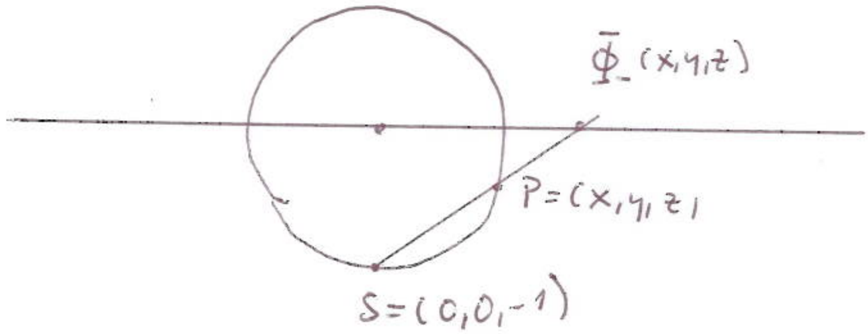
1.2



$$S + t(P-1) = (0,0,-1) + t(x,y,z+1) = (tx, ty, -1+t(z+1))$$

$$-1+t(z+1)=0 \Rightarrow t = \frac{1}{z+1}$$

$$\Phi_-(x,y,z) = \left(\frac{x}{z+1}, \frac{y}{z+1}\right)$$

$$\Phi_- \circ \Phi^{-1}(u,v) = \Phi_-\left(\frac{2u}{u^2+v^2+1}, \frac{2v}{u^2+v^2+1}, \frac{u^2+v^2-1}{u^2+v^2+1}\right)$$

$$= \left(\frac{2u/u^2+v^2+1}{1+\frac{u^2+v^2-1}{u^2+v^2+1}}, \frac{2v/u^2+v^2+1}{1+\frac{u^2+v^2-1}{u^2+v^2+1}}\right)$$

$$= \left(\frac{2u}{2(u^2+v^2)}, \frac{2v}{2(u^2+v^2)}\right) = \left(\frac{u}{u^2+v^2}, \frac{v}{u^2+v^2}\right) = \frac{z}{|z|^2}$$

Note that $\Phi_{-1} \circ \Phi^{-1}$ is defined on $\mathbb{R}^2 - \{(0,0)\} = \mathbb{C} - \{0\}$

1.4

a) Put $w = u+iv$, $f(w) = w^{-1} = \frac{\bar{w}}{|w|^2} = \frac{(u,-v)}{u^2+v^2}$

$$f\circ\Phi(x,y,z) = f\left(\frac{x}{1-z},\frac{y}{1-z}\right) = \frac{1}{(1-z)}(x,-y)\Big/\frac{x^2+y^2}{(1-z)^2}$$

$$= \frac{(1-z)}{x^2+y^2}(x,-y) = \frac{(1-z)}{(1-z^2)}(x,-y) = \frac{1}{1+z}(x,-y)$$

(since $x^2+y^2+z^2=1$ hence $1-z^2 = x^2+y^2$)

Using the formula $\Phi^{-1}(u,v) = \left(\frac{2u}{u^2+v^2+1},\frac{2v}{u^2+v^2+1},\frac{u^2+v^2-1}{u^2+v^2+1}\right)$ we get that

$$\Phi^{-1}\circ f\circ\Phi(x,y,z) = \Phi^{-1}\left(\frac{x}{1+z},\frac{-y}{1+z}\right)$$

$$= \left(\frac{2x}{1+z},\frac{-2y}{1+z},\frac{x^2+y^2}{(1+z)^2}-1\right)\Big/\left(\frac{x^2+y^2}{(1+z)^2}+1\right)$$

$$= \frac{(1+z)^2}{x^2+y^2+(1+z)^2}\left(\frac{2x}{1+z},\frac{-2y}{1+z},\frac{x^2+y^2-(1+z)^2}{(1+z)^2}\right)$$

$$= \frac{(1+z)^2}{1+2z+x^2+y^2+z^2}\left(\frac{2x}{1+z},\frac{-2y}{1+z},\frac{x^2+y^2-1-2z-z^2}{(1+z)^2}\right)$$

$$= \frac{(1+z)^2}{2(1+z)}\left(\frac{2x}{1+z},\frac{-2y}{1+z},\frac{-2z-2z^2}{(1+z)^2}\right)$$

$$= \frac{1+z}{2}\left(\frac{2x}{1+z},\frac{-2y}{1+z},\frac{-2z}{1+z}\right) = \underline{(x,-y,-z)}$$

This corresponds to a rotation by an angle π in the yz plane (leaving the x-axis fixed).

(b) Consider $S^2 - \{0,0,\pm 1\} \xrightarrow{f} S^2 - \{0,0,\pm 1\}$

defined by $f(x,y,z) = -(x,y,z)$

Let $g(w) = \Phi \circ f \circ \Phi^{-1} : \mathbb{C} - \{0\} \to \mathbb{C} - \{0\}$

Let $w = u + iv = (u,v)$

then $f \circ \Phi^{-1}(u,v) = f\left(\frac{2u}{u^2+v^2+1}, \frac{2v}{u^2+v^2+1}, \frac{u^2+v^2-1}{u^2+v^2+1}\right)$

$= \left(\frac{-2u}{u^2+v^2+1}, \frac{-2v}{u^2+v^2+1}, \frac{1-u^2-v^2}{u^2+v^2+1}\right)$

$\Phi \circ f \circ \Phi^{-1}(u,v) = 1\Big/\left(1 - \frac{1-u^2-v^2}{u^2+v^2+1}\right)\left(\frac{-2u}{u^2+v^2+1}, \frac{-2v}{u^2+v^2+1}\right)$

$= \frac{1}{2(u^2+v^2)}(-2u,-2v) = -\frac{(u,v)}{u^2+v^2} = \frac{-z}{|z|^2} = \underline{\underline{\frac{-1}{\bar{z}}}}$

2.2

Let $f(z) = \frac{1}{z}$, $f: \overline{\mathbb{C}} \to \overline{\mathbb{C}}$.

Then $f(C)$ is a $\mathbb{C}$-circle. $f(C) = l \cup \{\infty\}$, l a line,

when $\infty \in f(C)$. Since $f(0) = \infty$

$f(C) = l \cup \{\infty\}$ if and only if $0 \in C$, if and only

if $|z_0| = r$.

2.5

Assume $f(\mathbb{H}) = \mathbb{D}$.

Then $f(\overline{\mathbb{R}}) = \{z \mid |z| = 1\}$

So ~~we assume~~ let us first find f such that

$f(0) = 1$, $f(1) = i$, $f(\infty) = -1$

Let g be such that $g(0) = 1$, $g(1) = 0$, $g(\infty) = \infty$.

then $g(z) = -z+1$

Let h be such that $h(1) = 1$, $h(i) = 0$, $h(-1) = \infty$

then $h(z) = \dfrac{z-i}{z+1}\,\dfrac{2}{1-i} = \dfrac{2z-2i}{(1-i)z+(1-i)}$

and $h^{-1}(z) = \dfrac{(1-i)z+2i}{(i-1)z+2}$

Now $h^{-1} \circ g = f$, consider $\begin{bmatrix} 1-i & 2i \\ i-1 & 2 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} i-1 & 1+i \\ 1-i & 1+i \end{bmatrix}$ $\quad f(z) = \dfrac{(i-1)z + (1+i)}{(1-i)z + (1+i)} = \dfrac{-z+i}{z+i}$

Note with this f, we get that

$f(i) = 0$, so $f(\mathbb{H}) = \{z \mid |z| < 1\} = \mathbb{D}$.

(Suppose that we (by an other choice of points in $\overline{\mathbb{R}}$ and S^1) had ~~could~~ obtained that $|f(i)| > 1$, then $\cancel{\frac{1}{f(z)}}$ we could have defined $f(z) := \frac{1}{f(z)}$ and obtained an ELT mapping ~~$\mathbb{D}$ to~~ $\mathbb{H}$ to $\mathbb{D}$)

$$\text{Now} \quad \begin{bmatrix} -1 & i \\ 1 & i \end{bmatrix}^{-1} = \frac{1}{2} \begin{bmatrix} -1 & 1 \\ -i & -i \end{bmatrix}$$

so $f^{-1}(z) = \dfrac{-z+1}{-iz-i}$ and f^{-1} maps $\mathbb{D}$ to $\mathbb{H}$