t)(1/4) + t(1/2) = (1+t)/4 ,$$

then traverses g for

$$(1+t)/4 \le s \le (1-t)(1/2) + t(3/4) = (2+t)/4 ,$$

and finally traverses h for $(2+t)/4 \le s \le 1$:

$$F(s,t) = \begin{cases} f(4s/(1+t)) & \text{for } 0 \le s \le (1+t)/4 \\ g(4s-1-t) & \text{for } (1+t)/4 \le s \le (2+t)/4 \\ h((4s-2-t)/(2+t)) & \text{for } (2+t)/4 \le s \le 1 \end{cases}$$

[[ETC]] □

7.2 (§52) The Fundamental Group

7.2.1 The fundamental group

Definition 7.2.1. A path in X from x_0 to x_0 is called a loop in X based at x_0. Let

$$\pi_1(X, x_0)$$

be the set of path homotopy classes of loops in X based at x_0. For $[f], [g] \in \pi_1(X, x_0)$ note that

$$[f] * [g] = [f * g]$$

is an element of $\pi_1(X, x_0)$.

Lemma 7.2.2. *The set $\pi_1(X, x_0)$ with the composition operation $*$ is a group, called the fundamental group of X (based at x_0), with neutral element $e = [e_{x_0}]$ and group inverse $[f]^{-1} = \bar{f}$.*

Example 7.2.3. If $A \subset \mathbb{R}^n$ is convex, and $x_0 \in A$, then $\pi_1(A, x_0) = \{e\}$ is the trivial group.

7.2.2 Functoriality

Definition 7.2.4. Let $h \colon X \to Y$ be a map, $x_0 \in X$ and $y_0 = h(x_0)$. Write $h \colon (X, x_0) \to (Y, y_0)$. Let

$$h_* \colon \pi_1(X, x_0) \to \pi_1(Y, y_0)$$

be defined by $h_*([f]) = [h \circ f]$.

Lemma 7.2.5. *h_* is a group homomorphism, called the homomorphism induced by h.*

Theorem 7.2.6. *If $h \colon (X, x_0) \to (Y, y_0)$ and $k \colon (Y, y_0) \to (Z, z_0)$, then*

$$(k \circ h)_* = k_* \circ h_* .$$

If $i \colon (X, x_0) \to (X, x_0)$ is the identity map, then i_* is the identity homomorphism of $\pi_1(X, x_0)$.

Corollary 7.2.7. *If $h \colon (X, x_0) \to (Y, y_0)$ is a homeomorphism then*

$$h_* \colon \pi_1(X, x_0) \to \pi_1(Y, y_0)$$

is a group isomorphism.

In this sense the fundamental group is a topological invariant of based spaces (= spaces with a base point).

Let $s_0 = (1, 0) \in S^1 \subset \mathbb{R}^2$. We shall show that $\pi_1(S^1, s_0)$ is a nontrivial group.

Theorem 7.2.8 (Brouwer's fixed point theorem). *Each map $f \colon B^2 \to B^2$ has a fixed point, i.e., a point $p \in B^2$ such that $f(p) = p$.*

7.2.3 Dependence on base point

Definition 7.2.9. Let $\alpha \colon I \to X$ be a path from x_0 to x_1, with reverse path $\bar{\alpha}$ from x_1 to x_0. Define

$$\hat{\alpha} \colon \pi_1(X, x_0) \to \pi_1(X, x_1)$$

by $\hat{\alpha}([f]) = [\bar{\alpha} * f * \alpha]$.

Theorem 7.2.10. *$\hat{\alpha}$ is a group isomorphism.*

Lemma 7.2.11. *If $\alpha \simeq_p \beta$ then $\hat{\alpha} = \hat{\beta}$.*

Corollary 7.2.12. *If X is path connected and $x_0, x_1 \in X$ then there exists an isomorphism $\pi_1(X, x_0) \cong \pi_1(X, x_1)$.*

Note that different paths give conjugate isomorphisms, which will be different if $\pi_1(X, x_0)$ is not abelian.

[[ETC]]

[[$x_0 \in P \subset X$ path component, $\pi_1(P, x_0) \cong \pi_1(X, x_0)$]]

[[Define X simply-connected. If X is simply-connected then any two paths f, g from x_0 to x_1 are path homotopic.]]

7.3 (§53) Covering Spaces

Definition 7.3.1. Let $p\colon E \to B$ be a surjective map. We call E the *total space* and B the *base space*. An open subset $U \subset B$ is said to be *evenly covered* by p if the preimage $p^{-1}(U)$ is the disjoint union $\coprod_\alpha V_\alpha$ of a collection of open subsets $V_\alpha \subset E$, such that for each α the restricted map $p|V_\alpha\colon V_\alpha \to U$ is a homeomorphism. The open subspaces V_α are called the slices, or sheets, of the covering.

Definition 7.3.2. Let $p\colon E \to B$ be a surjective map. If each point $x \in B$ has a neighborhood U that is evenly covered (by p), then p is called a *covering map* and E is said to be a *covering space* of B.

Example 7.3.3. Let B be any space and let F be a discrete set. Let $p\colon E = B \times F \to B$ be the projection that takes (b, f) to b. Then B is evenly covered, since $p^{-1}(B) = E$ is the disjoint union of the sheets $V_\alpha = B \times \{\alpha\}$ for $\alpha \in F$, and each restricted map $p|V_\alpha\colon B \times \{\alpha\} \to B$ is a homeomorphism.

Lemma 7.3.4. *If $p\colon E \to B$ is a covering map, then for each $x \in B$ the preimage $p^{-1}(x)$ is a discrete subspace of E.*

Proof. By assumption, some neighborhood U of x is evenly covered, so $p^{-1}(x)$ is a subspace of $p^{-1}(U) \cong \coprod_\alpha V_\alpha$. Each point $y \in p^{-1}(x)$ lies in a unique V_α, and $\{y\} = p^{-1}(x) \cap V_\alpha$, hence $\{y\}$ is open in the subspace topology. □

Lemma 7.3.5. *Each covering map $p\colon E \to B$ is an open surjection, hence a quotient map.*

Proof. We prove that p is an open map. Let $A \subset E$ be open. To prove that $p(A) \subset B$ is open, consider any point $x \in p(A)$. There is an evenly covered neighborhood U of x, with $p^{-1}(U) \cong \coprod_\alpha V_\alpha$. Choose a point $y \in A$ with $p(y) = x$, and let V_β be the sheet containing y. The set $A \cap V_\beta$ is open in V_β, since A is open, hence $p(A \cap V_\beta)$ is open in U, since p is a homeomorphism. Hence $p(A \cap V_\beta)$ is a neighborhood of x contained in $p(A)$. □

Theorem 7.3.6. *The map $p\colon \mathbb{R} \to S^1$ given by*

$$p(t) = (\cos 2\pi t, \sin 2\pi t)$$

is a covering map.

Each unit interval $[n, n+1] \subset \mathbb{R}$ gets wrapped once around the circle.

Proof. Let $U \subset S^1$ be the open subset of $(x, y) \in S^1$ with $x > 0$. We claim that U is evenly covered. Note that

$$p^{-1}(U) = \coprod_{n\in\mathbb{Z}} (n - \frac{1}{4}, n + \frac{1}{4}) .$$

Let $V_n = (n - 1/4, n + 1/4)$. Then each V_n is open in $\mathbb{R}$, and we claim that $p|V_n : V_n \to U$ is a homeomorphism. A continuous inverse is given by

$$(x, y) \mapsto n + \frac{1}{2\pi} \arcsin y .$$

Alternatively, we may use that $\bar{V}_n = [n-1/4, n+1/4]$ is compact, $\bar{U}$ is Hausdorff, and $p|\bar{V}_n : \bar{V}_n \to \bar{U}$ is a continuous bijection, hence a homeomorphism. It restricts to the desired homeomorphism $p|V_n : V_n \to U$.

Similar arguments show that the parts of S^1 where $y > 0$, $x < 0$ or $y < 0$ are also evenly covered, and these four open subsets cover S^1. □

Definition 7.3.7. A map $p: E \to B$ is a *local homeomorphism* if each point $y \in E$ has a neighborhood V such that $p(V)$ is open in E and $p|V : V \to p(V)$ is a homeomorphism.

Lemma 7.3.8. *Each covering map $p: E \to B$ is a local homeomorphism.*

Proof. For each $y \in E$ the image $x = p(y)$ is contained in an evenly covered neighborhood $U \subset B$, and the sheet $V_\beta \subset E$ that contains y is then a neighborhood of y that p maps homeomorphically to U. □

Example 7.3.9. Let $E = (0, \infty) \subset \mathbb{R}$. The restricted map $q = p|E : E \to S^1$ is a local homeomorphism but not a covering map. No neighborhood of $x = (1, 0) \in S^1$ is evenly covered.

Example 7.3.10. Let $n \in \mathbb{N}$, and consider the map

$$p_n : S^1 \to S^1$$

given by $p_n(z) = z^n$ in complex coordinates $z = x + iy$,

$$p_n(\cos\theta, \sin\theta) = (\cos n\theta, \sin n\theta)$$

in polar coordinates, or

$$\begin{aligned} p_n(x, y) &= (\Re(x + iy)^n, \Im(x + iy)^n) \\ &= (\sum_{0\le 2k\le n} (-1)^k \binom{n}{2k} x^{n-2k} y^{2k}, \sum_{0\le 2k+1\le n} (-1)^k \binom{n}{2k+1} x^{n-2k-1} y^{2k+1}) \end{aligned}$$

in real coordinates. (For example, $p_2(x, y) = (x^2 - y^2, 2xy)$.) Each p_n is a covering map.

Proposition 7.3.11. *Let $p: E \to B$ be a covering map, and $B_0 \subset B$ a subspace. Let $E_0 = p^{-1}(B_0) \subset E$. Then $p_0 = p|E_0 : E_0 \to B_0$ is a covering map.*

Proof. Let $x \in B_0$. If U is an evenly covered neighborhood of x in B then $B_0 \cap U$ is an evenly covered neighborhood of x in B_0, since

$$p_0^{-1}(B_0 \cap U) = E_0 \cap p^{-1}(U) \cong E_0 \cap \coprod_\alpha V_\alpha \cong \coprod_\alpha E_0 \cap V_\alpha$$

and each $E_0 \cap V_\alpha$ is open in E_0, with the homeomorphism $p|V_\alpha : V_\alpha \to U$ restricting to a homeomorphism $E_0 \cap V_\alpha \to B_0 \cap U$. □

Proposition 7.3.12. *If $p\colon E \to B$ and $p'\colon E' \to B'$ are covering maps, then $p \times p'\colon E \times E' \to B \times B'$ is a covering map.*

Proof. Let $x \in B$ and $x' \in B'$. If $x \in U \subset B$ and $x' \in U' \subset B'$ with U and U' evenly covered, then $(x, x') \in U \times U' \subset B \times B'$ is evenly covered, since

$$(p \times p')^{-1}(U \times U') = p^{-1}(U) \times (p')^{-1}(U') \cong \coprod_\alpha V_\alpha \times \coprod_\beta V'_\beta \cong \coprod_{\alpha,\beta} V_\alpha \times V'_\beta$$

is a disjoint union of open subsets $V_\alpha \times V'_\beta \subset E \times E'$, and the restricted map

$$(p \times p')|(V_\alpha \times V'_\beta)\colon V_\alpha \times V'_\beta \to U \times U'$$

is the product of the homeomorphisms $p|V_\alpha\colon V_\alpha \to U$ and $p'|V'_\beta\colon V'_\beta \to U'$, hence is a homeomorphism. □

Example 7.3.13. Let $T^2 = S^1 \times S^1$ be the *torus*. The product map

$$p \times p\colon \mathbb{R}^2 = \mathbb{R} \times \mathbb{R} \to S^1 \times S^1 = T^2$$

is a product over covering map, hence is a covering map. Each unit square $[n, n+1] \times [m, m+1]$ gets wrapped once around the torus.

Example 7.3.14. Let $s_0 = (1, 0) \in S^1$ be a base point, and consider the one-point union

$$B_0 = (S^1 \times \{s_0\}) \cup (\{s_0\} \times S^1) \subset T^2$$

of two circles. Let $E_0 = (p \times p)^{-1}(B_0) \subset \mathbb{R}^2$. It is the subspace

$$E_0 = (\mathbb{R} \times \mathbb{Z}) \cup (\mathbb{Z} \times \mathbb{R}) \subset \mathbb{R}^2$$

consisting of $(x, y) \in \mathbb{R}^2$ with $x \in \mathbb{Z}$ or $y \in \mathbb{Z}$. The restricted map $p_0\colon E_0 \to B_0$ is a covering map of the figure-eight space B_0 by the infinite grid E_0.

Example 7.3.15. The complex exponential map

$$\exp\colon \mathbb{C} \to \mathbb{C} - \{0\}$$

given by $\exp(z) = e^z$ is given in real coordinates by

$$p\colon \mathbb{R}^2 \to \mathbb{R}^2 - \{(0, 0)\}$$

with $p(x, y) = (e^x \cos y, e^x \sin y)$. Under the homeomorphism

$$h\colon \mathbb{R} \times S^1 \cong \mathbb{R}^2 - \{(0, 0)\}$$

taking $(t, (x, y))$ to $(e^t x, e^t y)$ we see that p factors as $h \circ q$, where $q\colon \mathbb{R}^2 \to \mathbb{R} \times S^1$ is the product of two covering maps $\mathbb{R} \to \mathbb{R}$ and $\mathbb{R} \to S^1$. Hence exp and p are covering maps.

7.4 (§54) The Fundamental Group of the Circle

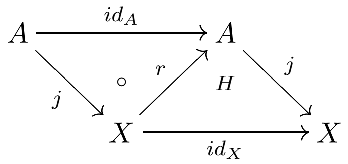
[[See Munkres.]]

7.5 (§55) Retractions and Fixed Points

[[See Munkres.]]

7.6 (§58) Deformation Retracts and Homotopy Type

Let $j\colon A \to X$ be the inclusion of a subspace. Recall that a *retraction* of X onto A is a map $r\colon X \to A$ such that $rj = id_A$ is the identity on A.



Definition 7.6.1. A *deformation retraction* of X onto A is a homotopy $H\colon X \times I \to I$ from id_X to $jr\colon X \to X$, such that $H(a,t) = a$ for all $a \in A$ and $t \in I$.

Example 7.6.2. For $n \geq 1$ let $A = S^n$ and $X = \mathbb{R}^{n+1} - \{\vec{0}\}$. Let $j\colon S^n \to \mathbb{R}^{n+1} - \{\vec{0}\}$ be the inclusion, and let $r\colon \mathbb{R}^{n+1} - \{\vec{0}\} \to S^n$ be given by $r(x) = x/\|x\|$. Clearly $rj(a) = a$ for each $a \in S^n$. The composite $jr\colon \mathbb{R}^{n+1} - \{\vec{0}\} \to \mathbb{R}^{n+1} - \{\vec{0}\}$ is homotopic to the identity map by the straight-line homotopy

$$H(x,t) = (1-t)x + tx/\|x\|\,,$$

and $H(a,t) = (1-t)a + ta/1 = a$ for all $a \in S^n$ and $t \in I$. Hence H is a deformation retraction of $\mathbb{R}^{n+1} - \{\vec{0}\}$ onto S^n.

Lemma 7.6.3. *Let $h, k\colon (X, x_0) \to (Y, y_0)$ be maps. If $H\colon X \times I \to Y$ is a homotopy from h to k, with $H(x_0, t) = y_0$ for all $t \in I$, then the induced homomorphisms*

$$h_* = k_*\colon \pi_1(X, x_0) \to \pi_1(Y, y_0)$$

are equal.

We might say that H is a homotopy from h to k *relative to* x_0.

Proof. Let $f\colon I \to X$ be a loop in X at x_0, so that $[f] \in \pi_1(X, x_0)$. Then $h_*([f]) = [hf]$ and $k_*([f]) = [kf]$, so we need to show that $hf, kf\colon I \to Y$ are path homotopic. The composite

$$H(f \times id)\colon I \times I \to X \times I \to Y$$

is such a path homotopy. □

Theorem 7.6.4. *Let A be a deformation retract of X. The inclusion map $j\colon A \to X$ induces an isomorphism*

$$j_*\colon \pi_1(A, a_0) \to \pi_1(X, a_0)$$

of fundamental groups, for any $a_0 \in A$.

Proof. Let $H\colon X \times I \to X$ be the deformation retraction, with associated retraction $r\colon X \to A$. Then $rj = id_A$ so $(rj)_* = r_* j_*$ is the identity on $\pi_1(A, a_0)$. We must prove that $(jr)_* = j_* r_*$ is the identity on $\pi_1(X, a_0)$. But $H\colon X \times I \to X$ is a homotopy from id_X to jr with $H(a_0, t) = a_0$ for all $t \in I$, so by the lemma above, $(id_X)_*$ and $(jr)_*$ are equal. □

Example 7.6.5. Let $X = \mathbb{C} - \{0\} \cong \mathbb{R}^2 - \{(0,0)\}$ be the *punctured plane*. The inclusion $S^1 \to \mathbb{C} - \{0\}$ induces an isomorphism

$$\pi_1(S^1, b_0) \cong \pi_1(\mathbb{C} - \{0\}, b_0)$$

for any $b_0 \in S^1$. Both of these groups are infinite cyclic, i.e., isomorphic to the additive group of integers $\mathbb{Z}$.

Example 7.6.6. Let $X = \mathbb{C} - \{+i, -i\} \cong \mathbb{R}^2 - \{(0,1), (0,-1)\}$ be the *doubly punctured plane.* Let $\mathbf{8} \subset X$ be the *figure eight space* given by the union of the circle of radius 1 with center at $+i$ and the circle of radius 1 with center at $-i$. The two circles meet at 0, so $\mathbf{8} = (+i+S^1) \cup (-i+S^1)$ is the one-point union of two circles, often denoted $S^1 \vee S^1$.

There exists a deformation retraction of $\mathbb{C} - \{+i, -i\}$ onto $\mathbf{8}$. Hence the inclusion induces an isomorphism of fundamental groups

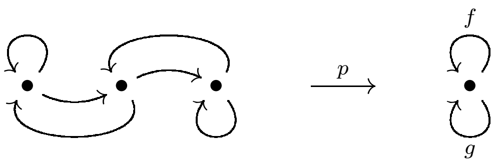
$$\pi_1(\mathbf{8}, 0) \cong \pi_1(\mathbb{C} - \{+i, -i\}, 0) .$$

Let $f \colon I \to \mathbf{8}$ be the loop winding once around $+i + S^1$, and let $g \colon I \to \mathbf{8}$ be the loop winding once around $-i + S^1$. Let $x = [f]$ and $y = [g]$ be their path homotopy classes in $\pi_1(\mathbf{8}, 0)$.

It is possible to prove that $\pi_1(\mathbf{8}, 0)$ is the free (non-abelian) group on the two generators x and y, often denoted $F_2 = \langle x, y \rangle$. Every element of F_2 can be written as a finite product of copies of x, x^{-1}, y and y^{-1}, and the only relations satisfied by these finite products are those required by the axioms of a group. We prove a little less:

Proposition 7.6.7. *The fundamental group of the figure eight space is not abelian.*

Proof. Let $B = \mathbf{8}$ be the figure eight space, with a loop f traversing one circle and a loop g traversing the other circle. It has a path-connected threefold covering space $p \colon E \to B$ where the part above each circle is a disjoint union of a onefold and a twofold covering space.



Starting at the intersection of the two twofold covering spaces, the lifts of the paths $f * g$ and $g * f$ have different endpoints. Hence the lifting function

$$\phi \colon \pi_1(\mathbf{8}, 0) \to p^{-1}(0)$$

takes different values at $[f] * [g]$ and $[g] * [f]$. In particular, $[f] * [g] \neq [g] * [f]$, so $\pi_1(\mathbf{8}, 0)$ is not an abelian group. □

7.7 (§59) The Fundamental Group of S^n

Theorem 7.7.1. *Suppose that $X = U \cup V$ is covered by two open, simply-connected subspaces U and V, and that $U \cap V$ is path connected. Then X is simply-connected.*

Proof. (It is clear that X is path connected.)

Choose a base point $x_0 \in U \cap V$. Consider any loop $f \colon I \to X$ in X based at x_0. The open subsets $f^{-1}(U)$ and $f^{-1}(V)$ cover the compact metric space I, so by the Lebesgue number lemma there exists a finite partition

$$0 = s_0 < s_1 < \cdots < s_n = 1$$

of I such that f maps each subinterval $[s_{i-1}, s_i]$ into U or V (or both). Consider any $0 < i < n$. If $f(s_i) \in X - V = U - U \cap V$ then f maps $[s_{i-1}, s_i]$ and $[s_i, s_{i+1}]$ into U, hence we can delete s_i from the finite partition. Likewise, if $f(s_i) \in X - U = V - U \cap V$ then we can delete s_i from the partition. Repeating as often as possible, we may assume that each $f(s_i) \in U \cap V$.

Choose a loop $g\colon I \to X$ with image in $U \cap V$ such that $g(s_i) = f(s_i)$ for each $0 \le i \le n$. This is possible since $U \cap V$ is path connected.

For each $1 \le i \le n$ the two paths $f|[s_{i-1}, s_i]$ and $g|[s_{i-1}, s_i]$ have the same beginning and the same end, and lie in one of the simply-connected spaces U and V. Hence there is a path homotopy

$$H_i\colon [s_{i-1}, s_i] \times I \to X$$

between these paths. The glued map

$$H\colon I \times I \to X$$

with $H|[s_{i-1}, s_i] \times I = H_i$ is then a path homotopy between f and g. But $[g]$ is in the image of the homomorphism induced by the inclusion $U \cap V \to X$, which factors through $\pi_1(U, x_0)$, which is the trivial group.

$$\pi_1(U \cap V, x_0) \to \pi_1(U, x_0) \to \pi_1(X, x_0)$$

Hence $[f] = [g] = e$. Since f was arbitrary, $\pi_1(X, x_0)$ is the trivial group. □

Let $S^2 \subset \mathbb{R}^3$ be the unit sphere.

Theorem 7.7.2. *The sphere S^2 is simply-connected.*

Proof. Let $N = (0, 0, +1)$ and $S = (0, 0, -1)$ be the north and south pole, respectively. Let $U = S^2 - \{S\}$ and $V = S^2 - \{N\}$ be open neighborhoods of the upper and lower hemisphere, respectively.

There is a homeomorphism

$$h\colon U \cong \mathbb{R}^2$$

given by straight-line projection from the south pole, called *stereographic projection*. In coordinates,

$$h(x_1, x_2, x_3) = \frac{1}{1 + x_3}(x_1, x_2)$$

for $(x_1, x_2, x_3) \in U$. The inverse h^{-1} is given by

$$h^{-1}(y_1, y_2) = (0, 0, -1) + t(y)(y_1, y_2, 1)$$

for $y = (y_1, y_2) \in \mathbb{R}^2$, $t(y) = 2/(1 + \|y\|^2)$.

Similarly, there is a homeomorphism $V \cong \mathbb{R}^2$. The intersection $U \cap V$ corresponds under h to $\mathbb{R}^2 - \{(0, 0)\}$, which is path connected. □

7.8 (§60) Fundamental Groups of Some Surfaces

Recall that a *surface* is the same as a 2-dimensional *manifold*, i.e., a second-countable Hausdorff space X such that each point p has a neighborhood U that is homeomorphic to $\mathbb{R}^2$.

7.8.1 Cartesian products

Definition 7.8.1. The cartesian product of two groups G and H is the set $G \times H$ with the product structure given by

$$(g, h) \cdot (g', h') = (g \cdot g', h \cdot h').$$

The neutral element is (e, e), and the group inverse is given by $(g, h)^{-1} = (g^{-1}, h^{-1})$.

Let (X, x_0) and (Y, y_0) be based spaces, with cartesian product $X \times Y$ based at (x_0, y_0). The projection maps

$$p = pr_X : (X \times Y, (x_0, y_0)) \to (X, x_0)$$

and

$$q = pr_Y : (X \times Y, (x_0, y_0)) \to (Y, y_0)$$

induce group homomorphisms

$$p_* : \pi_1(X \times Y, (x_0, y_0)) \to \pi_1(X, x_0)$$

and

$$q_* : \pi_1(X \times Y, (x_0, y_0)) \to \pi_1(Y, y_0) .$$

These are the components of a group homomorphism

$$\Phi = (p_*, q_*) : \pi_1(X \times Y, (x_0, y_0)) \to \pi_1(X, x_0) \times \pi_1(Y, y_0)$$

taking $[h]$ to

$$\Phi([h]) = (p_*([h]), q_*([h])) = ([ph], [qh])$$

for any loop $h: I \to X \times Y$ based at (x_0, y_0).

Theorem 7.8.2. $\Phi : \pi_1(X \times Y, (x_0, y_0)) \to \pi_1(X, x_0) \times \pi_1(Y, y_0)$ *is a group isomorphism.*

Proof. Any element of $\pi_1(X, x_0) \times \pi_1(Y, y_0)$ has the form $([f], [g])$, where $f : I \to X$ is a loop at x_0 and $g : I \to Y$ is a loop at y_0. Let $h : I \to X \times Y$ be given by $h(s) = (f(s), g(s))$. Then $ph = f$ and $qh = g$, so $\Phi([h]) = ([f], [g])$ and Φ is surjective.

If $h : I \to X \times Y$ is a loop at (x_0, y_0) with $\Phi([h]) = 0$ then $ph : I \to X$ is path homotopic to the constant loop e_{x_0} and $qh : I \to Y$ is path homotopic to the constant loop e_{y_0}. Let $F : I \times I \to X$ and $G : I \times I \to Y$ be such path homotopies. Then $H : I \times I \to X \times Y$ given by $H(s, t) = (F(s, t), G(s, t))$ is a path homotopy from $h = (ph, qh)$ to $e_{(x_0, y_0)} = (e_{x_0}, e_{y_0})$. Hence $[h] = e$ and Φ is injective. □

Example 7.8.3. The fundamental group of the torus T^2 is

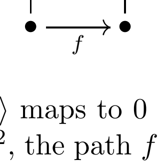
$$\pi_1(T^2) = \pi_1(S^1 \times S^1) \cong \pi_1(S^1) \times \pi_1(S^1) \cong \mathbb{Z} \times \mathbb{Z} = \mathbb{Z}^2$$

the free abelian group on two generators.

The inclusion $\mathbf{8} \cong S^1 \vee S^1 \subset S^1 \times S^1 = T^2$ induces the canonical surjection

$$F_2 = \pi_1(\mathbf{8}) \to \pi_1(T^2) = \mathbb{Z}^2$$

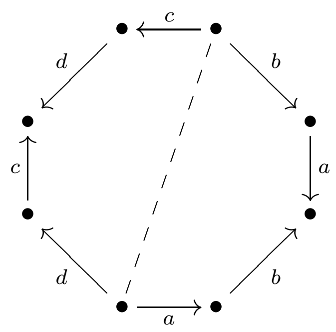
from the free group on two generators to the free abelian group on two generators.



The commutator $xyx^{-1}y^{-1}$ in $F_2 = \langle x, y \rangle$ maps to 0 (the additively neutral element), since in the model for T^2 as a quotient space of I^2, the path $f * g * \bar{f} * \bar{g}$ representing $xyx^{-1}y^{-1}$ is given by the boundary of I^2, and this loop is nullhomotopic in I^2.

7.8.2 A retraction

Let the *double torus* $T^2 \# T^2$ be the quotient space of the octagon with the following identifications along the boundary:



Cutting along the dashed line, each part becomes a torus with a disc removed. The dashed line becomes the boundary circle in each part. Gluing together again along the dashed line, we get the connected sum of the two tori.

Theorem 7.8.4. *The fundamental group of the double torus is not abelian.*

Proof. Let $X = T^2 \# T^2$. The subspace $A \subset X$ given by taking only the edges labeled a and c is homeomorphic to the figure eight space $A \cong \mathbf{8}$. Furthermore, there is a retraction $r \colon X \to A$ given by first collapsing the dashed line, to get the one-point union $T^2 \vee T^2$ of two tori, and then retracting each torus onto a circle, to get the one-point union $S^1 \vee S^1$ of two circles.

Hence the inclusion $j \colon \mathbf{8} \to T^2 \# T^2$ induces an injective homomorphism

$$\pi_1(\mathbf{8}) \to \pi_1(T^2 \# T^2)\,,$$

showing that $\pi_1(T^2 \# T^2)$ has a non-abelian subgroup. □

We can now distinguish between three different compact, connected surfaces.

Theorem 7.8.5. *The 2-sphere S^2, the torus T^2 and the double torus $T^2 \# T^2$ are all topologically distinct.*

Proof. The fundamental groups $\pi_1(S^2) = 1$ and $\pi_1(T^2) = \mathbb{Z}^2$ are abelian, while $\pi_1(T^2 \# T^2)$ is not abelian. Since these groups are pairwise not isomorphic, it follows that the spaces S^2, T^2 and $T^2 \# T^2$ are pairwise not homeomorphic. □

7.8.3 3-manifolds

Any compact 3-manifold is the disjoint union of a finite set of compact, connected 3-manifolds. A compact, connected 3-manifolds is said to be *irreducible* if each embedded sphere (S^2) bounds an embedded ball (B^3). Otherwise the 3-manifold is reducible and can be simplified by cutting it open along the embedded 2-sphere and gluing in a ball on each side. This process stops after finitely many steps.

Here is a post on irreducible 3-manifolds and their fundamental groups, by Bruno Martelli on mathoverflow.net:

> Perelman has proved Thurston's geometrization conjecture, which says that every irreducible 3-manifold decomposes along its canonical decomposition along tori into pieces, each admitting a geometric structure. A "geometric structure" is a nice riemannian metric, which is in particular complete and of finite volume.

There are eight geometric structures for 3 manifolds: three structures are the constant curvature ones (spherical, flat, hyperbolic), while the other 5 structures are some kind of mixing of low-dimensional structures (for instance, a surface Σ of genus 2 has a hyperbolic metric, and the three-manifold $\Sigma \times S^1$ has a mixed hyperbolic $\times S^1$ structure).

The funny thing is that geometrization conjecture was already proved by Thurston when the canonical decomposition is non-trivial, i.e., when there is at least one torus in it. In that case the manifold is a Haken manifold because it contains a surface whose fundamental group injects in the 3-manifold. Haken manifolds have been studied by Haken himself (of course) and by Waldhausen, who proved in 1968 that two Haken manifolds with isomorphic fundamental groups are in fact homeomorphic.

If the canonical decomposition of our irreducible manifold M is empty, now we can state by Perelman's work that M admits one of these 8 nice geometries. The manifolds belonging to 7 of these geometries are well-known and have been classified some decades ago (six of these geometries actually coincide with the well-known Seifert manifolds, classified by Seifert already in 1933). From the classification one can see that the only distinct manifolds with isomorphic fundamental groups are lens spaces (which belong to the elliptic geometry, since they have finite fundamental group).

The only un-classified geometry is the hyperbolic one. However, Mostow rigidity theorem says that two hyperbolic manifolds with isomorphic fundamental group are isometric, hence we are done. Some simple considerations also show that two manifolds belonging to distinct geometries have non-isomorphic fundamental groups.

Therefore now we know that the fundamental group is a complete invariant for irreducible 3-manifolds, except lens spaces.

— — — END OF NOTES — — —