2.3

Let $f(z)=\dfrac{az+b}{cz+d}$, $ad-bc\neq 0$

Assume first that $c\neq 0$. Then

$$f(z)=\frac{a}{c}+\frac{bc-ad}{c^2z+cd}=\left((bc-ad)\left(\frac{1}{c^2(z+\frac{d}{c})}\right)\right)+\frac{a}{c}$$

which is a composition of ~~(i), (i~~ maps of type (i), (ii) and (iii).

If $c=0$, then $d\neq 0$ and we can write

$f(z)=\left(\left(\frac{a}{d}\right)z\right)+\left(\frac{b}{d}\right)$ which is a composition of maps of type (i) and (ii)

To prove Lemma 2.1, it is enough to see that maps of type (i), (ii) and (iii) map $\overline{\mathbb{C}}$ circles to $\overline{\mathbb{C}}$ circles.

This is clear for maps of type (i).

Let $f(z)=kz$ be of type (ii).

Consider a circle in $\mathcal{C}$; $\mathbb{C}$ defined by $|z-z_0|=r$. ~~Let then~~ Let $w=f(z)=kz$

then $|w-kz_0|=|k||z-z_0|=|k|r$

so $f(\mathcal{C})$ is a circle with center kz_0 and radius $|k|r$.

Let $\mathcal{C}$ be a $\overline{\mathbb{C}}$ circle of type $l\cup\{\infty\}$ where l is a line in $\mathbb{C}$.

Writing $z = x+iy$, l must satisfy an equation of type $ax+by=c$. Let $w = f(z) = kx+iky$

then ~~k if $z \in l$, we must have~~

$$(ka)x + (kb)y = kc \Leftrightarrow ax+by=c$$

so Since $(ka)x+(kb)y = kc$
is an equation for another line l'
we see that $f(l) = l'$. We also have $f(\infty)=\infty$
so $f(l \cup \infty) = l' \cup \infty$.

Finally let $f(z) = \frac{1}{z}$

Note that there is a 1-1 correspondance between $\bar{\mathbb{C}}$-circles and circles on S^2.
(given by $\hat{\Phi}: S^2 \to \bar{\mathbb{C}}$, $\hat{\Phi}(p) = \Phi(p) \in \mathbb{R}^2 \cong \mathbb{C}$
if $p \in S^2 - \{(0,0,1)\}$ and $\hat{\Phi}((0,0,1)) = \infty$)

Now by exercise 1.4 a) ~~$\hat{\Phi} \circ f \circ \hat{\Phi}^{-1}$~~

$\hat{\Phi}^{-1} \circ f \circ \hat{\Phi}(x,y,z) = (x,-y,-z)^{=\rho(x,y,z)}$ (a rotation in the y-x plane an angle π). So $\hat{\Phi}^{-1} \circ f \circ \hat{\Phi}$ will rotate a circle in S^2 to another circle and therefore $f = \hat{\Phi} \circ \rho \circ \hat{\Phi}^{-1}$ will map a $\bar{\mathbb{C}}$-circle to another $\bar{\mathbb{C}}$-circle.

2.6 Let $f \in \text{Möb}(\mathbb{H})$.

First let $f(z) = \dfrac{az+b}{cz+d}$, $ad-bc=1$, $a,b,c,d \in \mathbb{R}$

If f maps $\{iy \mid y \in \mathbb{R}\}$ to itself and
f must map the $\mathbb{H}$-line $\{iy \mid y>0\}$ to itself
hence either $f(0)=0, f(\infty)=\infty$ or $f(0)=\infty, f(\infty)=0$.

If $f(0)=\frac{b}{d}=0 \Rightarrow b=0$ if $f(\infty)=\infty$ then $c=0$

so $f(z)=\frac{a}{d}z$ with $ad=1$ so $d=\frac{1}{a}$

and $\underline{f(z)= a^2 z}$.

If $f(0)=\infty$ then $d=0$, if $f(\infty)=0$ then $a=0$

so $f(z)=\dfrac{b}{cz}$ with $bc=-1$, $c=-\frac{1}{b}$

so $\underline{f(z)=-\dfrac{b^2}{z}}$.

Now let $f \in \text{Mob}^-(\mathbb{H})$ $f(z)=\dfrac{a\bar{z}+b}{c\bar{z}+d}$ $a,b,c,d \in \mathbb{R}$

$ad-bc=-1$. Again we must either have
$f(0)=0, f(\infty)=\infty$ or $f(0)=\infty, f(\infty)=0$

If $f(0)=0$ then $b=0$ and if $f(\infty)=\infty$ $c=0$

So $f(z)=\frac{a}{d}\bar{z}$ with $ad=-1$, $d=-\frac{1}{a}$, $\underline{f(z)=-a^2\bar{z}}$

If $f(0)=\infty$, and $f(\infty)=0$ we again have $d=a=0$

So $f(z)=\dfrac{b}{c\bar{z}}$ with $bc=1$, $c=\frac{1}{b}$ $\underline{f(z)=\dfrac{b^2}{\bar{z}}}$

2.7

a) Let (z_1, z_2, z_3) and (w_1, w_2, w_3) be two tripples of distinct points in $\overline{\mathbb{R}}$

By Corollary 2.5 (last sentence)

$\exists$ an F.L.T. $f(z) = \frac{az+b}{cz+d}$, with $a, b, c, d \in \mathbb{R}$

such that $f(z_i) = w_i$, $i = 1, 2, 3$.

If $ad - bc > 0$ we know that $f \in Möb^+(\mathbb{H}) \subset Möb(\mathbb{H})$

If $ad - bc < 0$, let $g(z) = \frac{a\bar{z}+b}{c\bar{z}+d}$. then $g(z_i) = w_i$

(since $z_i \in \overline{\mathbb{R}}$) and $g \in Möb^-(\mathbb{H}) \subset Möb(\mathbb{H})$

Now let (l, l') and (Γ, Γ') be two pairs of distinct $\mathbb{H}$-lines such that l, l' have a common end point and Γ, Γ' also have a common end point. Now let p be the common end point of l, l' and let p_1, p_2 be the other end point correspondingly of l and l' respectively. Let q, q_1, q_2 be endpoints of Γ, Γ' with q the common end point of Γ and Γ'.

Let $f \in Möb(\mathbb{H})$ be such that $f(p) = q$, $f(p_i) = q_i$ $i = 1, 2$.

then, since f maps $\mathbb{H}$-lines to $\mathbb{H}$-lines, and an $\mathbb{H}$-line is uniquely determined by its endpoints $f(l) = \Gamma$ and $f(l') = \Gamma'$, so $Möb(\mathbb{H})$ acts transitively on such pairs of $\mathbb{H}$-lines.

b) It is enough to show that given $(p,q) \in \overline{\mathbb{R}}$, $p \neq q$ $\exists$ $f \in \text{Möb}^+(\mathbb{H})$ such that $f(p)=1$, $f(q)=-1$.

Choose $r \in \overline{\mathbb{R}}$, $r \neq p$, $r \neq q$ and by Corollary 2.5 find $f \in \text{Möb}^+(\mathbb{C})$

$f(z) = \frac{az+b}{cz+d}$, $a,b,c,d \in \mathbb{R}$ such that

$f(p)=1$, $f(q)=-1$, $f(r)=0$

if $ad-bc>0$ $f \in \text{Möb}^+(\mathbb{H})$ so we are done.

If $ad-bc<0$ put $g(z) = \frac{1}{f(z)} = \frac{cz+d}{az+b}$

then $g(p) = \frac{1}{f(p)} = \frac{1}{1} = 1$, $g(q) = \frac{1}{f(q)} = \frac{1}{-1} = -1$

and $cb-ad>0$ so $g \in \text{Möb}^+(\mathbb{H})$

c) Consider the pairs $(i,2i)$, $(i,4i)$

Assume $\exists$ $h \in \text{Möb}(\mathbb{H})$ such that $h(i)=i$, $h(2i)=4i$.

Now the only $\mathbb{H}$-line containing the pairs $(i,2i)$ is $\{yi \mid y>0\} = \ell$ and ℓ is also the only $\mathbb{H}$-line containing $(i,4i)$. Since h maps $\mathbb{H}$-lines to $\mathbb{H}$-lines. We must then have $h(\ell)=\ell$. So h is of of the maps we classified in ex. 2.6.

So either $f(z)=a^2z$, $f(z)=-\frac{b^2}{z}$, $f(z)=-a^2\bar{z}$ or $f(z)=\frac{b^2}{\bar{z}}$

$a,b>0$. Now $f(i)=i$ gives in all cases $a=1$ or $b=1$ and then $f(2i)=4i$ is impossible.