MAT 4510 - 2008

Problem 1

a) $f(z) = \frac{z-1}{z+1} = \frac{\frac{z}{\sqrt{2}} - \frac{1}{\sqrt{2}}}{\frac{z}{\sqrt{2}} + \frac{1}{\sqrt{2}}}$, $a = d = \frac{1}{\sqrt{2}}, b = -c = -\frac{1}{\sqrt{2}} \in \mathbb{R}$

with $ad - bc = 1$ so $f \in Möb^+(\mathbb{H})$

Moreover $(a+d)^2 = \left(\frac{2}{\sqrt{2}}\right)^2 = 2 < 4$

So <u>f is of elliptic type</u>

b) $g \in Möb^+(\mathbb{H})$, $\underbrace{g \text{ elliptic and } g(i) = i}$ then $g(z) = \frac{\cos\theta z + \sin\theta}{-\sin\theta z + \cos\theta}$

for some $\theta \in (0, \pi)$.

The matrix associated to $G g G^{-1}$ is then

$$\begin{bmatrix} i & 1 \\ 1 & i \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} i & -1 \\ -1 & i \end{bmatrix}$$

$$= \begin{bmatrix} i & 1 \\ 1 & i \end{bmatrix} \begin{bmatrix} i\cos\theta - \sin\theta & -\cos\theta + i\sin\theta \\ -i\sin\theta - \cos\theta & \sin\theta + i\cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} -2\cos\theta - 2i\sin\theta & 0 \\ 0 & -2\cos\theta + 2i\sin\theta \end{bmatrix}$$

. This gives the same F.L.T as $\begin{bmatrix} \cos\theta + i\sin\theta & 0 \\ 0 & \cos\theta - i\sin\theta \end{bmatrix}$

$$= \begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix}$$

So $G g G^{-1}(z) = \frac{e^{i\theta} z}{e^{-i\theta}} = e^{i2\theta} z$

which is <u>a rotation of an angle 2θ</u>

Problem 2

a) $X(u,v) = (\cos u, \sin u, v)$

$X_u = (-\sin u, \cos u, 0)$

$X_v = (0, 0, 1)$

$E = X_u \cdot X_u = \cos^2 u + \sin^2 u = 1$

$F = X_u \cdot X_v = 0$

$G = X_v \cdot X_v = 1$

$ds^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2 = du^2 + dv^2$

$$N = \frac{X_u \times X_v}{\|X_u \times X_v\|} = \frac{\begin{vmatrix} i & j & k \\ -\sin u & \cos u & 0 \\ 0 & 0 & 1 \end{vmatrix}}{\| - \|} = \underline{\underline{(\cos u, \sin u, 0)}}$$

$e = N \cdot X_{uu} = N \cdot (-\cos u, -\sin u, 0) = -1$

$f = N \cdot X_{uv} = N \cdot (0,0,0) = 0$

$g = N \cdot X_{vv} = N \cdot (0,0,0) = 0$

$$K = \frac{eg - f^2}{EG - F^2} = \frac{-1 \cdot 0 - 0}{EG - F^2} = \underline{\underline{0}}$$

b) $y(u,v) = (\cos u(2+\cos v), \sin u(2+\cos v), \sin v)$

$y_u = (-\sin u(2+\cos v), \cos u(2+\cos v), 0)$

$y_v = (-\cos u \sin v, -\sin u \sin v, \cos v)$

$y_u \times y_v = (\cos u \cos v(2+\cos v), \sin u \cos v(2+\cos v), \sin v(2+\cos v))$

$$N = \frac{y_u \times y_v}{\|y_u \times y_v\|} = (\cos u \cos v, \sin u \cos v, \sin v)$$

Let $\alpha_{v_0}(t) = y(t, v_0)$, $\alpha_{v_0}'(t) = y_u(t, v_0) = \cancel{(-\sin t(2}$

$= (-\sin t(2+\cos v_0), \cos t(2+\cos v_0), 0)$. Note that $\|\alpha_v'(t)\|$

$= (2+\cos v_0)$ ($=$ constant)

$\alpha_{v_0}''(t) = (-\cos t(2+\cos v_0), -\sin t(2+\cos v_0), 0)$

So since $\alpha_{v_0}(t)$ is of constant speed.

$\alpha_{v_0}(t)$ is a geodesic, if and only if

$\alpha_{v_0}''(t)$ is parallel to N along $\alpha_{v_0}(t)$,

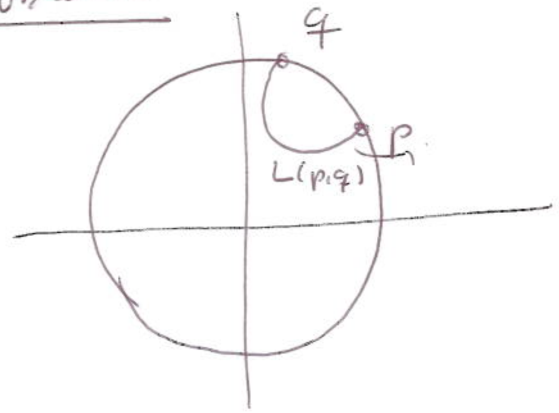
that is $\alpha_{v_0}''(t) \parallel = N(t, v_0)$

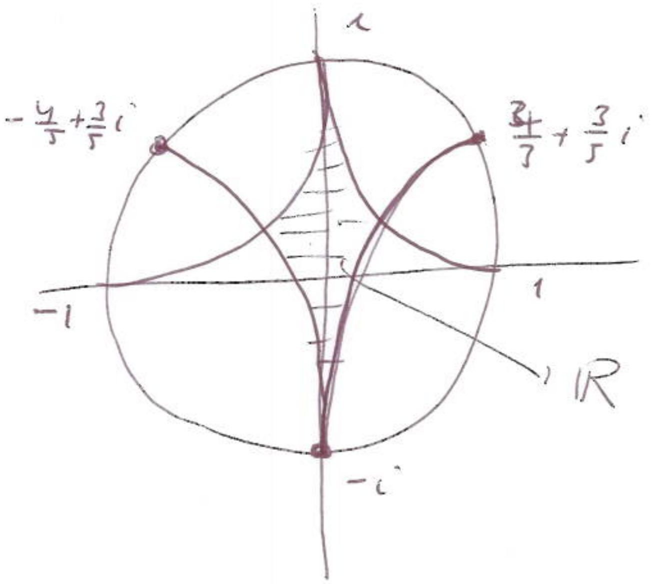
Now $N(t, v_0) = (\cos t \cos v_0, \sin t \sin v_0, \sin v_0)$

which is parallel to $\alpha_{v_0}''(t)$ if and only if

$\sin v_0 = 0$, that is when $v_0 = 0$ and $v_0 = \Pi$

Problem 3





a) $G(z) = \dfrac{iz+1}{z+i}$

$G^{-1}(z) = \dfrac{-iz+1}{z-i} = \dfrac{z+i}{iz+1}$

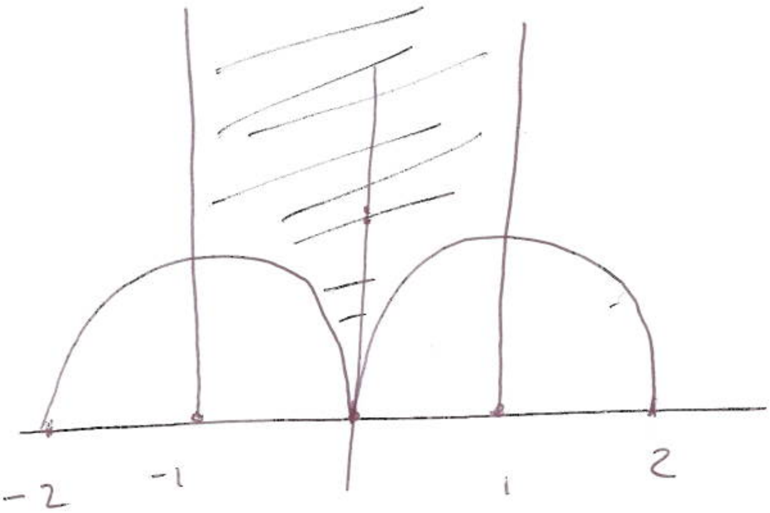
$G^{-1}(-1) = \dfrac{-1+i}{-i+1} = -1$, $G^{-1}(i) = \infty$, $G^{-1}(1) = 1$

$G^{-1}(\frac{4}{5}+\frac{3}{5}i) = \dfrac{\frac{4}{5}+\frac{8}{5}i}{\frac{4}{5}i+\frac{2}{5}} = 2$, $G^{-1}(-i) = 0$, $G^{-1}(-\frac{4}{5}+\frac{3}{5}i) = \dfrac{-\frac{4}{5}+\frac{8}{5}i}{-\frac{4}{5}i+\frac{2}{5}}$

$= -2$

The ~~hyperbolic area~~ area $G^{-1}(R)$

is therefore given by



$$\{ z \in \mathbb{H} \mid |Re(z)| \leq 1, \ |z+1| \geq 1, \ |z-1| \geq 1 \}$$

b) $G^{-1}(R)$ is the union of two doubly asymptotic triangles with vertices in $0, -1+i, \infty$ and $0, 1+i, \infty$ respectively. Both has angles equal $0, 0, \frac{\pi}{2}$. So the area is $2(\pi - (0+0+\frac{\pi}{2})) = \underline{\underline{\pi}}$

Problem 4

a) Since $\iint_{M_2} K\,dA > 0$ and C is a closed geodesic hence $\int_C k_g\,ds = 0$, Gauss-Bonnet theorem gives us that $\iint_{M_2} K\,dA = \chi(M_2) \geq 1$

(since $\chi(M_2) > 0$ and is an integer)

When we glue a disk D^2 to a space it is easy to see that the Euler charateristic will increase by 1, hence

$$\chi(M_2') = \chi(M_2) + 1 \geq 2.$$

M_2' is ~~homeom~~ a ~~topological~~ compact surface ~~so~~ and the only compact surface with $\chi(M_2') \geq 2$ is S^2 ~~and~~ so $\cancel{\chi}(M_2') \cong S^2$, hence $\chi(M_2') = \chi(S^2) = 2$

b) Now $M = M_1 \cup_C M_2$. But M_2 is equal M_2' with ~~the~~ an open disc removed. But since $M_2' \cong S^2$, and S^2 with an open disc removed is homeomorphic to a closed disc, M_2 is homeomorphic to a closed disc hence

$$M = M_1 \underset{C}{\cup} M_2 \cong M_1 \underset{C=\partial D^2}{\cup} D^2 = M_1'$$