3.1

$$f(z)=\frac{4z-3}{2z-1}, \quad (ad-bc=-4+6=2)$$

~~$a=4, d=-1$~~

$$=\frac{\frac{4}{\sqrt2}z-\frac{3}{\sqrt2}}{\frac{2}{\sqrt2}z-\frac{1}{\sqrt2}} \quad ; \quad a=\frac{4}{\sqrt2},\ b=-\frac{3}{\sqrt2}$$
$$c=\frac{2}{\sqrt2},\ d=-\frac{1}{\sqrt2}$$

$$(a+d)^2=\left(\frac{4}{\sqrt2}-\frac{1}{\sqrt2}\right)^2=\frac{9}{2}>4$$

Two real fixpoints.

$$z=\frac{a-d\pm\sqrt{(a+d)^2-4}}{2c}=\frac{\frac{5}{\sqrt2}\pm\sqrt{\frac12}}{\frac{4}{\sqrt2}}=\frac{5\pm1}{4}=\begin{cases}1\\ \frac32\end{cases}$$

(Check $f(1)=\frac{4-3}{2-1}=1$, $f\left(\frac32\right)=\frac{6-3}{3-1}=\frac32$)

Let $h\in \text{Möb}^+(\mathbb{H})$ mapping 1 to 0

and $\frac32$ to ∞.

Let $h(z)=-\frac{z-1}{z-\frac32}=\frac{1-z}{-\frac32+z}=\frac{(-2z)+2}{(0\,2z)-3}\in \text{Möb}^+(\mathbb{H})$

$$h^{-1}(z)=\frac{-3z-2}{-2z-2}$$

So since $\begin{bmatrix}-2 & 2\\ 2 & -3\end{bmatrix}\begin{bmatrix}4 & -3\\ 2 & -1\end{bmatrix}\begin{bmatrix}-3 & -2\\ -2 & -2\end{bmatrix}=\begin{bmatrix}4 & 0\\ 0 & 2\end{bmatrix}$

~~$=\begin{bmatrix}-2 & 2\\ 2 & -3\end{bmatrix}\begin{bmatrix}0 & 2\\ -4 & 0\end{bmatrix}=\begin{bmatrix}-4 & -4\\ 12 & 4\end{bmatrix}$~~

$g(z)=h\circ f\circ h^{-1}(z)=\frac{4z}{2}=2z=(\sqrt2)^2z$, and $f=h^{-1}\circ g\circ h$

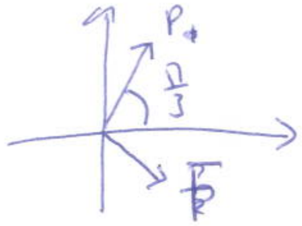
$f(z) = \frac{-1}{z-1}$, $\left((0)(-1)-(1)(-1) = 1\right)$, so $a=0$, $b=-1$, $c=1$, $d=-1$

$$(a+d)^2 = (0+(-1))^2 = 1 < 4$$

two complex fixed points

$$f(z) = z \Leftrightarrow z = \frac{(a-d) \pm \sqrt{(a+d)^2-4}}{2c} = \frac{1 \pm \sqrt{1-4}}{2}$$

$$= \frac{1 \pm \sqrt{3}i}{2}, \quad p = \frac{1}{2} + \frac{\sqrt{3}}{2} i$$



Let $h(z) = \frac{z - Re(p)}{Im\, p} =$

$$= \frac{z - \frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{2z-1}{\sqrt{3}} \quad \left(\text{note that } h\left(\tfrac{1}{2} + \tfrac{\sqrt{3}}{2} i\right) = i, \ h\left(\tfrac{1}{2} - \tfrac{\sqrt{3}}{2} i\right) = -i\right)$$

Now $\begin{bmatrix} 2 & -1 \\ 0 & \sqrt{3} \end{bmatrix}$, $\begin{bmatrix} 2 & -1 \\ 0 & \sqrt{3} \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2\sqrt{3}} \\ 0 & \frac{1}{\sqrt{3}} \end{bmatrix}$

$$\begin{bmatrix} 2 & -1 \\ 0 & \sqrt{3} \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2\sqrt{3}} \\ 0 & \frac{1}{\sqrt{3}} \end{bmatrix} =$$

$$= \begin{bmatrix} 2 & -1 \\ 0 & \sqrt{3} \end{bmatrix} \begin{bmatrix} 0 & -\frac{1}{\sqrt{3}} \\ \frac{1}{2} & -\frac{1}{2}\frac{1}{\sqrt{3}} \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2}\sqrt{3} \\ \frac{1}{2}\sqrt{3} & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

with $\theta = \frac{4\pi}{3}$, so $g(z) = \dfrac{-\frac{1}{2}z - \frac{1}{2}\sqrt{3}}{\frac{1}{2}\sqrt{3}z - \frac{1}{2}} = \dfrac{-z-\sqrt{3}}{\sqrt{3}z - 1}$

and $f(z) = h^{-1} \circ g \circ h(z)$ with $h(z) = \dfrac{2z-1}{\sqrt{3}}$

$\bar{p}$-

$f(z) = \frac{z}{z+1}$, $a=1, b=0, c=1, d=1$
$ad-bc=1, (a+d)^2=4$

so one-real fixpoint $z = \frac{a-d}{2c} = 0$

Let $q=0$, $p=1$. $f(p)=\frac{1}{2}$ and consider

$$h(z) = [z, f(p), p, q] = \left(\frac{z-1}{z}\right)(-1) = \frac{1-z}{z} = -\frac{z+1}{z}$$

~~here~~ $h \in \text{Möb}^+(\mathbb{C}) - \text{Möb}^+(\mathbb{H})$ (since $(-1)\cdot 0 - 1\cdot 1 = -1 < 0$)

so let us redefine h putting $h := -h = \frac{z-1}{z}$

Now consider $\begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}$

and the product $\begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

So $g = h \circ f \circ h^{-1}(z) = z - 1$

and $f(z) = h^{-1} \circ g \circ h(z)$

3.5

Put $f(z) = z+1$, $g(z) = z-1$, and $h(z) = -\bar{z}$

then $h^{-1}(z) = h(z)$ and

$$h \circ f \circ h^{-1}(z) = h \circ f \circ h(z) = h(f(-\bar{z})) = h(-\bar{z}+1)$$

$$= -\overline{(-\bar{z}+1)} = z-1 = g(z).$$

So f is conjugate to g in $\text{Möb}(\mathbb{H})$.

Assume f and g are conjugate in $\text{Möb}^+(\mathbb{H})$.

Then there exists $h(z) = \dfrac{az+b}{cz+d}$ with $a,b,c,d \in \mathbb{R}$ and $ad-bc=1$ such that $h \circ f \circ h^{-1} = g$ $*$

Now $h^{-1}(z) = \dfrac{dz-b}{-cz+a}$ and $*$ implies that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

So we must have $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} d-c & -b+a \\ -c & a \end{bmatrix}$

$$= \begin{bmatrix} ad-ac-bc & a^2 \\ c^2 & -bc+ac+ad \end{bmatrix} = \begin{bmatrix} 1-bc & a^2 \\ c^2 & 1+ac \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

From this follows that $c=0$ and $a^2=-1$ which is impossible.

3.6

Let $z \in \mathbb{H}$, $g_\theta(z) = \dfrac{\cos\theta\, z + \sin\theta}{-\sin\theta\, z + \cos\theta}$

~~Assume~~

If $z = i$ then $g_\theta(i) = i$ for each θ,
so $\theta \to g_\theta(z)$ is contained in any circle
passing through i.
So let $z \neq i$. ~~Note that~~ Put $\tau_z = \{g_\theta(z) \mid \theta \in \mathbb{R}\}$.

Note that ~~$g_{\theta+\Phi}(z$~~ $g_{\theta+\Phi}(z) = g_\theta(g_\Phi(z))$

and $g_\pi(z) = g_0(z) = z$. This gives

that if $z' = g_\phi(z) \in \tau_z$ then $\tau_{z'} = \tau_z$

and also $i \notin \tau_z$ if $z \neq i$. ~~(since~~

Let us write $g_\theta(z) = x(\theta) + y(\theta) i$

note that $g_{\frac{\pi}{2}}(z) = \dfrac{-1}{z}$. So

$x(0) = \mathrm{Re}(z)$ and $x(\frac{\pi}{2}) = \mathrm{Re}(-\frac{1}{z}) = \mathrm{Re}\left(\frac{-\bar{z}}{|z|^2}\right)$

$= -\dfrac{\mathrm{Re}(z)}{|z|^2} = \dfrac{-x(0)}{|z|^2}$. So either $z = ai$ with $a \neq 1$

or $x(0)$, $x(\frac{\pi}{2})$ are non-zero with different
signs.

So since $X(\theta)$ obviously is continuous, there exists θ_0 such that $X(\theta_0) = 0$

So $g_{\theta_0}(z) = ai \in \mathcal{T}_z, a \neq 1$.

Since $g_{\theta_0 + \frac{\pi}{2}}(z) = \dfrac{-\sin\theta_0 z + \cos\theta_0}{-\cos\theta_0 z - \sin\theta_0} = -\dfrac{1}{ai} = \dfrac{1}{a} i$

either $a > 1$ or $\frac{1}{a} > 1$.

Since $\mathcal{T}_z = \mathcal{T}_{z'}$ for any $z' \in \mathcal{T}_z$

we may assume that $z = ai$ with $a > 1$.

Let $z_0 = \frac{1}{2}(ai + \frac{1}{a} i) = \dfrac{a^2+1}{2a} i$

and $r = \left|\frac{1}{2}(ai - \frac{1}{a} i)\right| = \dfrac{a^2-1}{2a}$

Consider $\left| \dfrac{\cos\theta z + \sin\theta}{-\sin\theta z + \cos\theta} - z_0 \right|$ $\quad \cos\theta \neq 0$

$$= \left| \frac{z + \tan\theta}{-\tan\theta z + 1} - z_0 \right| = \left| \frac{z + t}{-tz + 1} - z_0 \right| , t = \tan\theta$$

$$= \left| \frac{(ai+t)}{(-tai+1)} - \frac{a^2+1}{2a} i \right| = \left| \frac{(ai+t) \cdot 2a - (a^2+1)i(-tai+1)}{2a(-tai+1)} \right|$$

$$= \left| \frac{(at - a^3 t) + (a^2-1)i}{2a(-tai+1)} \right| = \left| \frac{(a^2-1)(at+i)}{2a(-tai+1)} \right| = \frac{a^2-1}{2a}$$

This proves that C_z is contained, in the circle $|z-z_0|=r$, actually, from the algebraic properties of the family $\{g_\theta \mid \theta\in\mathbb{R}\}$ it is easy to see that C_z is the whole circle.

A more simple arguments, which however gives less ~~in~~ specific information about the circle C_z is the following:

Consider $g_\theta(z) = \dfrac{\cos\theta\, z + \sin\theta}{-\sin\theta\, z + \cos\theta}$, $\theta\in[0,\pi)$

If $\cos\theta\neq 0$ ($\theta\neq\frac{\pi}{2}$), we may write (by dividing with $\cos\theta$)

$$g_\theta(z) = \frac{z+\tan\theta}{-\tan\theta\, z+1} = \frac{z+t}{-tz+1} \quad \text{with } t = \tan\theta,$$

so $t\in\mathbb{R}$. (Since $\tan\theta$ takes all real values when $\theta\in[0,\pi)$, $\theta\neq\frac{\pi}{2}$)

Let $f(w)=\dfrac{z+w}{-wz+1}$ then f is an F.L.T.

So $\{g_\theta(z) \mid \theta\in[0,\pi),\ \theta\neq\frac{\pi}{2}\} = \{f(w)\mid w\in\mathbb{R}\}$

Also note that $f(\infty) = \lim_{|w|\to\infty} \dfrac{\frac{z}{w}+1}{-z+\frac{1}{w}} = -\dfrac{1}{z} = g_{\frac{\pi}{2}}(z)$

So $\{g_\theta(z)\mid\theta\in[0,\pi)\} = f(\overline{\mathbb{R}})$. Since $\overline{\mathbb{R}}$ is a $\overline{\mathbb{C}}$ circle, $f(\overline{\mathbb{R}})$ is a $\overline{\mathbb{C}}$ circle, and since $g_\theta(z)\in\mathbb{H}$, $\infty\notin f(\overline{\mathbb{R}})$ hence $f(\overline{\mathbb{R}})$ is a circle in $\mathbb{C}$.