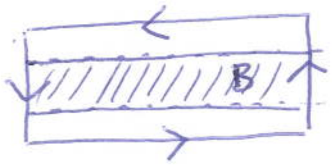
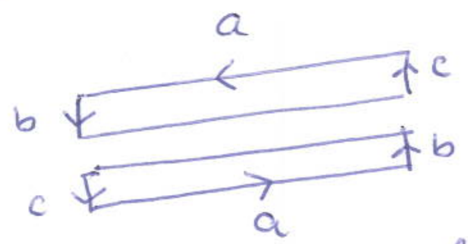
Problem

1) Consider $\mathbb{P}^2$: 

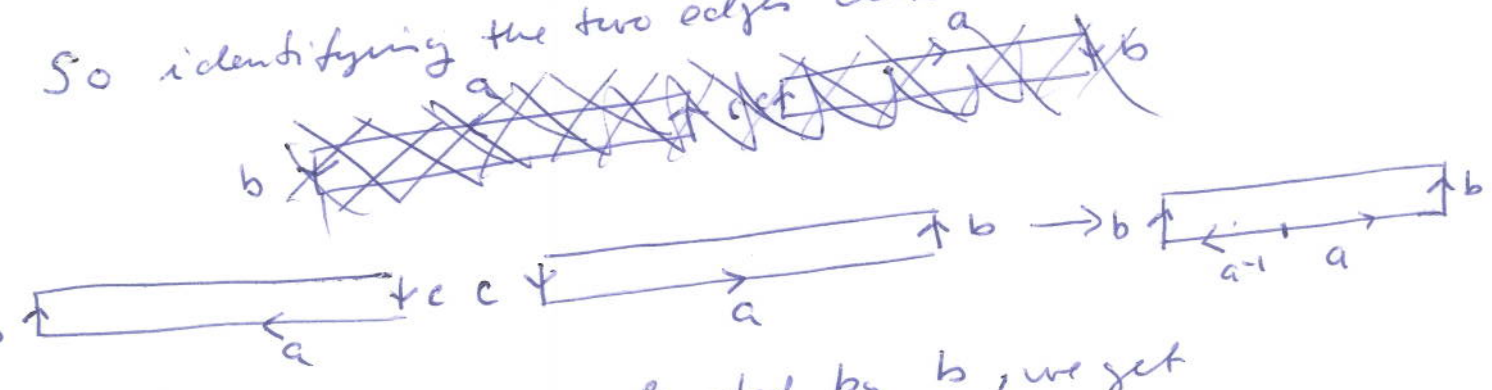
The shaded area  form B

a closed Möbius band, with a circle $S = S^1$

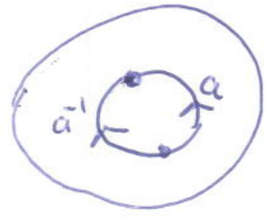
Removing the corresponding open Möbius band  from $\mathbb{P}^2$, we are left with an open subset of $\mathbb{P}^2$ which is the quotient space of the space given below (where the identifications are given).



So identifying the two edges denoted by c



and the two edges denoted by b, we get



Which is ~~topologically~~ homeomorphic to a closed half-sphere  and ~~thus~~ thus to $\mathbb{D}^2$.

and identifying the boundary of $\mathbb{D}^2$, S^1, with the boundary S^1 of the removed Möbius band B, we get back $\mathbb{P}^2$.

So $\mathbb{P}^2 = B \cup_{S^1} \mathbb{D}^2$. To see that $K^2 \approx B \cup_C B$ (see page 10)

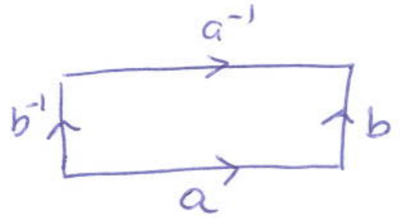
~~[illegible]~~

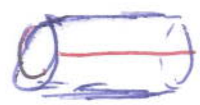
2) We have $\mathbb{T}^2 = D^2/aba^{-1}b^{-1}$ and $\mathbb{P}^2 = \mathbb{D}^2/cc$ and repeated application of Lemma 2 give us that

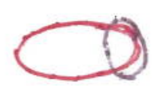
$$D^2/[a_1,b_1]\cdots[a_m,b_m]c_1^2\cdots c_n^2$$

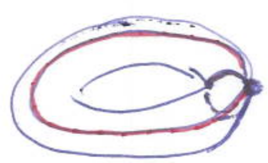
$$= \mathbb{D}^2/a_1b_1a_1^{-1}b_1^{-1} \# \cdots \# \mathbb{D}^2/a_mb_ma_m^{-1}b_m^{-1} \# \mathbb{D}^2/c_1c_1 \# \cdots \# \mathbb{D}^2/c_nc_n$$

$$= \underbrace{\mathbb{T}^2 \# \cdots \# \mathbb{T}^2}_{m} \# \underbrace{\mathbb{P}^2 \# \cdots \# \mathbb{P}^2}_{n} = S(m,n)$$

3) $\mathbb{T}^2 = \mathbb{D}^2/aba^{-1}b^{-1}$ = 

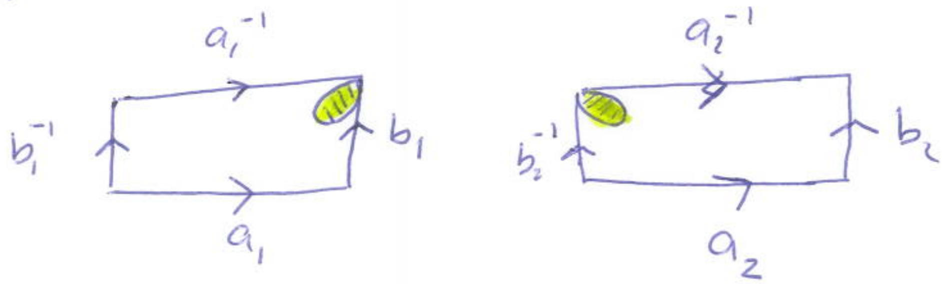
~ 

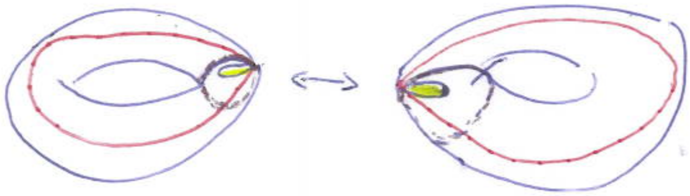
and the curves that we get identifying the edges denoted by a, a^{-1} and b, b^{-1} becomes 

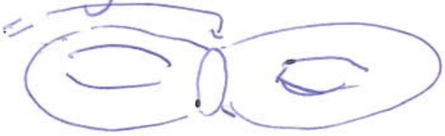
~ 

Now $S(2,0) = \mathbb{T}^2 \# \mathbb{T}^2$

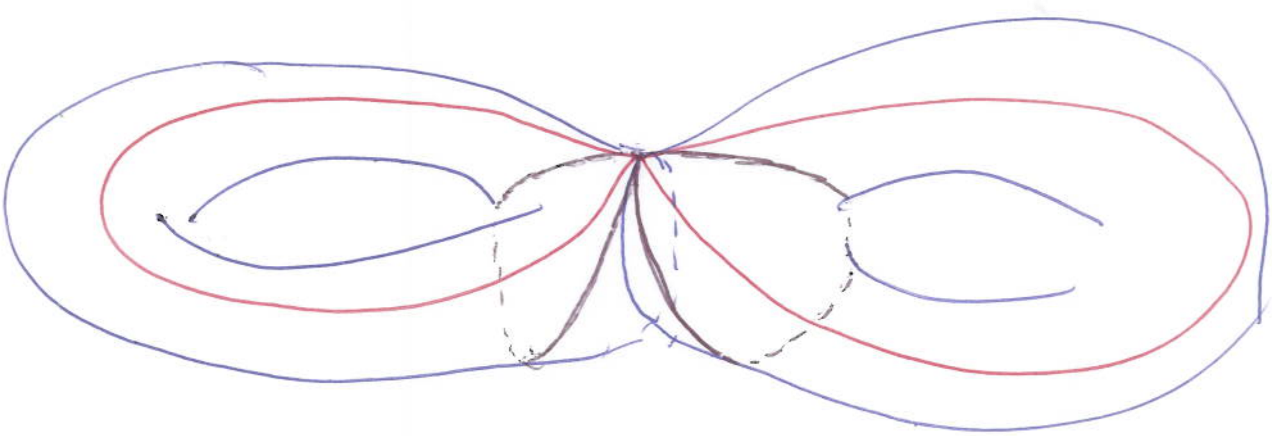
and we may form $\mathbb{T}^2 \# \mathbb{T}^2$ by cutting out and open disk from each $\mathbb{T}^2$ and gluing along the boundary





Doing this and deforming such that the boundary circle becomes 

we get the picture



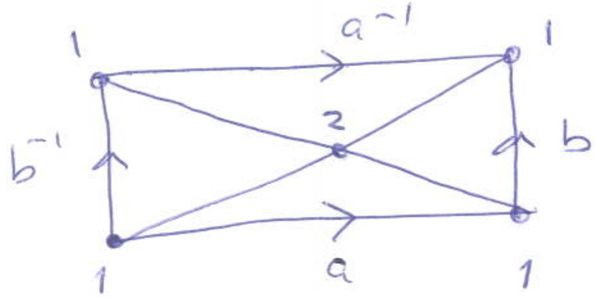
Problem 4

$$S(m,n) = \underbrace{\mathbb{T}^2 \# \cdots \# \mathbb{T}^2}_{m} \# \underbrace{\mathbb{P}^2 \# \cdots \# \mathbb{P}^2}_{n}$$

$$\cancel{\simeq \mathbb{T}^2 \# \mathbb{P}^2 \# S(m-1, n-1)}$$

Let us calculate $\chi(\mathbb{T}^2)$ and $\chi(\mathbb{P}^2)$

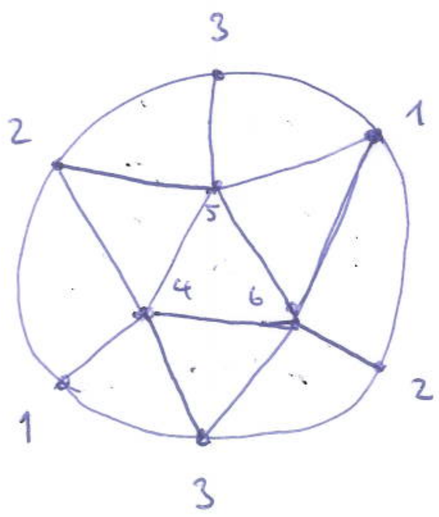
A triangulation of $\mathbb{T}^2$ is given by



Here $s = 4$, $e = 6$, $V = 2$

so $\chi(\mathbb{T}^2) = 4 - 6 + 2 = 0$

A triangulation of $\mathbb{P}^2$ is given by



Here $s = 10$

$e = 15$

$V = 6$

so $\chi(\mathbb{P}^2) = 10 - 15 + 6 = 1$

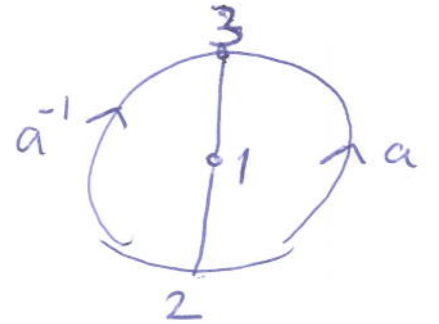
Now using problem 7, we get that (if $m+n>1$)

$$\chi(S(m,n)) = \sum_{i=1}^{m} \chi(\mathbb{T}^2) + \sum_{i=1}^{n} \chi(\mathbb{P}^2) - 2(m+n-1)$$

$$= n - 2(m+n-1) = \underline{2-2m-n}$$

So $\chi(\mathbb{K}^2) = \chi(\mathbb{P}^2 \# \mathbb{P}^2) = \chi(0,2) = 2-2 = \underline{0}$

A triangulation of $\mathbb{S}^2$ is given by



Here $s=2$, $e=3$, $v=3$

so $\chi(\mathbb{S}^2) = 2-3+3 = \underline{2}$

Note that if M is not homeomorphic to $\mathbb{S}^2$, then M is homeomorphic to

$S(m,n)$ where $m+n \geq 1$

~~No~~ Then $\chi(S(m,n)) = \chi(M) = 2-2m-n$

But ~~it~~ the only pairs (m,n) such that $2-2m-n \geq 0$ is $(1,0), (0,1), (0,2)$

So if M is not homeomorphic to $\mathbb{S}^2$

M must be homeomorphic to either

$S(1,0) = \mathbb{T}^2$, $S(0,1) = \mathbb{P}^2$ or $S(0,2) = \mathbb{P}^2 \# \mathbb{P}^2 = \mathbb{K}^2$

Problem 5

$$M = \mathbb{D}^2/(abc^{-1}bdacd^{-1}) = \mathbb{D}^2/cd^{-1}abc^{-1}bda$$

$$= \mathbb{D}^2/w_1 a w_2 a \quad (\text{with } w_1 = cd^{-1},\ w_2 = bc^{-1}bd)$$

$$\simeq \mathbb{D}^2/w_1 w_2^{-1} aa \simeq \mathbb{D}^2/w_1 w_2^{-1} \# \mathbb{P}^2 = \cancel{\mathbb{D}^2/cd^{-1}bc^{-1}bd \# \mathbb{P}^2}$$

$$\cancel{= \mathbb{D}^2/acd^{-1}bc^{-1}b \# \mathbb{P}}$$

$$= \left(\mathbb{D}^2/cd^{-1}d^{-1}b^{-1}cb^{-1}\right) \# \mathbb{P}^2 = \cancel{\mathbb{D}^2/cb^{-1}cb^{-1} \#}$$

$$= \left(\mathbb{D}^2/b^{-1}cb^{-1}cd^{-1}d^{-1}\right) \# \mathbb{P}^2 = \left(\mathbb{D}^2/b^{-1}cb^{-1}c\right) \# \mathbb{P}^2 \# \mathbb{P}^2$$

$$= \mathbb{D}^2/b^{-1}bcc \# \mathbb{P}^2 \# \mathbb{P}^2 = \left(\mathbb{D}^2/b^{-1}b\right) \# \mathbb{P}^2 \# \mathbb{P}^2 \# \mathbb{P}^2$$

$$= S^2 \# \mathbb{P}^2 \# \mathbb{P}^2 \# \mathbb{P}^2 \cong \mathbb{P}^2 \# \mathbb{P}^2 \# \mathbb{P}^2 \simeq \underline{\underline{\mathbb{T}^2 \# \mathbb{P}^2}}$$

Since we have the factor $\mathbb{P}^2$, M is not orientable and $\chi(M) = 2-2-1 = \underline{-1}$

Problem 6

Assume $M_1 \simeq M_1 \# M_2$

then $\chi(M_1) = \chi(M_1) + \chi(M_2) - 2$

$\Rightarrow \chi(M_2) = 2$ and by problem 4

$M_2 \cong S^2$.

Problem 7

Let us show that $\chi(M \# N) = \chi(M) + \chi(N) - 2$

Proof

Consider triangulations T and T' of M and N.

Let Δ be a triangle in M and Δ' a triangle in N. We may form $M \# N$ by removing the interiour of Δ and Δ' from M and N and glue together along pair of edges in Δ and Δ'. ~~Then we obtain a triangu~~ Let s, s', e, e', v, v' be the numbers of triangles, edges and vertices in T and T' respectivly. Now we get a triangulation of $M \# N$ where ~~we~~ the triangles are the triangles in T and T' with Δ and Δ' removed.

So the number of triangles is $s+s'-2$.

The number of ~~vertices~~ edges becomes ~~the~~ $e+e'-3$

(because we remove no edges by removing the interiour of Δ and Δ', but the three edges in Δ becomes identified with the three edges in Δ')

Also the three vertices in Δ and Δ' are identified, hence the number of vertices becomes $v+v'-3$, so $\chi(M \# N) = (s+s'-2) - (e+e'-3) + (v+v'-3)$

$= (s-e+v) + (s'-e'+v') - 2 = \underline{\chi(M) + \chi(N) - 2}$

Problem 7 continued:

Assume M is irreducible, but M is not homeomorphic to $\mathbb{S}^2$.

Then $M \simeq S(m,n)$ with $m+n \geq 1$

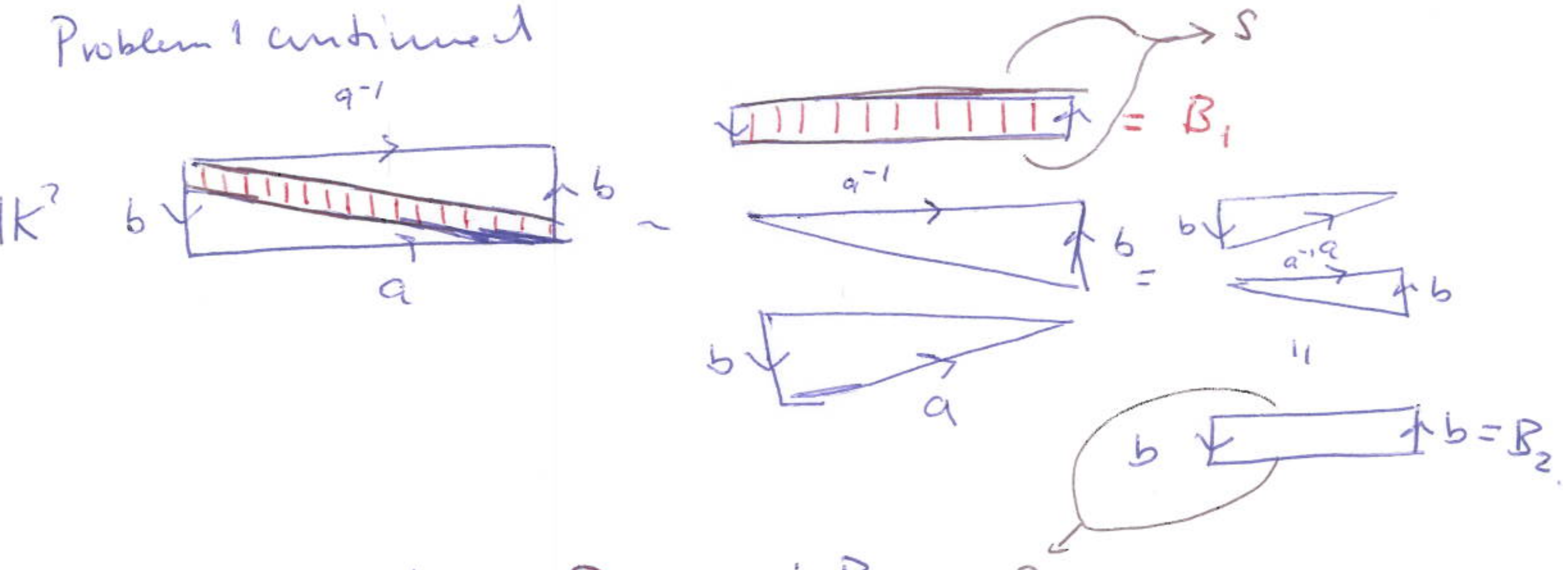
Now if $m+n > 1$ we either have $(m,n) = (1,1)$ and then $M \simeq S(1,1) = \mathbb{T}^2 \# \mathbb{P}^2$ and M is not irreducible or, $m > 1$ and then $M \simeq \mathbb{T}^2 \# S(m-1,n)$ and $S(m-1,n)$ is not homeomorphic to $\mathbb{S}^2$, hence M is not irreducible. or $n > 1$ so $M \simeq \mathbb{P}^2 \# S(m,n-1)$ and M is not irreducible. So $m+n=1$ hence either $m=1, n=0$ and $M \simeq \mathbb{T}^2$ or $m=0, n=1$ and $M \simeq \mathbb{P}^2$.

If M is reducible, $M \cong M_1 \# M_2$ where $M_1 \simeq S(m_1,n_1)$ $(m_1+n_1 \geq 1)$ and $M_2 \cong S(m_2,n_2)$ $(m_2+n_2 \geq 1)$

So $M \cong S(m_1,n_1) \# S(m_2,n_2) = S(m_1+m_2, n_1+n_2)$

So $m_1+m_2+n_1+n_2 \geq 2$ and therefore $S(m_1+m_2, n_1+n_2)$ is different from $\mathbb{T}^2 = S(1,0)$ and $\mathbb{P}^2 = S(0,1)$

Problem 1 continued



Then gluing B_1 and B_2 together along S, we obtain IK^2 and we see that $IK^2 = B \cup_S B$.