## 2.9

Let $g \in \text{Möb}^-(\mathbb{H})$. Then we may write

$g(z) = \overline{h(z)}$ where $h$ is an F.L.T

$h(z) = \dfrac{az+b}{cz+d}$ with $ad-bc<0$.

Note that from the proof of 2.3 it follows that

$$\overline{[z, z_1, z_2, z_3]} = [\bar{z}, \bar{z}_1, \bar{z}_2, \bar{z}_3]$$

Now from 2.10 (iii) we thus get that

$$[g(z), g(z_1), g(z_2), g(z_3)] = [\overline{h(z)}, \overline{h(z_1)}, \overline{h(z_2)}, \overline{h(z_3)}]$$

$$= \overline{[h(z), h(z_1), h(z_2), h(z_3)]} = \overline{[z, z_1, z_2, z_3]}$$

5.1

Assume $f: (1,\infty) \to \mathbb{R}$, $f$ not the zero-function. ~~(0,\infty)~~
is differentiable at a point $a \in (1,\infty)$
and that $f(st) = f(s) + f(t) \ \forall \ s,t \in (1,\infty)$
we will first prove that $f$ is differentiable everywhere.

~~Note that $f(1) = f(1\cdot 1) = f(1)+f(1)$~~

~~hence $f(1)=0$.~~

Assume first that $b \in (1,\infty)$ and $b > a$

then we may write $f(t) = \cancel{f(\frac{bt}{a}\frac{a}{b}}$

$= f(\frac{ta}{b}\frac{b}{a}) = f(\frac{b}{a}) + f(\frac{a}{b}t)$

(note that when $t$ is close to $b$ $\frac{a}{b}t$ is close to $a$ hence $\frac{a}{b}t > 1$ (since $a > 1$) and $f(\frac{a}{b}t)$ is defined). Now put $h(t) = f(\frac{a}{b}t)$

$h$ is a composition of $t \to \frac{a}{b}t = u$ which is differentiable everywhere, and $f(u)$ which is differentiable at $a$, so by the chainrule we get that $h$ is differentiable at $b$ with $h'(b) = f'(a)\frac{a}{b}$, so $f'(b) = f'(a)\frac{a}{b}$

Next assume that $1 < b < a$.

Then we may write $f(t) = f(\frac{tb}{a}) + f(\frac{a}{b})$

Now $f(\frac{tb}{a}) = f(t) - f(\frac{a}{b})$ is differentiable at $a$

Put $g(t) = f\left(\frac{tb}{a}\right)$

then $g'(a) = \frac{d}{dt}\left(f(t) - f\left(\frac{a}{b}\right)\right) = f'(a)$

Let $h(u) = \frac{a}{b}u$, then $h(b) = a$ and

$f(u) = g(h(u))$ and by the chain rule

$f$ is differentiable at at $b$ and

$f'(b) = g'(a)h'(b) = \frac{a}{b} f'(a)$

So $f$ is differentiable at any $t \in (0,1)$

and $f'(t) = af'(a)\frac{1}{t} = \frac{c}{t}$ with $c = af'(a)$

It also follows that $f$ is continuous.

Let $x_n$ be a sequence such that $x_n \longrightarrow 1$

Since $|f'(t)| = \left|\frac{c}{t}\right| \leq C$ it follows that

$$|f(x_n) - f(x_m)| \leq C|x_n - x_m|$$

and we easily deduce that $\{f(x_n)\}$ is a Cauchy sequence so $f(x_n)$ is convergent.
On the other hand this implies that each such sequence $\{f(x_n)\}$ must tend to the same limit $a$, but then $f(x_n^2) = f(x_n) + f(x_n) \to 2a = a$ hence $a = 0$ (since $x_n^2 \to 1$), so we may extend $f$ to a continuous map, putting $f(1) = 0$

So $f(t) = \int_1^t f'(s)ds = \int_1^t \frac{c}{s}ds = C\ln t$.

7.8 $\cosh x = \frac{1}{2}(e^x + e^{-x})$, $\sinh x = \frac{1}{2}(e^x - e^{-x})$

(a) $\cosh^2 x - \sinh^2 x = \frac{1}{4}((e^x + e^{-x})^2 - (e^x - e^{-x})^2)$

$= \frac{1}{4}((\cancel{e^{2x}} + 2 + \cancel{e^{-2x}}) - (\cancel{e^{2x}} - 2 + \cancel{e^{-2x}}))$

$= \frac{1}{4} \cdot 4 = \underline{1}$

(b) $\sinh^2 x = \dfrac{\sinh^2 x}{\cosh^2 x - \sinh^2 x} = \dfrac{\frac{\sinh^2 x}{\cosh^2 x}}{\frac{\cosh^2 x}{\cosh^2 x} - \frac{\sinh^2 x}{\cosh^2 x}} = \dfrac{\tanh^2 x}{1 - \tanh^2 x}$

(c) $\sinh 2x = \frac{1}{2}(e^{2x} - e^{-2x}) = \frac{1}{2}((e^x)^2 - (e^{-x})^2)$

$= \frac{1}{2}((e^x + e^{-x})(e^x - e^{-x})) = \frac{1}{2}(2\cosh x \cdot 2\sinh x)$

$= 2 \sinh x \cosh x$

(d) ~~$\cosh 2x = \frac{1}{2}$~~

$\cosh^2 x + \sinh^2 x = \frac{1}{4}(e^x + e^{-x})^2 + \frac{1}{4}(e^x - e^{-x})^2$

$= \frac{1}{4}(e^{2x} + 2 + e^{-2x} + e^{2x} - 2 + e^{-2x}) = \frac{1}{2}(e^{2x} + e^{-2x})$

$= \cosh 2x$

(e) $\cosh 2x = \cosh^2 x + \sinh^2 x = \dfrac{\cosh^2 x + \sinh^2 x}{\cosh^2 x - \sinh^2 x} \dfrac{\frac{1}{\cosh^2 x}}{\frac{1}{\cosh^2 x}}$

$= \dfrac{1 + \frac{\sinh^2 x}{\cosh^2 x}}{1 - \frac{\sinh^2 x}{\cosh^2 x}} = \dfrac{1 + \tanh^2 x}{1 - \tanh^2 x}$

(f) $\sinh 2x = \dfrac{2\sinh x \cosh x}{\cosh^2 x - \sinh^2 x} = \dfrac{2\frac{\sinh x}{\cosh x}}{1 - \frac{\sinh^2 x}{\cosh^2 x}} = \dfrac{2\tanh x}{1 - \tanh^2 x}$