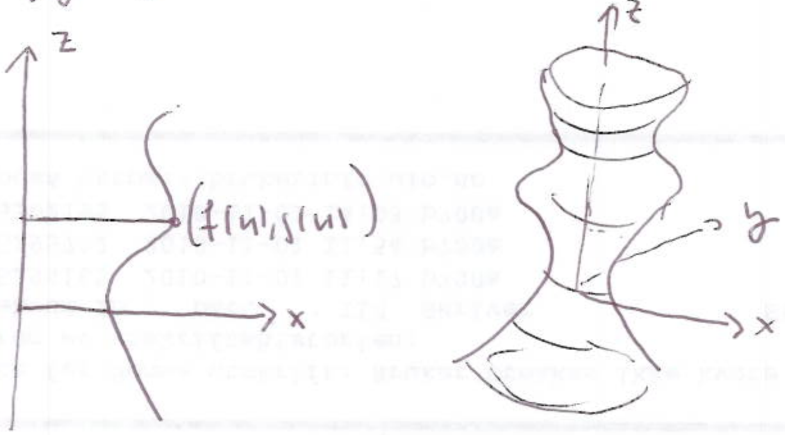
1.4

$\alpha(u) = (f(u), g(u))$, $u \in [a,b]$, $f(u) > 0$



Fixing u, and rotating the point $(f(u), g(u))$ around the z-axis, we get a circle in the plane $z = g(u)$, with centre at $(0,0,g(u))$ and radius $f(u)$, this circle may be parametrized by $x = f(u)\cos v$, $y = f(u)\sin v$, $z = g(u)$.
So the surface of revolution is parametrised by $(u,v) \to (f(u)\cos v, f(u)\sin v, g(u))$

The Jacobean of this parametrization is given by

$$\begin{bmatrix} f'(u)\cos v & -f(u)\sin v \\ f'(u)\sin v & f(u)\cos v \\ g'(u) & 0 \end{bmatrix}$$

This matrix has rank 2 at points where $f(u)f'(u) \neq 0$, hence if $f'(u) = 0$ ~~we see that we also have rank 2 if $f(u)\neq 0$~~ ~~and~~ if $f'(u)=0$ we see that the matrix has rank 2 if $g'(u) \neq 0$. So if $\alpha'(u) \neq 0$ then S is a regular surface. If $f(u) < 0 \ \forall u$ then the conclusion becomes the same. On the other hand, we see that if $f(u) = 0$ then the matrix has rank 1, so f must be either positive or negative for all u.

3.1

We had $X_u = \begin{pmatrix} f'(u)\cos v \\ f'(u)\sin v \\ g'(u) \end{pmatrix}$, $X_v = \begin{pmatrix} -f(u)\sin v \\ f(u)\cos v \\ 0 \end{pmatrix}$

So $E = X_u \cdot X_u = f'(u)^2 + g'(u)^2$; $F = 0$, $G = f(u)^2$

So $E\,du^2 + 2F\,du\,dv + G\,dv^2 =$

$= (f'(u)^2 + g'(u)^2)\,du^2 + f(u)^2\,dv^2$

5.1

$$N = \frac{X_u \times X_v}{\sqrt{EG-F^2}} = \frac{\begin{vmatrix} i & j & k \\ f'(u)\cos v & f'(u)\sin v & g'(u) \\ -f(u)\sin v & f(u)\cos v & 0 \end{vmatrix}}{f(u)\sqrt{f'(u)^2+g'(u)^2}} = -g'(u)f(u)\cos v$$

$$= \frac{1}{f(u)\sqrt{f'(u)^2+g'(u)^2}} \begin{pmatrix} -f(u)\cos v \\ -f(u)\sin v \\ f(u)f'(u) \end{pmatrix} = \frac{-\begin{pmatrix} \cos v \\ \sin v \\ f'(u) \end{pmatrix}}{\sqrt{f'(u)^2+g'(u)^2}}$$

$$e = N \cdot X_{uu} = \frac{-1}{\sqrt{EG-F^2}} \begin{pmatrix} \cos v \\ \sin v \\ f'(u) \end{pmatrix} \cdot \begin{pmatrix} f''(u)\cos v \\ f''(u)\sin v \\ g''(u) \end{pmatrix}$$

$$= \frac{-1}{\sqrt{EG-F^2}} \left(f'' + f'g'' \right), \quad f = N \cdot X_{uv} =$$

$$= \frac{1}{\sqrt{EG-F^2}} \begin{pmatrix} \cos v \\ \sin v \\ f'(u) \end{pmatrix} \cdot \begin{pmatrix} -f'(u)\sin v \\ f'(u)\cos v \\ 0 \end{pmatrix} = 0$$

$$g = \frac{-1}{\sqrt{EG-F^2}} \begin{pmatrix} \cos v \\ \sin v \\ f'(u) \end{pmatrix} \cdot \begin{pmatrix} -f(u)\cos v \\ -f(u)\sin v \\ 0 \end{pmatrix} = \frac{f(u)}{\sqrt{EG-F^2}}$$

5.1

$r(u,v) = (g(u)\cos v, g(u)\sin v, h(u))$

$r_u = (g'\cos v, g'\sin v, h')$

$r_v = (-g\sin v, g\cos v, 0)$

$$r_u \times r_v = \begin{vmatrix} i & j & k \\ g'\cos v & g'\sin v & h' \\ -g\sin v & g\cos v & 0 \end{vmatrix} = (-h'g\cos v, -h'g\sin v, gg')$$

So $N = \dfrac{(-h'\cos v, -h'\sin v, g')}{\sqrt{(g')^2+(h')^2}}$

$$e = N \cdot r_{uu} = \frac{1}{\sqrt{(g')^2+(h')^2}} \begin{pmatrix} -h'\cos v \\ -h'\sin v \\ g' \end{pmatrix} \cdot \begin{pmatrix} g''\cos v \\ g''\sin v \\ h'' \end{pmatrix}$$

$$= \frac{g'h''-h'g''}{\sqrt{(g')^2+(h')^2}}, \quad f = N \cdot r_{uv} = N \cdot \begin{pmatrix} -g'\sin v \\ g'\cos v \\ 0 \end{pmatrix} = 0$$

$$g = N \cdot r_{vv} = \frac{1}{\sqrt{(g')^2+(h')^2}} \begin{pmatrix} -h'\cos v \\ -h'\sin v \\ g' \end{pmatrix} \cdot \begin{pmatrix} -g\cos v \\ -g\sin v \\ 0 \end{pmatrix} =$$

$$= \frac{h'g}{\sqrt{(g')^2+(h')^2}}. \quad K = \frac{eg-f^2}{EG-F^2} =$$

$$= \frac{h'g(g'h''-h'g'')}{g^2((g')^2+(h')^2)^2} = \frac{h'(g'h''-h'g'')}{g((g')^2+(h')^2)}$$

5.2

Here we use 5.1 interchanging u and v with $g(v) = a\cos v + b$, $h(v) = a\sin v$

Then we get

$$K = \frac{a\cos v\left((-a\sin v)(-a\sin v) - (-a\cos v)(a\cos v)\right)}{(a\cos v + b)\left((-a\sin v)^2 + (a\cos v)^2\right)^2}$$

$$= \frac{a\cos v \cdot a^2}{(a\cos v + b)\, a^4} = \frac{\cos v}{a(a\cos v + b)}$$

Here $0 < a < b$ so $a\cos v + b > 0 \ \forall v$

so K is positive when $v \in [0, \frac{\pi}{2}) \cup (\frac{3\pi}{2}, 2\pi)$

and negative when $v \in (\frac{\pi}{2}, \frac{3\pi}{2})$

The curvature is zero when $v = \frac{\pi}{2}$ and $3\frac{\pi}{2}$