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INTRODUCTION TO DIFFERENTIABLE MANIFOLDS

Kompendium til bruk i MA 252

UNIVERSITETET I OSLO
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Forord

Disse forelesningsnotater gir sammen med oppgavesettene en resymé-aktig fremstilling av den elementære teori for differensiable manfoldigheter slik denne foreleses i kurset Ma 352.

De er ment å være til hjelp for studentene når de gjennomgår stoffet på egen hånd, men fremstillingen er for kortfattet til å være egnet til selvstudium. Den bør i så fall suppleres med en fyldigere tekstbok, Spivaks bok: "Calculus on Manifolds" og de fire første kapitler av Spivak: "A Comprehensive Introduction to Differential Geometry".

Stoffet er presentert på en måte som legger spesiell vekt på sammenhengen med Ma 101 og Ma 104. Henvisninger til Apostol II er til 2. bind av Apostol: Calculus, Second Edition. Notasjonen er mest mulig i overensstemmelse med Spivak: "A Comprehensive Introduction ...". Etter å ha gjennomgått det stoff som presenteres her, kan man fortsette direkte med Chapter 5 i Spivaks bok. Henvisninger til oppgaver gjelder oppgavesamlingen for kurset hvis intet annet er presisert. Viktige deler av stoffet (f.eks. rangeteoremet, Sards teorem) finnes bare i oppgavesamlingen.

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Chapter 1

Calculus in Several Variables

1.1 Differentiable mappings

$\mathbf{R}^m$ is Euclidean m-space, its points are denoted by $(x^1, \cdots, x^m)$. An open domain in $\mathbf{R}^m$ is an open, connected subset of $\mathbf{R}^m$. Let U be an open domain in $\mathbf{R}^m$ and let $f : U \to \mathbf{R}^n$ be a function.

Definitions

a) f is differentiable at a point p in U if there exists a linear map $Df(p)$ from $\mathbf{R}^m$ to $\mathbf{R}^n$ and a function $E(p, v)$ such that the first-order Taylor formula: $f(p + v) = f(p) + Df(p)(v) + \|v\|E(p, v)$ holds for all v in a small neighbourhood of the origin, where $E(p, v) \to 0$ as $v \to 0$.

b) Let $f = (f', \cdots, f^n)$. Then f is continuously differentiable at p if all partial derivatives $\frac{\partial f^i}{\partial x^j}$ exist in a neighbourhood of p and are continuous at p. $i = 1, \cdots, n, j = 1, \cdots, m$.

   See Apostol II, 8.11, 8.13, 8.19 and Problem 1.

In particular the matrix of $Df(p)$ is the Jacobian $(\frac{\partial f^i}{\partial x^j})(p)$. If a mixed partial derivative $\frac{\partial^2 f^k}{\partial x^i \partial x^j}$ is continous in U, then $\frac{\partial^2 f^k}{\partial x^i \partial x^j} = \frac{\partial^2 f^k}{\partial x^j \partial x^i}$. (Apostol II, 8.23).

If f is continuously differentiable at all points of U, we say that f is in $C^1(U, \mathbf{R}^n)$. Then $\frac{\partial f^i}{\partial x^j}$ are continuous functions on $U, i = 1, \cdots, n, j = 1, \cdots, m$. If they are all continously differentiable at all points of U, we say that f is in $C^2(U, \mathbf{R}^n)$, etc. f is in $C^\infty(U, \mathbf{R}^n)$ if f is in $C^k(U, \mathbf{R}^n)$ for all $k \geq 1$. If f is in $C^\infty(U, \mathbf{R}^n)$ all higher order mixed partial derivatives are independent of the order of differentiation. $C^\infty(U, \mathbf{R})$ is often denoted simply by $C^\infty(U)$. It is some times advantageous to change our viewpoint slightly. Let T_pU be the vector-space of vectors starting at the point p. There is a canonical identification of all these vectorspace with $\mathbf{R}^m$, which is often made in calculus books. (For example, if the tangent vector at p of a curve through p in U is considered as a vector in T_pU, then tangent vectors at different points belong to different vector spaces. In order to compute the derivative of the tangent vector (the accelleration) it is first necessary to make the above identification such that all the tangent vectors may be considered as lying in the same vector space). Let $q = f(p)$. Then $Df(p)$ may be considered as a linear map from T_pU to $T_q\mathbf{R}^n$; the constant term of the first-order Taylor's formula becomes irrelevant and this formula expresses the fact that this linear map is a "good" approximation of f in a small neighbourhood of p.

1.2 Curvilinear coordinate systems

It is often desirable to work with other coordinates than the usual Cartesian ones; polar coordinates in the plane and spherical or cylinder-coordinates in $\mathbf{R}^3$ are well-known from calculus. More general coordinates are often used to simplify the computation of multiple integrals, they are also used extensively in mathematical physics. A change of coordinates should satisfy certain regularity conditions; we formulate these as follows (see Apostol II, p. 408): A C^k-chart in $\mathbf{R}^m$ is a C^k-mapping from an open set U in $\mathbf{R}^m$ onto an open set V in $\mathbf{R}^m$ whose Jacobian determinant never vanishes and such that there is an inverse C^k-mapping from V to U. $(1 \le k \le \infty)$. The map can be expressed by: $(x^1, \cdots, x^m) \to (y^1, \cdots, y^m)$ where $y^i = y^i(x^1, \cdots, x^m), i = 1, \cdots, m$; we then think of the y^i's as new coordinates. By the chain rule for derivatives it follows that the Jacobian of the inverse mapping is the inverse of the Jacobian $(\frac{\partial y^i}{\partial x^j})$, hence the condition that the Jacobian determinant never vanishes is automatically true for the inverse mapping.

If we are given m functions $y^i = y^i(x^1, \cdots, x^m), i = 1, \cdots, m$, we may check the condition on the Jacobian by computation. The following important theorem is helpful to check the other conditions.

1.3 Inverse Function Theorem

Let f be a C^k-map $(1 \le k \le \infty)$ from an open set of $\mathbf{R}^m$ into $\mathbf{R}^m$. If the Jacobian determinant of f at a point p is non-zero, then there is an open neighbourhood U of p such that the restriction of f to U defines a C^k-chart. (In particular $f(U)$ is open).

Remark: If $m = 1$ you know the inverse function theorem already. See Apostol I, 6.20 or K. Sydsæter: Matematisk Analyse, 6.2.

Remark: In Apostol II these conditions for a C^1-chart could be relaxed for a set of measure zero. This is because coordinate changes were discussed mainly in connection with integration theory. However, it is clear that the differential calculus can be discussed for general coordinates. We will do this and stick to the above definition of a C^k-chart.

Example Polar coordinates in the plane are defined by $x^1 = r\cos\theta$, $x^2 = r\sin\theta$. It is clear that these do not define a C^k-chart around the origin. Even if the origin is removed, they cannot possibly define a chart. If we follow a circle around the origin and the angle θ varies continously, it cannot get back to its initial value. But polar coordinates do define a C^∞-chart on $\mathbf{R}^2 \setminus \{(x^1, 0) | x^1 \geq 0\}$ (difference of sets).

1.4 Differentiation Along a Curve. Local derivations.

Let $f \in C^\infty(U)$ and let $\bar{X} \in T_pU$. The derivative of f along $\bar{X}$, is defined by $f'(p, \bar{X}) = \lim \frac{1}{t}(f(p + t\bar{X}) - f(p))$. (Apostol II, 8.6). If $\bar{X}$ is the i-th basis vector $\bar{e}_i$, this is simply the i-th partial derivative of f at p. From the first order Taylor's formula it follows that $f'(p, \bar{X}) = Df(p)(\bar{X})$. (Apostol II, Theorem 8.9.)

In particular, $f'(p, \bar{X})$ depends linearly on $\bar{X}$. If $\bar{X} = a^1\bar{e}_1 + \cdots + a^n\bar{e}_n$, then $f'(p, \bar{X}) = a^1\frac{\partial f}{\partial x^1}(p) + \cdots + a^m\frac{\partial f}{\partial x^m}(p)$. Differentiation along the vector $\bar{X}$ is thus a first order differential operator which can be written as $a^1\frac{\partial}{\partial x^1}|_p + \cdots + a^m\frac{\partial}{\partial x^m}|_p$. If $g(t) = f(p + t\bar{X})$, then $f'(p, \bar{X}) = g'(0)$. It follows that $f'(p, \bar{X})$ depends linearly on f; i.e.

$$(f_1 + f_2)'(p, \bar{X}) = f_1'(p, \bar{X}) + f_2'(p, \bar{X}) \quad \text{and} \quad (\alpha f)'(p, \bar{X}) = \alpha f'(p, \bar{X}).$$

Thus differentiation along $\bar{X}$ is a linear differential operator. The product rule for derivatives implies an important property for the differential operator determined by $\bar{X}$: Let $f_3(q) = f_1(q) \cdot f_2(q), q \in U$. Let $g_i(t) = f_i(p + t\bar{X}), i = 1, 2, 3$.

Then

$$f_3'(p, \bar{X}) = g_3'(0) = g_1(0)g_2'(0) + g_2(0)g_1'(0) = f_1(p)f_2'(p, \bar{X}) + f_2(p)f_1'(p, \bar{X}).$$

Definition: Let X be a linear functional on the vector space $C^\infty(U)$. X is a local derivation at p if the following product rule is satisfied:

$$X(f_1 f_2) = f_1(p)X(f_2) + f_2(p)X(f_1).$$

It was shown above that differentiation along $\bar{X}$ is a local derivation at p.

More generally, we may consider differentiation of a function f along a curve.

Definition: A curve in U with initial point at p is a C^∞-mapping c from an open interval $(-\epsilon, \epsilon)$ to U such that $c(0) = p$ (thus we always consider parameterized curves). The tangent vector of c at p is $c'(0) \in T_pU$. If $c(t) = (c^1(t), \cdots, c^m(t))$, then $c'(0) = (c^{1'}(0), c^{2'}(0), \cdots, c^{m'}(0))$. (Components with respect to the standard basis $\bar{e}_1, \cdots, \bar{e}_m$).

Derivatives of functions along curves are discussed in Apostol II, 8.15. Let $f \in C^\infty(U)$ and let c be a curve with initial point p.

Definition: The derivative of f along c at the point p is $\lim_{t\to 0} \frac{1}{t}(f(c(t)) - f(c(0))) = g'(0)$ where $g(t) = f(c(t))$.

It is clear that differentiation of functions along a fixed curve is a linear functional on $C^\infty(U)$.

Proposition 1. The derivative of f along the curve c is equal to the derivative of f along the vector $c'(0)$.

Proof: Let $g(t) = f(c(t))$ and $h(t) = f(p + tc'(0))$. The chain rule gives: $g'(0) = Df(c(0))[c'(0)]$ and $h'(0) = Df(p)[c'(0)]$; hence $g'(0) = h'(0)$. But by definition $g'(0)$ and $h'(0)$ are the derivatives of f along c and along the vector $c'(0)$ respectively.

We would also write out the components in the proof. Let $c(t) = (c^1(t), \cdots, c^m(t))$, then $g(t) = f(c^1(t), \cdots, c^m(t))$ and $h(t) = f(p + (tc^{1'}(0), tc^{2'}(0), \cdots, tc^{m'}(0)))$. The classical chain rule gives:

$$g'(0) = \sum_{i=1}^{m} \frac{\partial f}{\partial x^i}(p)c^{i'}(0) \quad \text{and} \quad h'(0) = \sum_{i=1}^{m} \frac{\partial f}{\partial x^i}(p)c^{i'}(0),$$

hence $g'(0) = h'(0)$.

q.e.d.

Remark: The form of the chain rule we have used above corresponds to the chain rule for vector fields in Apostol II, Theorem 8.11 and is a better form than the version in Theorem 8.8 in Apostol II. Notice that we do not talk about the gradient of a function or scalar (inner) products of vectors here. As you know from Ma 104 the form of the scalar product depends on the basis. The expression is different in non-Euclidean coordinate systems. But we want all we do here to be valid in general coordinate systems. This will become clearer later.

We notice that differentiation along a vector $\bar{X}$ is a special case of differentiation along curves. (Let the curve be $c(t) = p + t\bar{X}$). On the other hand the proposition shows that differentiation along a curve c is the same as differentiation along the tangent vector $c'(0)$. Differentiation along a curve is always a local derivation of $C^\infty(U)$. Thus the set of all curves with initial point at p and the set of tangent vectors at p, T_pU, determine the same set of local derivations at p.

Definition: The curves $c_1(t)$ and $c_2(t)$ with initial point at p are equivalent if they define the same local derivation at p.

Corollary. The curves $c_1(t)$ and $c_2(t)$ with initial point at p are equivalent if and only if $c_1'(0) = c_2'(0)$; i.e. they have the same tangent vector at p.

Proof: If $c_1'(0) = c_2'(0)$ the two curves are equivalent by the proposition. Conversely, if $c_1'(0) - c_2'(0) = X \neq 0$ it is easy to see that there must exist a function f in $C^\infty(U)$ such that $f'(p, \bar{X}) \neq 0$. Then the curves must define different local derivations.

q.e.d.

From the corollary it follows that we may consider an equivalence class of curves with initial point at p as a vector in $T_pU \simeq \mathbf{R}^m$; hence the set of equivalence classes of curves is identified with a vector space. (The set of curves is not a vector space).

We have seen that a vector $\bar{X} = a^1\bar{e}_1 + \cdots + a^m\bar{e}_m$ in T_pU determines a local derivation $j(\bar{X}) = a^1\frac{\partial}{\partial x^1} + \cdots + a^m\frac{\partial}{\partial x^m}$ in $C^\infty(U)^*$ (the dual vector space of $C^\infty(U)$). ("Differentiation along the vector $\bar{X}$"). It is clear that j is a linear map from T_pU to $C^\infty(U)^*$, which has infinite dimension (see Problem 14), whereas $j(T_pU)$ can have dimension at most equal to m. By the corollary it follows that j is injective (different vectors determine different local derivations), hence $j(T_pU)$ actually has dimension m. Moreover, we know that $j(T_pU)$ is included in the subspace of $C^\infty(U)^*$ which consists of local derivations at p. We will now show that this subspace has dimension m and hence must be equal to $j(T_pU)$.

Lemma 1. Let $f, g \in C^\infty(U)$ and let X be a local derivation of $C^\infty(U)$ at p. Assume that $f = g$ in a neighbourhood of p. Then $X(f) = X(g)$.

Proof: Let $h = f - g$, then $h = 0$ in a neighbourhood V of p. Let k be a C^∞-function on U such that $k = 0$ outside V and $k(p) = 1$ (See Problem 14). Then $h \cdot k = 0$, and

$$0 = X(hk) = h(p)X(k) + k(p)X(h) = 0 + X(h), \quad \text{i.e. } X(f) = X(g).$$

q.e.d.

Lemma 2. Let V be a convex, open neighbourhood of 0 in $\mathbf{R}^m$. Let $f \in C^\infty(V)$ and $f(0) = 0$. Then there exist functions $g_i \in C^\infty(V), i = 1, \cdots, m$ such that:

$$(1) \quad f(x^1, \cdots, x^m) = \sum_{i=1}^{m} x^i g_i(x^1, \cdots, x^m) \quad \text{for } x = (x^1, \cdots, x^m) \in U$$

$$(2) \qquad g_i(0) = \frac{\partial f}{\partial x^i}(0), i = 1, \cdots, m.$$

Proof: Let $x \in V$ and let $h_x(t) = f(tx)$ for $t \in [0,1]$. Then

$$f(x) = f(x) - f(0) = h_x(1) - h_x(0) = \int_0^1 h_x'(t)dt$$

$$= \int_0^1 \sum_{i=1}^{m} \frac{\partial f}{\partial x^i}(tx)x^i dt = \sum_{i=1}^{m} x^i \int_0^1 \frac{\partial f}{\partial x^i}(tx^1, \cdots, tx^m)dt$$

q.e.d.

Proposition 2. Let U be an open domain in $\mathbf{R}^m$ and let $p \in U$. Then the set

$\{\frac{\partial}{\partial x^1}|_p, \cdots, \frac{\partial}{\partial x^m}|_p\}$ is a basis for the vector space of local derivations of $C^\infty(U)$ at p.

Proof: Since $\frac{\partial}{\partial x^i}|_p = j(\bar{e}_i)$ (differentiation along the i-th basis vector) it follows that the $\frac{\partial}{\partial x^i}|_p$ are linearly independent, $i = 1, \cdots, m$. It remains only to show that if X is a local derivation at p, there exists constants $a^1, \cdots, a^m$ such that

$$X = a^1 \frac{\partial}{\partial x'}\Big|_p + \cdots + a^m \frac{\partial}{\partial x^m}\Big|_p; \quad \text{i.e. } X(f) = a^1 \frac{\partial f}{\partial x'}(p) + \cdots + a^m \frac{\partial f}{\partial x^m}(p)$$

for all $f \in C^\infty(U)$. Now $X(1) = X(1 \cdot 1) = 1 \cdot X(1) + X(1) \cdot 1 = 2X(1)$, hence $X(1) = 0$ and $X(c) = 0$ for any constant function c.

By translation we may assume that p is the origin. By Lemma 1 we may assume that U is a convex neighbourhood of 0. Then $X(f) = X(f - f(0))$, and we may use Lemma 2 on the function $f - f(0)$.

We have:

$$\begin{aligned} X(f) &= X(f - f(0)) = X(\sum_{i=1}^{m} x^i g_i(x^1, \cdots, x^m)) \\ &= \sum_{i=1}^{m} X(x^i) g_i(0, 0, \cdots, 0) + 0 = \sum_{i=1}^{m} X(x^i) \frac{\partial f}{\partial x^i}(0, 0, \cdots, 0). \end{aligned}$$

Hence

$$X(f) = \sum_{i=1}^{m} a^i \frac{\partial f}{\partial x^i}(p), \quad \text{where} \quad a^i = X(x^i), i = 1, \cdots, m.$$

q.e.d.

Hence the vector space of local derivations of $C^\infty(U)$ at p is now identified with the vector space T_pU (by the isomorphism j).

It is clear that the definitions here are independent of what kind of coordinates I use. The derivative of the function f along the curve c is $\frac{d}{dt} f(c(t))$; and the coordinates do not enter here. It is only when I want to actually compute this that I must choose a coordinate chart. If I choose Euclidean coordinates as above, the curve $c(t)$ may be expressed in terms of its components: $c(t) = (x^1(t), \ldots, x^m(t))$. Suppose I have another coordinate chart (V, y) around $p = c(0)$, then $y = (y^1, \cdots, y^m)$ where $y^i = y^i(x^1, \cdots, x^m)$, $i = 1, 2, \cdots, m$.

Obviously, we must now express the function f as a function f' of the new coordinates $(y^1, \cdots, y^m)$; i.e. $f'(y^1, \cdots, y^m) = f(x^1(y^1, \cdots, y^m), x^2(y^1, \cdots, y^m), \cdots, x^m(y^1, \cdots, y^m))$, the curve $c(t)$ as $y^i = y^i(t), i = 1, \cdots, m$, where $y^i(t) = y^i(x^1(t), \cdots, x^m(t))$ and compute $\frac{d}{dt}_{t=0} f'(y^1(t), \cdots, y^m(t)) = \sum \frac{\partial f'}{\partial y^i}(y(p)) y^{i'}(0)$.

Hence when I have a coordinate system (V, y) around p, I can compute the derivative of functions along curves. I do not have to know in advance what kind of coordinates I work with. This is an important property of the differential calculus developed here. It is not shared by the usual integral calculus. Let I be the integral of f over U; i.e.

$$I = \int_U f(x^1, \cdots, x^m) dx^1 \cdots dx^m .$$

Let (V, y) be another chart on U. To compute I with respect to the new coordinates, it is not correct to simply compute $\int_{y(U)} f'(y^1, \cdots, y^m) dy^1 \cdots dy^m$.

The change of variable formula tells us that

$$I = \int_{y(U)} f'(y^1, \cdots, y^m) | \det(\frac{\partial x^i}{\partial y^j}) | dy^1 \cdots dy^m.$$

(Apostol II, 11.32).

In a later chapter on integration on manifolds we consider integration of differential forms, which can be computed by the same rule in all coordinate systems.

Chapter 2

Differentiable Manifolds

2.1 The sphere as a differentiable manifold

Let S^2 be the 2-sphere in $\mathbf{R}^3$, i.e.

$$S^2 = \{(x^1, x^2, x^3) | (x^1)^2 + (x^2)^2 + (x^3)^2 = 1\}.$$

The 2-sphere can also be given parametric representations. Spherical coordinates in $\mathbf{R}^3$ are defined by: $x^1 = \rho \sin\phi \cos\theta$, $x^2 = \rho \sin\phi \sin\theta$, $x^3 = \rho \cos\phi$. These define a C^∞-chart for certain domains in $\mathbf{R}^3$, f.eks. for the first octant: $\{(x^1, x^2, x^3) | x^1, x^2, x^3 > 0\}$. S^2 is then described by the equation $\rho = 1$; i.e. ϕ and θ are parameters. The part W of S^2 in the first octant is described by $(\phi, \theta) \epsilon \langle 0, \frac{\pi}{2}\rangle \times \langle 0, \frac{\pi}{2}\rangle$. A function on W may then be expressed as a function $g(\phi, \theta)$, $(\phi, \theta) \epsilon \langle 0, \frac{\pi}{2}\rangle \times \langle 0, \frac{\pi}{2}\rangle$, and may be differentiated with respect to ϕ or θ. Just as in the case of planar polar coordinates however, ϕ and θ do not define a satisfactory coordinate system for the whole sphere. We now describe two parametric representations which together will cover all the points on S^2. Let N be the north-pole: $N = (0, 0, 1)$ and S the south-pole: $S = (0, 0, -1)$. Let $U = S^2 \backslash N$ and $V = S^2 \backslash S$. Stereographic projection from the north-pole maps a point $P = (x^1, x^2, x^3)$ in U to the intersection point of the line through N and P with the (x^1, x^2)-plane. A simple picture shows that stereographic projection from the north pole is defined by:

$$(x^1, x^2, x^3) \to u = (u^1, u^2) = \frac{1}{1 - x^3}\left(x^1, x^2\right).$$

Similarly, stereographic projection from the south pole maps V onto R^2 by:

$$(x^1, x^2, x^3) \to v = (v^1, v^2) = \frac{1}{1 + x^3}\left(x^1, x^2\right)$$

All points on S^2 are described by these coordinates ($U \cup V = S^2$). The points on $U \cap V = S^2 \backslash (N \cup S)$ are given two sets of coordinates: (u^1, u^2) and (v^1, v^2), both in $\mathbf{R}^2 \backslash (0,0)$. The coordinate transformation formula are given by $v \cdot u^{-1}$ on $\mathbf{R}^2 \backslash (0,0)$. Here

$$u = (u^1, u^2) = \frac{1}{1-x^3}(x^1, x^2), v = (v^1, v^2) = \frac{1}{1+x^3}\left(x^1, x^2\right).$$

Hence $v \cdot u^{-1}$ is given by

$$(u^1, u^2) \to (v^1, v^2) = \frac{1}{1+x^3}\left(x^1, x^2\right) = \frac{1-x^3}{1+x^3}\left(u^1, u^2\right).$$

Here

$$\frac{1-x^3}{1+x^3} = \frac{(1-x^3)^2}{1-(x^3)^2} = \frac{(1-x^3)^2}{(x^1)^2+(x^2)^2} = \frac{1}{(u^1)^2+(u^2)^2} = \frac{1}{\|u\|^2},$$

and $(v^1, v^2) = \frac{1}{\|u\|^2}(u^1, u^2)$. The coordinate transformation formula (from U- to V-coordinates) are given by the C^∞-map $(u^1, u^2) \to \frac{1}{(u^1)^2+(u^2)^2}\left(u^1, u^2\right)$ from $\mathbf{R}^2 \backslash (0,0)$ to $\mathbf{R}^2 \backslash (0,0)$. This map is its own inverse, hence it is also a C^∞-diffeomorphism.

Let f be a function in a neighbourhood of a point P in $U \cap V$. f can then be expressed as a function $f_U(u^1, u^2)$, (the local expression of a f in (u^1, u^2) coordinates) or as a function $f_V(v^1, v^2)$. Then

$$f_V(v^1, v^2) = f_V\left(\frac{u^1}{(u^1)^2+(u^2)^2} \cdot \frac{u^2}{(u^1)^2+(u^2)^2}\right) = f_U(u^1, u^2);$$

i.e. $f_U = f_V \cdot v \cdot u^{-1}$ or $f_U \cdot u = f_V \cdot v = f$.

Since $v \circ u^{-1}$ is a C^∞-diffeomorphism, f_V is a C^k-map in a neighbourhood of $u(P)$ if and only if f_V is a C^k-map in a neighbourhood of $v(P)$. If this is true, we say that f is a C^k-map in a neighbourhood of P. In order of this to be well defined, it was necessary that the coordinate transformation formula $v \circ u^{-1}$ was a C^k-diffeomorphism (Otherwise the local expression of f could be a C^k-map in one coordinate system, but not in another).

A C^k-chart is a homeomorphism $g = (g^1, g^2)$ from an open connected subset G of S^2 to an open domain in $\mathbf{R}^2$ such that the local expression of $g, g \circ u^{-1}$ and $g \circ v^{-1}$ in $G \cap U$ and $G \cap V$ respectively, are C^k-diffeomorphisms. If f is a function from S^2 to $\mathbf{R}$, then f is a

C^k-function if and only if the local expression of f with respect to any C^k-chart is a C^k-function. (Check that $(W, (\phi, \theta))$ is a C^k-chart).

In calculus the tangent plane to S^2 through a point P in S^2 is defined as the two-dimensional vector space of all tangent-vectors to all curves in S^2 with initial point at P. (Apostol II, 12.3). This is a subspace of $T_p\mathbf{R}^3$, and its definition depends on regarding S^2 as a subset of $\mathbf{R}^3$. An *intrinsic* definition of tangent space should refer only to the space S^2 itself. Another point is that we cannot define directly what the derivative of a C^k-function f along a tangent vector should be, since f is not in general defined for points outside S^2. But we can define the derivative of f along curves in S^2 with initial point at P.

A (C^∞) curve in S^2 with initial point at P is a map c from an interval $(-\epsilon, \epsilon)$ to S^2 such that the local expression of c with respect to any C^∞-chart $(G, (g^1, g^2))$ is a C^∞-map and such that $c(0) = P$. (The local expression is $t \to (g^1(c(t)), g^2(c(t)))$, but this is often written simply as $t \to (g^1(t), g^2(t))$ if no confusion can arise). A curve with initial point at P defines a local derivation X at P of $C^\infty(S^2, \mathbf{R})$ (the C^∞-functions from S^2 to $\mathbf{R}$); i.e. a linear map X from $C^\infty(S^2, \mathbf{R})$ to $\mathbf{R}$ such that $X(f \cdot g) = f(p)X(g) + g(p)X(f)$. Here $X(f)$ is defined as $\frac{d}{dt}_{t=0} f(c(t))$. (Use the product rule for derivatives as in Chapter I). Two such curves are equivalent if they define the same local derivation at P.

The tangent vector of the curve $c(t) = (x^1(t), x^2(t), x^3(t))$ is $(x^{1'}(0), x^{2'}(0), x^{3'}(0))$. Let the local expression of $c(t)$ be: $t \to (g^1(t), g^2(t))$ as above; then

$$c(t) = (x^1(g^1(t), g^2(t)), x^2(g^1(t), g^2(t)), x^3(g^1(t), g^2(t)))$$

and

$$c'(0) = \left(\sum_{i=1}^{2} \frac{\partial x^1}{\partial g^i} g^{i'}(0), \sum_{i=1}^{2} \frac{\partial x^2}{\partial g^i} g^{i'}(0), \sum_{i=1}^{2} \frac{\partial x^3}{\partial g^i} g^{i'}(0) \right)$$

A function f can be expressed as a function of g^1 and g^2 locally; i.e. $\bar{f}(g^1, g^2)$. Then

$$\frac{d}{dt}_{t=0} f(c(t)) = \frac{d}{dt}_{t=0} \bar{f}(g^1(t), g^2(t)) = \frac{\partial \bar{f}}{\partial g^1} g^{1'}(0) + \frac{\partial \bar{f}}{\partial g^2} g^{2'}(0).$$

It follows that two such curves are equivalent if and only if they have the same tangent vector at P. The tangent plane of S^2 at P may be defined intrinsically as the vector space of local derivations at P of $C^\infty(S^2, \mathbf{R})$ or the space of equivalence classes of curves with initial point at P. It follows from Chapter I that these vector spaces are isomorphic.

2.2 Differentiable Manifolds

We now generalize the ideas in the preceding section to define differentiable manifolds. Roughly, a differentiable manifold is a space which can be covered by charts such that the coordinate transformation formulas are C^∞-diffeomorphisms.

Let M be a Hausdorff topological space.

Definition: M is locally Euclidean of dimension m at a point p if there exists an open neighbourhood V of p in M and a homeomorphism from V onto an open subset of $\mathbf{R}^m$. M is a (topological) manifold of dimension m if M is locally Euclidean of dimension m at every point p.

A chart on a manifold M of dimension m consists of a pair (U, x) where U is an open subset of M and x a homeomorphism from U onto an open subset of $\mathbf{R}^m$. An atlas is a family of charts (U_i, x_i) such that $\cup_i U_i = M$. By definition any manifold of dimension m has an atlas. If (U, x) and (V, y) are two charts, we have coordinate transformations: $y \cdot x^{-1} : x(U \cap V) \to y(U \cap V)$. Obviously these are homeomorphisms between open subsets of $\mathbf{R}^m$.

We do not discuss topological manifolds here, but specialize to differentiable manifolds. If f is a function from M to $\mathbf{R}$, f has local expressions as a function of $x = (x', \cdots, x^m)$ for any chart (U, x). We would like to define f to be a differentiable function if these local expressions are always differentiable. If the coordinate transformations formulas were not differentiable, it would be possible for the local expression of a function around a point to be differentiable in one chart and non-differentiable in another.

Definition: Two charts (U, x) and (V, y) are C^∞-related if the coordinate transformations $y \circ x^{-1}$ and $x \circ y^{-1}$ are C^∞-mappings (between open subset in $\mathbf{R}^m$). A C^∞-atlas is an atlas which consists of C^∞-related charts.

Definition: A differentiable manifold of dimension m is a topological manifold of dimension m together with a C^∞-atlas.

It is often an advantage to consider the maximal C^∞-atlas which consists of all charts which are C^∞-related to the charts in the given C^∞-atlas.

Let $(M, \mathcal{A})$ be a differentiable manifold where $\mathcal{A}$ is a C^∞-atlas for M. From now on a chart on M will denote a chart which is C^∞-related to $\mathcal{A}$, i.e. an element in the corresponding maximal atlas.

Definition: Let f be a function from M to $\mathbf{R}$. f is differentiable if the local expression $f \circ x^{-1}$ of f is a C^∞-function on $x(U)$ for any chart (U, x).

Example 1. Let U be an open domain in $\mathbf{R}^m$. The identity function: $x \to x$ defines a chart on U, which obviously is a C^∞-atlas and makes U a differentiable manifold.

Example 2. Let $M = \mathbf{R}$ (as a topological space). Let $f(x) = x^3$. Then f is a homeomorphism from M to $\mathbf{R}$; i.e. a (topological) chart, which defines M as a differentiable manifold $(M, \mathcal{A}_2)$. Let $(M, \mathcal{A}_1)$ be the differentiable manifold defined by the identity function on M: $x \to x$, as in Ex.1. These are different differentiable manifolds (with the same underlying topological space). For example, the function g from M to $\mathbf{R}$ defined by $g(x) = x$ is differentiable on $(M, \mathcal{A}_1)$; its local expression is simply: $x \to x$. But with respect to the second chart, the local coordinate of a point x is x^3. Hence the local expression of g is: $x^3 \to x$ or: $y \to y^{1/3}$ which is not differentiable at zero.

Let M and N be differentiable manifolds and let f be a mapping from M to N.

Definition: f is differentiable if for any point p in M there exists a chart (U, x) around p and chart (V, y) around $q = f(p)$ in N such that $f's$ local expression: $y \circ f \circ x^{-1}$ is a C^∞-function.

The space of C^∞-mappings from M to N is denoted by $C^\infty(M, N)$. $C^\infty(M, \mathbf{R})$ (where $\mathbf{R}$ has the standard differentiable structure defined in Example 1) is simply the space of differentiable functions on M as defined previously, and is denoted by $C^\infty(M)$.

Definition: A differentiable mapping $f : M \to N$ is a diffeomorphism if there is an inverse mapping from N to M which is also differentiable. The manifolds M and N are diffeomorphic if there exists a diffeomorphism from M to N.

Diffeomorphism is the equivalence relation which is relevant for the theory of differentiable manifolds. (Just as homeomorphism for topological spaces). If two manifolds are identified by a diffeomorphism, the whole theory will look the same. Consider the manifolds $(M, \mathcal{A}_1)$ and $(M, \mathcal{A}_2)$ in Example 2. These are diffeomorphic differentiable manifolds.

The identity map from M to M is not a diffeomorphism when it is considered as a map from $(M, \mathcal{A}_1)$ to $(M, \mathcal{A}_2)$. (Its local expression is $x \to x^3$, which does not have a differentiable inverse). But consider the map g from M to M defined by $g(x) = x^{1/3}$ as a map from $(M, \mathcal{A}_1) \to (M, \mathcal{A}_2)$. In local coordinates this is: $x \to x$ which is obviously a diffeomorphism from $\mathbf{R}$ to $\mathbf{R}$. Hence g is a diffeomorphism from $(M, \mathcal{A}_1)$ to $(M, \mathcal{A}_2)$.

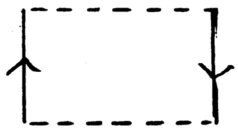
Example 3. The n-sphere $S^n = \{(x^1, \cdots, x^{n+1}) | (x^1)^2 + \cdots + (x^{n+1})^2 = 1\} \subset \mathbf{R}^{n+1}$. There are stereographic projections from the north pole and the south pole to $\mathbf{R}^n$, which define the n-sphere as a differentiable structure on S^n. Our discovery in Example 2 that the real line $\mathbf{R}$ has different differentiable structures, was not sensational, because these were diffeomorphic. In 1956 John Milnor made the sensational discovery that the topological space S^7 has a differentiable structure which is not diffeomorphic to the standard differentiable structure (he was awarded the Fields medal in 1962).

It has been long known that for $n \neq 4$; any differentiable structure on $\mathbf{R}^n$ is diffeomorphic to the usual one. In 1982 Simon Donaldson, while still a graduate student, stunned the mathematical world, by proving that $\mathbf{R}^4$ has "exotic" differentiable structures (not diffeomorphic to the usual one, for these there exist compact sets in $\mathbf{R}^4$ which cannot be contained inside any differentiable 3-sphere). For this Donaldson used methods from differential geometry and theoretical physics (Yang-Mills theory) in 1986 he was awarded the Fields medal.

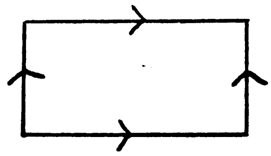
Example 4. Consider the closed rectangle $[0,1] \times [0,1]$ with the two horizontal edges removed. Identify points on the vertical edges which have the same x^2-coordinate, as indicated on the figure. The quotient space M is a differentiable manifold, it is simply the open cylinder. A neighbourhood around the point in M corresponding to the pair $(0,a), (1,a)$ is given by half-disks around $(0,a)$ and $(1,a)$ as indicated on the figure.

Example 5. More details for the following examples may be found in Spivak's book:

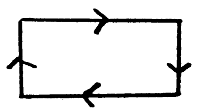
The Mobius strip:



The torus:
The Klein bottle:



The projective plane:



This can also be obtained by attaching a two-disc to the Möbius-strip along the boundaries.

Example 5. The projective n-space P^n is obtained from S^n by identifying antipodal points p and $-p$. There is a projection π from S^n to P^n sending a point p onto the equivalence class $[p, -p]$ in P^n. Clearly there is a chart (U, x) around any point p in S^n on which π is injective; hence we may consider this as a chart for the open set $\pi(U)$ in P^n. This defines a differentiable structure on P^n. The inverse image of any point in P^n under π consists of two points in S^n. π is an example of a "two-fold covering map". For another definition of the differentiable structure on P^n, see Problem 21.

Example 6. If M and N are differentiable manifolds, the product $M \times N$ is also a differentiable manifold. If (U, x) and (V, y) are charts on M and N, we define a chart $(U \times V, x \times y)$ on $M \times N$, where $(x \times y)(u, v) = (x(u), y(v))$. It is easy to check that these charts define a differentiable structure on $M \times N$.

We notice that we may always consider a differentiable manifold M

to be obtained by an identification process. Let (U_α, x_α) be an atlas for $M, \alpha \in A$. Let X be the disjoint union of the $x_\alpha(U_\alpha)$. We define an element $v \in x_\alpha(U_\alpha)$ to be equivalent to an element $w \in x_\beta(U_\beta)$ if $v \in x_\alpha(U_\alpha \cap U_\beta)$ and $w = x_\beta \circ x_\alpha^{-1}(v)$. The quotient space of X under this equivalence relation may be identified with M.

Example: On S^2 we have an atlas defined by stereographic projections: (U, u) and (V, v). Here $u(U) = v(V) = \mathbf{R}^2$ and $u(U \cap V) = v(U \cap V) = \mathbf{R}^2 \setminus (0,0)$. The coordinate transformation $v \cdot u^{-1}$ on $\mathbf{R}^2 \setminus (0,0)$ is given by $(u^1, u^2) \rightarrow \frac{1}{(u^1)^2+(u^2)^2}(u^1, u^2)$. We obtain S^2 by considering two copies of $\mathbf{R}^2$ and identifying points in $\mathbf{R}^2 \setminus (0,0)$ by this coordinate transformation.

2.3 The Tangent Space and the Derivative of a Mapping

Let M be a differentiable manifold and $p \in M$. The intrinsic definition of the tangent space of the two-dimensional sphere shows how we can define the tangent space of M at p.

Definition. A C^∞-curve in M with initial point at p is a C^∞-map $c : (-\epsilon, \epsilon) \rightarrow M, c(0) = p$. A C^∞-curve with initial point at p defines a local derivation of $C^\infty(M)$ at $p : f \rightarrow \frac{d}{dt}_{t=0} f(c(t))$. Two such curves are said to be equivalent if they define the same local derivation. Let (U, x) be a chart around $p, x = (x^1, \cdots, x^m)$. Then the curve $c(t)$ has a local expression around $p : t \rightarrow (x^1(t), \cdots, x^m(t))$. (We have written $x^i(t)$ for $x^i(c(t))$), and the equivalence class of c is determined by $(x^{1\prime}(0), \cdots, x^{m\prime}(0))$.

Definition. The tangent space of M at p, M_p, is the vector space of local derivations of $C^\infty(M)$ at p.

If $f, g \in C^\infty(M)$ and $f = g$ in a neighbourhood of p, then $X(f) = X(g)$ for any local derivation X of $C^\infty(M)$ at p. This follows because the proof of lemma 1 in Chapter I is valid in a differentiable manifold.

Lemma 1. Let U be an open neighbourhood of p in M and let $f \in C^\infty(U)$. Than there is a function $\tilde{f} \in C^\infty(M)$ such that $\tilde{f} = f$ on some neighbourhood of p.

Proof: Let V_1 and V_2 be neighbourhoods of p such that $p \in V_1 \subseteq \overline{V_1} \subseteq V_2 \subseteq \overline{V_2} \subseteq U$ and g a function in $C^\infty(M)$ such that $g \equiv 1$ on $V_1, g \equiv 0$ outside V_2. (See Problem 14). Then $f \cdot g \equiv 0$ outside V_2 and may be extended to a function $\tilde{f}$ on M which is zero outside V_2 and equals f on V_1. q.e.d.

Corollary. The vector space of local derivations of $C^\infty(U)$ at p is naturally isomorphic to the vector space of local derivations of $C^\infty(M)$ at p.

Proof: If Y is a local derivation of $C^\infty(U)$ at p, we can define a local derivation $\tilde{Y}$ of $C^\infty(M)$ at p by $\tilde{Y}(f) = Y(f|U)$. The linear map $Y \to \tilde{Y}$ has an inverse, which may be defined as follows: Let X be a local derivation of $C^\infty(M)$ at p, let $f \in C^\infty(U)$ and let $\tilde{f}$ be as in Lemma 1. Define $X'(f) = X(\tilde{f})$. It then follows that $X'(f)$ is independent of the choice of $\tilde{f}$. $X \to X'$ is inverse to $Y \to \tilde{Y}$. q.e.d.

We may take (U, x) to be a chart on M with $p \in U$. Since $x(U)$ is an open subset of $\mathbf{R}^m$, it follows from Proposition 2 in Chapter I that the set of equivalence classes of C^∞-curves with initial point at p is identified with the tangent space M_p. Also, the set $\left(\frac{\partial}{\partial x^1}|_p, \cdots, \frac{\partial}{\partial x^m}|_p\right)$ constitutes a basis for M_p.

We recall from Chapter I that if $f : U \to \mathbf{R}^n$ is a differentiable mapping from an open domain U in $\mathbf{R}^m$ to $\mathbf{R}^n$, the "derivative of f at p", $Df(p)$ is a linear map from T_pU to $T_{f(p)}\mathbf{R}^n, p \in U$. In this case T_pU and $T_{f(p)}\mathbf{R}^n$ were identified with $\mathbf{R}^m$ and $\mathbf{R}^n$ in a natural way, and $Df(p)$ was given by the Jacobian matrix $\left(\frac{\partial f^i}{\partial x^j}\right) 1 \le i \le n, 1 \le j \le m$, (relative to the bases $(\bar{e}_1|_p, \cdots, \bar{e}_m|_p)$ and $(\bar{e}_1|_{f(p)}, \cdots, \bar{e}_n|_{f(p)})$ for T_pU and $T_{f(p)}\mathbf{R}^n$). If I use other coordinate systems around p and $f(p)$, however, the local expression of f around p will change, and we get a new Jacobian matrix. Let $x = (x^1, \cdots, x^m)$ and $y = (y^1, \cdots, y^n)$ be the natural coordinate systems in U and $\mathbf{R}^n$. Let (U', u) and (V', v) be charts around p and $f(p)$ respectively. The new local expression of f around p is now given by $\tilde{f} = v \circ f \circ u^{-1}$. Let $f : U \to \mathbf{R}^n$ be given by $y^i = y^i(x^1, \cdots, x^n), i = 1, \cdots, n$. Then

$$\tilde{f}(u^1, \cdots, u^m) = [v^1\{y^1[x^1(u^1, \cdots, u^m), \cdots, x^m(u^1, \cdots, u^m)], \cdots, y^n [x^1(u^1, \cdots, u^m), \cdots, x^m(u^1, \cdots, u^m)]\}, \cdots, v^n\{y^1[x^1(u^1, \cdots, u^m), \cdots, x^m(u^1, \cdots, u^m)], \cdots, y^n[x^1(u^1, \cdots, u^m), \cdots, x^m(u^1, \cdots, u^m)]\}]$$

The v^i's are then composite functions of the u^i's, we can write

$$\tilde{f}(u^1, \cdots, u^m) \quad \text{as} \quad v^i = v^i(u^1, \cdots, u^m), i = 1, \cdots, n.$$

Then the Jacobian

$$\frac{\partial(v^1,\cdots,v^n)}{\partial(u^1,\cdots,u^m)} = \frac{\partial(v^1,\cdots,v^n)}{\partial(y^1,\cdots,y^n)} \cdot \frac{\partial(y^1,\cdots,y^n)}{\partial(x^1,\cdots,x^m)} \cdot \frac{\partial(x^1,\cdots,x^m)}{\partial(u^1,\cdots,u^m)}$$

In a more compact notation: $\frac{\partial v}{\partial u} = \frac{\partial v}{\partial y} \cdot \frac{\partial y}{\partial x} \cdot \frac{\partial x}{\partial u}$ - this is simply the chain rule. But this is the matrix of the linear map $Df(p)$ with respect to new bases in T_pU and $T_{f(p)}\mathbf{R}^n$ obtained by change of bases with matrices $\frac{\partial u}{\partial x}$ and $\frac{\partial v}{\partial y}$ respectively. (See Appendix and Problem 7). In elementary calculus one usually consider Cartesian coordinates to be a prefered coordinate system. It now becomes an advantage, however, to think of U as an abstract space and $x = (x^1,\cdots,x^m)$ as just an arbitrary way of assigning coordinates to points in U, which is no better than other coordinate system. (Think of an abstract vector space V; each choice of basis assigns coordinates in $\mathbf{R}^m$ to elements of V, but no choice is prefered over others).

When dealing with differentiable manifolds this point of view becomes imperative. Hence the tangent space of M at p should be defined as a vector space independent of any particular coordinate system around p, this we have already done by defining M_p as the vector space of local derivations at p. It was seen in Chapter I that U_p is naturally isomorphic to T_pU. It is then desirable to define the derivative of f at $p, f_*(p)$, directly as a linear map from U_p to $\mathbf{R}^n_{f(p)}$ such that $f_*(p)$ corresponds to $Df(p)$ under the natural isomorphisms:

$$\begin{array}{ccc} T_pU & \xrightarrow{Df(p)} & T_{f(p)}\mathbf{R}^n \\ \simeq & & \simeq \\ U_p & \xrightarrow{f_*(p)} & \mathbf{R}^n_{f(p)} \end{array} \qquad \text{(the diagram commutes).}$$

Let $X \in U_p$, i.e. X is a local derivation at p. To define $f_*(p)(X)$ as a local derivation at $f(p)$, we must define $f_*(p)(X)(g)$ for $g \in C^\infty(\mathbf{R}^n)$. The natural way to define this is by $f_*(p)(X)(g) = X(g \cdot f)$. ($X$ can be applied to $g \cdot f$, which is a function on U). It is straightforward to verify that $f_*(p)$ is a linear map from U_p to $\mathbf{R}^n_{f(p)}$. (See Problem 8a)). Under the natural isomorphism between T_pU and U_p the basis $\bar{e}_1|_p,\cdots,\bar{e}_m|_p$ corresponds to $\frac{\partial}{\partial x^1}|_p,\cdots,\frac{\partial}{\partial x^m}|_p$, similarly the basis $\bar{e}_1|_{f(p)},\cdots,\bar{e}_n|f_{f(p)}$ for $T_{f(p)}\mathbf{R}^n$ corresponds to $\frac{\partial}{\partial y^1}|_p,\cdots,\frac{\partial}{\partial y^n}|_p$. To show that $f_*(p)$ corresponds to $Df(p)$, it is then sufficient to show that the matrix of $f_*(p)$ with respect to the bases $\frac{\partial}{\partial x^1}|_p,\cdots,\frac{\partial}{\partial x^m}|_p$ and $\frac{\partial}{\partial y^1}|_{f(p)},\cdots,\frac{\partial}{\partial y^n}|_p$ is the Jacobian $\left(\frac{\partial f^i}{\partial x^j}\right)(p)$. ($f^i = y^i \circ f$) (problem 8b).)

Let M and N be differentiable manifolds and $f \in C^\infty(M,N)$. Let $p \in M$ and $q = f(p)$.

Definition: The derivative of f at p, $f_*(p)$, is the linear map from M_p to N_q defined by $f_*(p)(X)(g) = X(g \circ f)$ for $X \in M_p$, $g \in C^\infty(N)$.

If (U, x) is a chart around p and (V, y) is a chart around q, let $B = \left(\frac{\partial}{\partial x^1}|_p, \cdots, \frac{\partial}{\partial x^m}|_p\right)$ and $C = \left(\frac{\partial}{\partial y^1}|_q, \cdots, \frac{\partial}{\partial y^n}|_q\right)$ be the corresponding bases for M_p and N_q respectively. Then the corresponding matrix $f_*(p)^C_B$ is the Jacobian matrix for f's local expression at p.

To visualize the derivative, $f_*(p)$, it is often useful to think of tangent vectors as equivalence classes of curves. If $c(t)$ is a C^∞-curve with initial point at p, then $f(c(t))$ is a C^∞-curve in N with initial point at q. If $c_1(t)$ is equivalent to $c_2(t)$, then $f(c_1(t))$ is equivalent to $f(c_2(t))$. Hence f induces a map between equivalence classes of curves, it is easy to check that this must be $f_*(p)$. For let X be the local derivation defined by $c(t)$; i.e. $X(g) = \frac{d}{dt}_{t=0} g(c(t)), g \in C^\infty(M)$. Let $h \in C^\infty(N)$. Then $f_*(p)(X)(h) = X(h \cdot f) = \frac{d}{dt}_{t=0}(h \cdot f)(c(t)) = \frac{d}{dt}_{t=0} h(f(c(t)))$, i.e. $f_*(p)(X)$ is the local derivation defined by the curve $f(c(t))$. Hence the derivative $f_*(p)$ just corresponds to taking the image of C^∞-curves with initial point at p under the mapping f. If (U_α, x_α) is a chart around p, we can define the i-th coordinate curve λ^i_α through p by its local expression:

$$t \to (x^1_\alpha(p), \cdots, x^{i-1}_\alpha(p), x^i_\alpha(p) + t, x^{i+1}_\alpha(p), \cdots, x^m_\alpha(p)), i = 1, \cdots, m.$$

(i.e. we let the i-th coordinate vary and keep the other ones fixed). The equivalence classes of these curves define the basis in M_p corresponding to $\left(\frac{\partial}{\partial x^1_\alpha}|_p, \cdots, \frac{\partial}{\partial x^m_\alpha}|_p\right)$.

If $\bar{v}$ is a vector in M_p then $\bar{v}$ has a column vector $\bar{v}^\alpha$ relative to the basis $\left(\frac{\partial}{\partial x^1_\alpha}|_p, \cdots, \frac{\partial}{\partial x^m_\alpha}|_p\right)$,

$$\bar{v}^\alpha = \begin{pmatrix} v^1 \\ \vdots \\ v^m \end{pmatrix} \qquad \text{where} \quad \bar{v} = \sum_{i=1}^m v^i \frac{\partial}{\partial x^i_\alpha}|_p.$$

If (U_β, x_β) is another chart around p, then $\frac{\partial}{\partial x^i_\alpha}|_p = \sum_{j=1}^m \frac{\partial x^j_\beta}{\partial x^i_\alpha} \frac{\partial}{\partial x^j_\beta}|_p$. Hence the matrix for the change of basis is the Jacobian $\left(\frac{\partial x^j_\beta}{\partial x^i_\alpha}\right)$, and $\bar{v}^\beta = \left(\frac{\partial x^j_\beta}{\partial x^i_\alpha}\right)\bar{v}^\alpha$. $\bar{v}$ is called a contravariant vector. In classical books a vector was defined by its components. A set of n components was assigned for each coordinate system, in order that these define a contravariant vector they must transform according to the above rule under change of coordinates.

Proposition 1. Let $f \in C^\infty(M,N)$ and $g \in C^\infty(N,P)$. Then $(g \cdot f)_*(p) = g_*(f(p)) \cdot f_*(p)$.

2.4 The Cotangent Space and the Differential of a Function

From linear algebra we know there is a bilinear pairing between a vector space V and its dual V^* which we denote by $\langle v, v^* \rangle$, (See Appendix). Each $v \in V$ determines a linear functional e_v on V^* defined by $e_v(v^*) = v^*(v)$. Let M be a differentiable manifold and $p \in M$.

M_p is a subspace of the dual space of $C^\infty(M)$, hence each function f in $C^\infty(M)$ defines a linear functional on M_p: Let λ be a C^∞-curve with initial point at p, and define $\langle \lambda, f \rangle = \frac{d}{dt}_{t=0} f(\lambda(t))$, i.e. the local derivation defined by λ applied to f. Let $H_p = \{f \in C^\infty(M) | \langle \lambda, f \rangle = 0$ for all curves λ with initial point at $p\}$, i.e. H_p is the vector space of functions f such that all directional derivatives of f at p are zero.

Definitions: The cotangent space of the differentiable manifold M at p, M_p^* is the quotient vector space $C^\infty(M)/H_p$. Hence M_p^* is a vector space of equivalence classes of functions, two functions being equivalent if all their directional derivatives at p are identical. If (U, x) is a chart around p, f is equivalent to g if $\frac{\partial f}{\partial x^i}(p) = \frac{\partial g}{\partial x^i}(p), i = 1, \cdots, m$. Also, f and g are equivalent if they define the same linear functional on M_p.

Definition: The equivalence class of f in M_p^* is called the differential of f at p and denoted by df_p. p is a critical point for f is $df_p = 0$. p is a regular point for f if it is not critical.

It is immediately clear that if $f = g$ in a neighbourhood of p, then $df_p = dg_p$. Let V be an open subset of M, let $p \in M$ and let f be a C^∞-function defined on V. Let $\tilde{f}$ be as in Lemma 1, and define df_p as $d\tilde{f}_p$, it is clear that this is well-defined.

Proposition 2.

$$d(f+g)_p = df_p + dg_p$$

$$d(\alpha f)_p = \alpha df_p \cdot (\alpha \in \mathbf{R})$$

$$d(f \cdot g)_p = f(p) dg_p + g(p) df_p \,.$$

Proof: The two first relations are obvious. Let λ be a curve with initial point at p. The proposition follows from the relation

$$\frac{d}{dt}_{t=0}\{f(\lambda(t))\cdot g(\lambda(t))\} = f(p)\frac{d}{dt}_{t=0}g(\lambda(t)) + g(p)\frac{d}{dt}_{t=0}f(\lambda(t)).$$

Proposition 3. Let (U,x) be a chart around p. Then $dx_p^1,\cdots,dx_p^m$ is a basis for M_p^*.

Proof: Suppose a function f has the local expression $\bar{f}(x^1,\cdots,x^m)$ and λ is a curve with initial point at p. Then

$$\begin{aligned}\langle\lambda,f\rangle &= \frac{d}{dt}_{t=0}\bar{f}(x^1(\lambda(t)),\cdots,x^m(\lambda(t)))\\ &= \sum_{i=1}^{m}\frac{\partial\bar{f}}{\partial x^i}\frac{d}{dt}x^i(\lambda(t)) = \sum_{i=1}^{m}\frac{\partial\bar{f}}{\partial x^i}\langle\lambda,x^i\rangle,\end{aligned}$$

hence $df_p = \sum\frac{\partial\bar{f}}{\partial x^i}dx_p^i$. It only remains to prove that the dx_p^i are linearly independent. We can evaluate dx_p^i on the basis vectors of M_p corresponding to the coordinate curves $\lambda_j : \langle\lambda_j,x^i\rangle = \frac{d}{dt}_{t=0}x^i(\lambda_j(t))$.

But $\lambda_j(t)$ has local coordinates $(x^1(p),\cdots,x^j(p)+t,\cdots,x^m(p))$ hence $x^i(\lambda_j(t)) = x^i(p)$ if $j\neq i, x^i(\lambda_i(t)) = x^i(p)+t$. Hence $\langle\lambda_j,x^i\rangle = \delta_j^i = \begin{cases}1 & \text{if } j=i\\ 0 & \text{if } j\neq i\end{cases}$

It follows that $dx_p^1,\cdots,dx_p^m$ are linearly independent.

Corollary. The cotangent space M_p^* is the dual vector space of M_p. The basis

$\{dx_p^1,\cdots,dx_p^m\}$ is the dual basis of $\{\frac{\partial}{\partial x^1}|_p,\cdots,\frac{\partial}{\partial x^m}|_p\}$.

This is clear since the dimension of M_p^* has been shown to be m. Also: $\langle\frac{\partial}{\partial x^j}|_p, dx_p^i\rangle = \langle\lambda_j,x^i\rangle = \delta_j^i$.

Let $f\in C^\infty(M)$. The computation in the proof of Proposition shows that

$df_p = \sum_{i=1}^{m}\frac{\partial f}{\partial x_i}(p)dx_p^i$. This gives precise meaning to a classical formula for the differential of a function. If (V,y) is another chart around p, then

$$df_p = \sum_{j=1}^{m}\frac{\partial f}{\partial y^j}dy_p^j = \sum_{j=1}^{m}\sum_{i=1}^{m}\frac{\partial f}{\partial y^j}\frac{\partial y^j}{\partial x^i}dx^i = \sum_{i=1}^{m}\Big(\sum_{j=1}^{m}\frac{\partial y^j}{\partial x^i}\frac{\partial f}{\partial y^j}\Big)dx_p^i.$$

Hence, by a change from y- to x-coordinates, the components of the vector $df_p\in M_p^*$ is changed by the matrix

$$\begin{pmatrix}\frac{\partial y^1}{\partial x^1} & \frac{\partial y^2}{\partial x^1} & \frac{\partial y^m}{\partial x^1}\\ \frac{\partial y^1}{\partial x^2} & \cdots & \\ \cdots & & \frac{\partial y^m}{\partial x^m}\end{pmatrix},$$

which is the transpose of the Jacobian matrix for the change from X- to Y-coordinates. Of course we could also find this matrix by expressing the old basis in terms of the new ones: $dy^i_p = \sum_j \frac{\partial y^i}{\partial x^j} dx^j_p$.

We see that the transformation matrix for M^*_p is the transpose of the inverse of the transformation matrix for M_p when we change from x- to y-coordinates. Vectors in M^*_p are called covariant vectors. Thus the somewhat mysterious definitions in classical texts which prescribes that the components of a covariant vectors transforms in a different way from the components of a contravariant vector is given a natural explanation, they belong to different vector spaces, and the bases associated with a coordinate system are dual of each other. In Problem 19 the relation between the transformation matrices for pairs of such dual bases is studied in general.

Let $f \in C^\infty(M,N)$ and $p \in M, q = f(p) \in N$. Let $g_i \in C^\infty(N)$, then $g_i \circ f \in C^\infty(M)$. If $dg_{1q} = dg_{2q}$, it follows easily that $d(g_1 \circ f)_p = d(g_2 \circ f)_p$, hence the map $g \to f \circ g$ from $C^\infty(N)$ to $C^\infty(M)$ induces a linear map from N^*_q to M^*_p which we call $f^*(p)$. (Notice that we cannot call this $f^*(q)$; if $w \in N^*_g$ and $p_1, p_2 \in f^{-1}(q)$, we would not know if $f^*(q)(w)$ should belong to M_{p1} or M_{p2}). Let λ be a curve with initial point at p, and $[\lambda]$ the equivalence class of λ in M_p. Then

$$\langle[\lambda], f^*(p)dg\rangle = \langle[\lambda], d(g \circ f)\rangle = \frac{d}{dt}_{t=0} g(f(\lambda(t)) = \langle f_*(p)[\lambda], dg\rangle.$$

Hence $f^*(p): N^*_q \to M^*_p$ is the adjoint of the linear map $f_*(p): M_p \to N_q$.

Let (U,x) be a chart around p and (V,y) a chart around q. Let the local expression of f around p be given by $y^i = y^i(x^1, \cdots, x^m), i = 1, \cdots, n$. Then

$$f^*(p)(dy^i_q) = d(y^i(x^1, \cdots, x^m))_p = \sum_{j=1}^m \frac{\partial y^i}{\partial x^j} dx^j_p,$$

hence the matrix of $f^*(p)$ with respect to the bases defined by these charts is the transpose of the Jacobian matrix $\left(\frac{\partial y^i}{\partial x^j}\right)$. (Compare Problem 19b)).

Proposition 4. Let $f \in C^\infty(M,N)$ and $g \in C^\infty(M,P)$. Then $(g \circ f)^*(p) = f^*(p) \circ g^*(f(p))$.

Proof: Let $h \in C^\infty(P)$. Then $(g \circ f)^*(p)(dh) = d(h \circ (g \circ f))_p . g^*(f(p))(dh) = d(h \circ g)_{f(p)}$ and $f^*(p)(d(h \circ g)_{f(p)}) = d((h \circ g) \circ f)_p = d(h \circ (g \circ f))_p$.

q.e.d.

2.5 Immersions and Submanifolds

Let M, N be differentiable manifolds with $m = \dim M, n = \dim N$. Let $f : M \to N$ be differentiable.

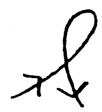
Definition. f is an immersion if $f_*(p)$ is injective for all p in M.

It follows that $f_*(p)$ is an isomorphism from M_p to an m-dimensional subspace of $N_{f(p)}$; i.e. the tangent space M_p is injectively embedded in $N_{f(p)}$. The rank theorem (Problem 15) gives information on the local structure of f around p. Since the rank of f must equal m in a neighbourhood of p, there are charts (U, x) around p and (V, y) around $f(p)$ such that the local expression of f in U is: $(x^1, \cdots, x^m) \to (x^1, \cdots, x^m, 0, \cdots, 0)$. In particular, f is locally injective.

Examples.

1. The standard injection of $\mathbf{R}^m$ into $\mathbf{R}^n$ $(n \geq m)$.

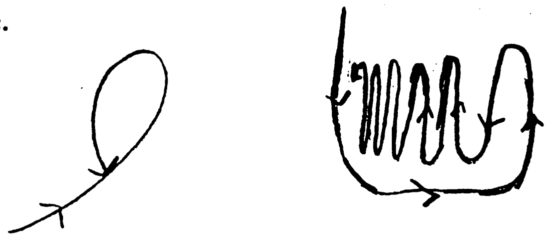
2. An immersion is not always injective. The following figure illustrates a non-injective immersion of $\mathbf{R}$ into $\mathbf{R}^2$.



3. An injective differentiable map is not always an immersion. Example: Let $f : \mathbf{R} \to \mathbf{R}$ be given by $f(x) = x^3$. Then $f_*(0)$ has zero rank.

Let $f : M \to N$ be an *injective* immersion of M into N. Since f maps M bijectively onto the image $f(M)$, we may consider $f(M)$ as a differentiable manifold. (The differentiable structure on $f(M)$ transported from M by f).

In addition $f(M)$ has the structure of a topological space as a subspace of N. The injective immersions of $\mathbf{R}$ in $\mathbf{R}^2$ indicated by the following figures show that these structures on $f(M)$ are not always compatible.

4. 

(e.g. consider neighbourhoods of p in these two topologies).

In general, let M be a differentiable manifold, and M_1 a subset of M which is a differentiable manifold in some way. M_1 is an *immersed submanifold* of M if the inclusion map $i : M_1 \to M$ is an immersion. In this case, if $p \in M_1$ it follows that there existsa neighbourhood U of p in M_1 (manifold topology) and a chart (V, y) around p in M such that $U \cap V = \{q \in V, y^{k+1}(q) = \cdots = y^m(q) = 0\}$. This shows that a neighbourhood U of p in M_1 is embedded in M as $\mathbf{R}^k$ in $\mathbf{R}^m$, but it does not tell us how M_1 lies in M in a neighbourhood of p. This is because $M_1 \cap V$ can be much bigger than $U \cap V$. In Problem 36 it is shown that the real line $\mathbf{R}$ can be a *dense*, immersed submanifold of the torus T^2 (although each little neighbourhood of $\mathbf{R}$ is embedded in T^2 as $\mathbf{R}$ in $\mathbf{R}^2$).

Definition. $f : M \to N$ is an imbedding if it is an injective immersion and a homeomorphism onto the image $f(M)$ with the subspace topology induced from N.

In this case $f(M)$ as a topological subspace of N has a differentiable structure.

The subset M_1 with a manifold structure of M is a *submanifold* of M if the inclusion map $i : M_1 \to M$ is an imbedding. In this case a manifold neighbourhood U of p in M_1 is also a neighbourhood in the subspace topology, and consequently there is a chart (V, y) around p in M such that $V \cap M_1 = \{q \in V : y^m(q) = 0\}$. In this case we can truely say that "locally" M_1 lies in M as $\mathbf{R}^k$ in $\mathbf{R}^m$.

If M_1 is also closed in M, it is called a *closed submanifold* of M.

Let $f \in C^\infty(M, N)$ and $p \in M, q \in N$.

Definition. p is a *critical point* for f if the rank of $f_*(p)$ is less than $n(= \dim N)$. p is a *regular point* for f it is not a critical point. q is a *critical value* for f if there exists a p in M such that p is a critical point for f and $q = f(p)$. q is a *regular value* for f if it is not a critical value.

Note that if q is a regular value, then all points in $f^{-1}(q)$ are regular points. Also, if $q \notin f(M)$, then q is always a regular value for f (although q is not a value for f at all!).

If $M = \mathbf{R}^m$ and $N = \mathbf{R}, m = 1, 2$ we know quite a bit about critical points already (local maxima and minima, hyperbolic points, etc.).

Theorem. Let $f \in C^\infty(M, \mathbf{R})$ and assume that q is a regular value for f. Then $f^{-1}(q)$ is a closed submanifold of M of codimension one or empty.

(Codimension one in M means dimension one less than M).

Proof: Let $p \in f^{-1}(q)$. Then the rank of $f_*(p)$ is one by continuity it follows that the rank of f_* is one in a neighbourhood of p. By the rank theorem there is a local chart (U, x) around p in M such that $f's$ local expression is $(x^1, \cdots, x^{m-1}, x^m) \to x^m$. Hence the local coordinates of the points in $f^{-1}(q) \cap U$ are those which are of the form $(x^1, \cdots, x^{m-1}, q)$. We may then consider $U \cap f^{-1}(q)$ as a local chart around p in $f^{-1}(q)$, with coordinates $x^1, \cdots, x^{m-1}$. In this way we get an open covering of $f^{-1}(q)$ by charts, and we only have to check that these are C^∞-related. Let (V, y) be a different chart around p in M as above; i.e. $f's$ local expression is $(y^1, \cdots, y^{m-1}, y^m) \to y^m$ and $f^{-1}(q) \cap V$ consists of the points with local coordinates $(y^1, \cdots, y^{m-1}, q)$. The coordinate transformations in M are given by C^∞-functions: $y^i = y^i(x^1, \cdots, x^m), i = 1, \cdots, m$. The coordinate transformations in $f^{-1}(q)$ are just the restrictions of these functions to those points with x^m-coordinate constant equal to q; here $y^i = y^i(x^1, \cdots, x^{m-1}), i = 1, \cdots, m-1$, which are obviously C^∞-function. $f^{-1}(q)$ is obviously closed since f is continuous.

q.e.d.

Chapter 3

Vector bundles

3.1 The Tangent Bundle

A vector field X on a differentiable manifold M is a correspondence which to each point p in M assigns a vector $X(p)$ in the tangent space M_p. The tangent spaces $M_p, p \in M$ are different vector spaces; hence, in order to express X as a function in the normal way, it is necessary to form the union of all M_p. Let $TM = \bigcup\limits_{p \in M} M_p$; (disjoint union) then X is a function from M to TM with the additional requirement that $X(p) \in M_p$ for all $p \in M$.

Let U be an open domain in $\mathbf{R}^m$ and X a vector field on U. Since each $X(p)$ lies in a different vector space U_p (or T_pU), it is in principle not clear what it means that $X(p)$ varies continuously or differentiably with p. In this case this problem is quickly solved however, when we recall that there is a standard identification of these vectors spaces by means of the bases $\left(\frac{\partial}{\partial x^1}|_p, \cdots, \frac{\partial}{\partial x^m}|_p\right)$ (or $\bar{e}_1|_p, \cdots, \bar{e}_m|_p$). The vector spaces $U_p(T_pU)$ are all identified with a fixed vector space $\mathbf{R}^m$ (hence $TU \cong U \times \mathbf{R}^m$), and we may regard the $X(p)'s$ as lying in the same vector space $\mathbf{R}^m$. We may write $X(p) = \sum\limits_{i=1}^{m} a^i(p) \frac{\partial}{\partial x^i}|_p$ (or $X(p) = \sum\limits_{i=1}^{m} a^i(p) \bar{e}_i|_p$) this vector field is continuous (differentiable) if and only if $a^i(p)$ is continuous (differentiable). (Apostol II, 8.4 and 8.18). In calculus books this vector field is often written simply as $X(p) = (a^1(p), \cdots, a^m(p)))$. If we give $TU \cong U \times \mathbf{R}^m$ the differentiable structure it inherits as a subset of $\mathbf{R}^{2m}$, it is clear that X is a continuous (differentiable) vector field if and only if X is continuous (differentiable) as a function from U to TU.

From our present point of view this description is not completely satisfactory, however, because the way we identify the vector spaces

U_p and U_q with $\mathbf{R}^m$ $(p \neq q)$ depends on the coordinate system $x = (x^1, \cdots, x^m)$. Let $y = (y^1, \cdots, y^m)$ be another coordinate system on U, then $\frac{\partial}{\partial x^i}|_p = \sum_{j=1}^{m} \frac{\partial y^j}{\partial x^i}(p) \frac{\partial}{\partial y^j}|_p$ and

$$X(p) = \sum_{i=1}^{m} a^i(p) \frac{\partial}{\partial x^i}\Big|_p = \sum_{i=1}^{m}\sum_{j=1}^{m} \frac{\partial y^j}{\partial x^i}(p) a^i(p) \frac{\partial}{\partial y^j}\Big|_p = \sum_{j=1}^{m} b^j(p) \frac{\partial}{\partial y^j}\Big|_p$$

with $b^j(p) = \sum_{i=1}^{m} \frac{\partial y^j}{\partial x^i}(p) a^i(p)$. Hence $(b^1(p), \cdots, b^m(p))$ is continuous (differentiable) if and only if $(a^1(p), \cdots, a^m(p))$ is continuous (differentiable); and although the identification of the tangent spaces at various points depends on the coordinate system, the continuity (differentiability) of the component functions of the vector field does not.

Example. Let U be the upper half plane in $\mathbf{R}^2$, and consider polar coordinates (r, θ). The basis $\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial \theta}\right)$ is given by

$$\frac{\partial}{\partial r} = \frac{\partial x^1}{\partial r}\frac{\partial}{\partial x^1} + \frac{\partial x^2}{\partial r}\frac{\partial}{\partial x^2} = \cos\theta \frac{\partial}{\partial x^1} + \sin\theta \frac{\partial}{\partial x^2}$$

and

$$\frac{\partial}{\partial \theta} = \frac{\partial x^1}{\partial \theta}\frac{\partial}{\partial x^1} + \frac{\partial x^2}{\partial \theta}\frac{\partial}{\partial x^2} = -r\sin\theta \frac{\partial}{\partial x^1} + r\cos\theta \frac{\partial}{\partial x^2}$$

If we think of T_pU instead of U_p, we can write the basis as:

$$(\cos\theta \bar{e}_1|_p + \sin\theta \bar{e}_2|_p, -r\sin\theta \bar{e}_1|_p + r\cos\theta \bar{e}_2|_p).$$

Let $p = (0,1)$ and $q = (1,1)$. Under the identification corresponding to polar coordinates $\frac{\partial}{\partial \theta}|_p = -\frac{\partial}{\partial x^1}|_p$ is identified with

$$\left(\frac{\partial}{\partial \theta}\right)_q = -\sqrt{2} \cdot \tfrac{1}{2}\sqrt{2} \frac{\partial}{\partial x^1}\Big|_q + \sqrt{2} \cdot \tfrac{1}{2}\sqrt{2} \frac{\partial}{\partial x^2}\Big|_q;$$

(i.e. $-\bar{e}_1|_p$ is identified with $-\bar{e}_1|_q + \bar{e}_2|_q$, instead of with $-\bar{e}_1|_q$).

Let $X(p)$ be a vector field which is expressed in Cartesian coordinates as

$$X(p) = a^1(x^1, x^2) \frac{\partial}{\partial x^1}\Big|_p + a^2(x^1, x^2) \frac{\partial}{\partial x^2}\Big|_p$$

with differentiable component functions $a^1(x^1, x^2)$ and $a^2(x^1, x^2)$. We have:

$$\frac{\partial}{\partial x^1}\Big|_p = \cos\theta \frac{\partial}{\partial r}\Big|_p - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\Big|_p, \quad \frac{\partial}{\partial x^2} = \sin\theta \frac{\partial}{\partial r}\Big|_p + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta}\Big|_p, \quad p \in U.$$

Then

$$X(p) = \left(a^1(r\cos\theta, r\sin\theta)\cdot\cos\theta + a^2(r\cos\theta, r\sin\theta)\cdot\sin\theta\right)\frac{\partial}{\partial r}\Big|_p$$
$$+\left(-a^1(r\cos\theta, r\sin\theta)\cdot\frac{\sin\theta}{r} + a^2(r\cos\theta, r\sin\theta)\frac{\cos\theta}{r}\right)\frac{\partial}{\partial\theta}\Big|_p$$
$$= b^1(r,\theta)\frac{\partial}{\partial r}\Big|_\theta + b^2(r,\theta)\frac{\partial}{\partial\theta}\Big|_p$$

where the new component functions

$$b^1(r,\theta) = a^1(r\cos\theta, r\sin\theta)\cdot\cos\theta + a^2(r\cos\theta, r\sin\theta)\cdot\sin\theta$$

and

$$b^2(r,\theta) = -\frac{1}{r}a^1(r\cos\theta, r\sin\theta)\cdot\sin\theta + \frac{1}{r}a^2(r\cos\theta, r\sin\theta)\cdot\cos\theta$$

are again seen to be differentiable.

Returning to the general situation, let M be a differentiable manifold and $p \in M$. Locally around p the manifold looks like an open domain U in a Euclidean space, hence it is easy to define continuity and differentiability of a vector field X near p. In fact, let (U, x) be a chart around p, which determines bases $\frac{\partial}{\partial x^1}\big|_q, \cdots, \frac{\partial}{\partial x^m}\big|_q$ for $M_q, q \in U$. Then $X(q) = \sum_{i=1}^m a^i(q)\frac{\partial}{\partial x^i}\big|_q$, and X is continuous (differentiable) in U if all the components functions $a^i(q)$ are, $i = 1, \cdots, m$. It follows from the above that this is independent of the coordinates $x = (x^1, \cdots, x^m)$ on U. Let $TM = \bigcup_{q\in M} M_q$ as before and let π be the projection from TM onto M which sends M_q to q. Then $\pi^{-1}(U) = \bigcup_{q\in U} M_q$ can be identified with $x(U) \times \mathbf{R}^m$ by a map ϕ_U which is defined as follows: Let $\bar{t} \in M_q$, $q \in U$, and expand $\bar{t}$ in the basis $\frac{\partial}{\partial x^1}\big|_q, \cdots, \frac{\partial}{\partial x^m}\big|_q$, i.e. $\bar{t} = \sum_{i=1}^m t^i \frac{\partial}{\partial x^i}\big|_q$. Then $\phi_U(\bar{t}) = (x^1(q), \cdots, x^m(q), t^1, \cdots, t^m)$. The tangent space $M_q = \pi^{-1}(q)$ is called the fibre over q in TM; thus the first m coordinates tells us which fibre $\bar{t}$ belongs to, while the last m coordinates "the fibre coordinates", are the components of $\bar{t}$ with respect to the basis in M_q determined by the coordinate system. Let (V, y) be another chart on M with $U \cap V \neq \varnothing$. Then the elements of $\pi^{-1}(U) \cap \pi^{-1}(V) = \bigcup_{q\in U\cap V} M_q$ are coordinatized in two ways: $\phi_V(\bar{t}) = (y^1(q), \cdots, y^m(q), s^1, \cdots, s^m)$ where $\bar{t} = \sum_{i=1}^m s^i \frac{\partial}{\partial y^i}\big|_q$. It follows that $s^i = \sum_{j=1}^m \frac{\partial y^i}{\partial x^j}(x(q))t^j$.

Hence the coordinate transformation formula in $\pi^{-1}(U) \cap \pi^{-1}(V)$ is given by

$$\phi_V \cdot \phi_U^{-1} : (x^1, \cdots, x^m, t^1, \cdots, t^m) \to (y^1(x^1, \cdots, x^m), \cdots, y^m$$

$$(x^1,\cdots,x^m),\sum_j \frac{\partial y^1}{\partial x^j}(x^1,\cdots,x^m)t^j,\cdots,\sum_j \frac{\partial y^m}{\partial x^j}(x^1,\cdots,x^m)t^j).$$

(Notice that the last m coordinates $s^i, i = 1,\cdots,m$ depend not only on the t^i, $i=1,\cdots,m$, but also on the point q with coordinates $(x^1,\cdots,x^m)$, because the matrix for the change of basis (the Jacobian) varies from point to point). It is clear that $\phi_V \cdot \phi_U^{-1}$ is a diffeomorphism from $x(U\cap V)\times \mathbf{R}^m$ to $y(U\cap V)\times \mathbf{R}^m$. A topology can be defined in TM by requiring the $\pi^{-1}(U)$ to be open subset with the topology defined by the maps ϕ_U, here U varies over an atlas of M. Since the coordinate transformation formulas $\phi_V\circ\phi_U^{-1}$ are diffeomorphisms, $\{(\pi^{-1}(U),\phi_U)\}$ defines a C^∞-atlas on TM. Hence TM is a differentiable manifold of dimension $2m$. The local expression of the projection π with respect to a chart $(\pi^{-1}(U),\phi_U)$ in TM and the chart (U,x) in M is given by $(x^1,\cdots,x^m,t^1,\cdots,t^m)\to(x^1,\cdots,x^m)$; hence π is differentiable.

A section of TM is a map s from M to TM such that $\pi\cdot s =$ identity map. This means that $\pi(s(p)) = p$, hence $s(p)\in M_p$; i.e. a section of TM is simply a vector field on M. By a section we will usually mean a differentiable section. With respect to the charts (U,x) and $(\pi^{-1}(U),\phi_U)$ the local expression of a section is given by

$$s(x^1,\cdots,x^m) = (x^1,\cdots,x^m,a^1(x^1,\cdots,x^m),\cdots,a^m(x^1,\cdots,x^m))$$

(i.e. $s(p) = \sum_{i=1}^m a^i \frac{\partial}{\partial x^i}\Big|_p$), hence the section is differentiable if and only if the functions a^i are differentiable, $i = 1,\cdots,m$.

The tangent bundle TM is called trivial if there is a diffeomorphism ϕ from TM to $M\times\mathbf{R}^m$ such that the restriction of ϕ to each fibre M_p is a linear isomorphism of M_p onto $\{p\}\times\mathbf{R}^m$. It is clear that if U is an open domain in $\mathbf{R}^m$, then TU is trivial. In general, it is an important problem to decide if TM is trivial for a manifold M. If TM is trivial there must exist a differentiable vector field on M which is not zero at any points of M. (Take $\phi^{-1}(\bar v)$ where $\bar v$ is a non-zero vector of $\mathbf{R}^m$). An important result in this course. says that there is no such vector field on S^2, hence TS^2 is non-trivial. In general TM is trivial if and only if there exist m differentiable vector fields $X_1,\cdots,X_m$ such that $X_1(p),\cdots,X_m(p)$ is a basis for M_p for each $p\in M$. (Problem 20b)). (Intuitively, we have then an identification of the $M_p's$ with $\mathbf{R}^m$ which varies "differentiably" from point to point and is globally defined for all of M).

We can sum up the data for the tangent bundle of a differentiable manifold M: TM is a differentiable manifold and the projection π is a differentiable mapping from TM onto M. Each fibre $M_p = \pi^{-1}(p)$ is a vector space. There is an atlas (U_α, x_α) on M and a trivialization ϕ_α of $\pi^{-1}(U_\alpha)$, i.e. a diffeomorphism from $\pi^{-1}(U_\alpha)$ to $x_\alpha(U_\alpha) \times \mathbf{R}^m$ such that the restriction $\phi_{\alpha,p}$ of ϕ_α to M_p is a linear isomorphism from M_p to $\{p\} \times \mathbf{R}^m, p \in U_\alpha$. Hence the following diagram commutes:

$$\begin{array}{ccc} \Pi^{-1}(U_\alpha) & \xrightarrow{\phi_\alpha} & x_\alpha(U_\alpha) \times \mathbf{R}^m \\ \pi \downarrow & & \pi_1 \downarrow \\ U_\alpha & \longrightarrow & x_\alpha(U_\alpha) \end{array}$$

where π_1 is projection on the first factor. For $p \in U_\alpha \cap U_\beta$ there are coordinate transformations $\phi_{\alpha\,\beta}(p)$ from β-coordinates to α-coordinates on $\mathbf{R}^m$ induced by $\phi_{\alpha,p} \cdot (\phi_{\beta,p})^{-1}$; these are simply the usual change of coordinates on the tangent space from β- to α-coordinates given by the Jacobian matrix $\left(\frac{\partial x_\alpha^i}{\partial x_\beta^j}\right)(p) = \phi_{\alpha,\beta}(p)$. Hence we have differentiable maps $\phi_{\alpha,\beta} : U_\alpha \cap U_\beta \to GL(\mathbf{R}^m)$.

Composition of Jacobian matrices shown that:

(i) $\phi_{\alpha,\beta} = (\phi_{\beta,\alpha})^{-1}$

(ii) $\phi_{\alpha,\gamma}(p) = \phi_{\alpha,\beta}(p) \cdot \phi_{\beta,\gamma}(p)$ for $p \in U_\alpha \cap U_\beta \cap U_\gamma$.

We recall that the manifold M could always be obtained by an identification process (se p. 13). We may also consider TM as obtained by an identification process. Let Y be the disjoint union of the $x_\alpha(U_\alpha) \times \mathbf{R}^m$. An element (u, s) in $x_\alpha(U_\alpha) \times \mathbf{R}^m$ is equivalent to an element (v, t) of $x_\beta(U_\beta) \times \mathbf{R}^m$ if $u \in x_\alpha(U_\alpha \cap U_\beta)$ and $v = x_\beta \cdot x_\alpha^{-1}(u), t = \phi_{\beta,\alpha}(x_\alpha^{-1}(u))(s)$.

The quotient space of Y under this equivalence relation can be identified with TM. Given a manifold M and an atlas $\{U_\alpha, x_\alpha\}$ we see that the maps $\phi_{\alpha,\beta} : U_\alpha \cap U_\beta \to GL(\mathbf{R}^m)$ are the essential data which determine the tangent bundle (the Jacobians tells us how to identify points in $x_\alpha(U_\alpha \cap U_\beta) \times \mathbf{R}^m$ with points in $x_\beta(U_\alpha \cap U_\beta) \times \mathbf{R}^m$ when we patch together the tangent bundle). See also Problem 31.

Let M and N be differentiable manifolds and $f \in C^\infty(M, M)$. Then there is a map f_* from TM to TN defined by $f_*|M_p = f_*(p)$. Let (U, x) be a chart around p and (V, y) a chart around $f(p)$ in N. Then the local expression of f_* around p is given by:

$$(x^1, \cdots, x^m, u^1, \cdots, u^m) \to (y^1, \cdots, y^n, v^1, \cdots, v^n)$$

where $y^i = y^i(x^1, \cdots, x^m)$, $i = 1, \cdots, n$ is the local expression of f around p and $v^i = \sum_{j=1}^{m} \frac{\partial y^i}{\partial x^j}(x^1, \cdots, x^m)u^j$. Hence f_* is a differentiable map from TM to TN which maps each fibre M_p linearly into $N_{f(p)}$.

We express this by saying that f_* is a "bundle map" from TM to TN. The following diagram commutes:

$$\begin{array}{ccc} TM & \xrightarrow{f_*} & TN \\ \downarrow \pi & & \downarrow \pi \\ M & \xrightarrow{f} & N \end{array}$$

Chapter 4

Oppgaver

4.1 Oppgaver

Oppgave 1

La U være et åpent område i $\mathbf{R}^m$ og la $f = (f^1, \cdots, f^n)$ være en funksjon fra U til $\mathbf{R}^n$. La $p \in U$.

Definisjon 1) f er deriverbar i p hvis det eksisterer en lineær avbildning $Df(p)$ fra $\mathbf{R}^m$ til $\mathbf{R}^n$ og en funksjon $\epsilon(\bar{v})$ fra en omegn $B_r(p) = \{p+\bar{v} | \|\bar{v}\| < r\}$ om p til $\mathbf{R}^n$ slik at $f(p+\bar{v}) = f(p) + Df(p)(\bar{v}) + \|\bar{v}\|\epsilon(\bar{v})$ der $\epsilon(\bar{v}) \to 0$ når $\bar{v} \to 0$. (1. ordens Taylor formel). $Df(p)$ kalles den (totalt) deriverte av f i p. Se Apostol II.8.18.

2. f er kontinuerlig deriverbar i p, følger det at f er deriverbar i p. (Apostol II, Th.8.7).

a) Vis at hvis f er kontinuerlig deriverbar i p, så er matrisen til $Df(p)$ lik Jacobimatrisen $\left(\frac{\partial f^i}{\partial x^j}\right)$.

b) La $L(\mathbf{R}^m, \mathbf{R}^n)$ være vektorrommet av lineære avbildninger fra $\mathbf{R}^m$ til $\mathbf{R}^n$. Dette er isomorft med vektorrommet av $m \times n$-matriser og derfor også med $\mathbf{R}^{mn}$.

La f være deriverbar for alle punkt i U. Definer funksjonen Df fra U til $L(\mathbf{R}^m, \mathbf{R}^n) : p \to Df(p)$. (Den "deriverte" til f). Vis at f er kontinuerlig deriverbar i p hvis og bare hvis Df er kontinuerlig i p. (Topologien i $L(\mathbf{R}^m, \mathbf{R}^n)$ er gitt ved identifikasjonen med $\mathbf{R}^{mn}$).

Oppgave 2

La $f : \mathbf{R}^3 \to \mathbf{R}^2$ og $g : \mathbf{R}^3 \to \mathbf{R}^3$ være gitt ved: $f(x,y,z) = (x^2 + y + z)\bar{\imath}$ $+2(x + y + z^2)\bar{\jmath}$, $g(u,v,w) = uv^2w^2\bar{\imath} + w^2 \sin \bar{\jmath} + u^2e^v\bar{k}$.

a) Regn ut Jacobimatrisene $Df(x,y)$ og $Dg(u,v,w)$.

b) Finn $h(u,v,w) = f(g(u,v,w))$.

c) Finn Jacobimatrisen $Dh(u,0,w)$.

Oppgave 3

a) La $\mathbf{R}$ være tallinjen. Definerer funksjonen $f(x) = x^3$ et (differensiabelt) kart på $\mathbf{R}$?

b) La A være området i (r,θ)-planet gitt ved $0 < r < 2$, $0 < \theta < 8$. Definerer ligningene $x = r\cos\theta$, $y = r\sin\theta$ et kart på A?

c) La U være første oktant i $\mathbf{R}^3$, dvs.

$$U = \{(x^1, x^2, x^3) | x^1, x^2, x^3 > 0\}.$$

La ρ, θ, ϕ være de vanlige sfæriske koordinater i U definert ved $x^1 = \rho \sin\phi\cos\theta, x^2 = \rho\sin\phi\sin\theta, x^3 = \rho\cos\phi$. Avgjør om disse definerer et kart på U (svaret skal begrunnes, men helst uten å finne ρ, ϕ, θ eksplisitt ved x, y, z). Kan sfæriske koordinater defineres slik at de bestemmer et kart på $\mathbf{R}^2 \setminus \{0\}$ (vi bruker en skråstrek for mengdeteorisk differens).

Oppgave 4

a) La U være et åpent område i $\mathbf{R}^n$ og la $p \in U$. Vis at det finnes åpne omegner V_1 og V_2 om p slik at $p \in V_1 \subseteq \overline{V_1} \subseteq V_2 \subseteq \overline{V_2} \subseteq U$ og en funksjon f i $C^1(U, \mathbf{R})$ slik at $f = 1$ på V_1 og $f = 0$ utenfor V_2.

b) Vis at $C^1(U, \mathbf{R})$ har uendelig dimensjon.

Oppgave 5

La avbildningen $f : \mathbf{R}^3 \sim \mathbf{R}^3$ være gitt ved $f(\bar{v}) = A\bar{v}$ der A er matrisen $\begin{pmatrix} 1 & 0 & a \\ 0 & 2 & -3 \\ 1 & 0 & 1 \end{pmatrix}$. Når definerer f et kart på $\mathbf{R}^3$? Hva er den deriverte av f i punktet $(1.0.2)$?

Oppgave 6

La $p, g \in \mathbf{R}^m$. La $\bar{t}_{p,q}$ være vektoren fra p til q, og la t_{pq} være translasjonsoperatoren $\bar{v} \to \bar{v}+\bar{t}_{pq}$. La T_{pq} være translasjonsoperatoren på $C^\infty(\mathbf{R}^m)$ gitt ved $T_{pq}(f) = f \circ t_{pg}$ (dvs. $T_{pq}(f)(\bar{v}) = f(\bar{v}+\bar{t}_{pq})$. La V_p og V_q være vektorrommene av lokale derivasjoner i p og q henholdsvis. Definér $\tilde{T}_{pq}(X_p) = X_p \circ T_{pq}$ (dvs. $\tilde{T}_{pq}(X_p)(f) = X_p(f \circ t_{pq})$). Vis at $\tilde{T}_{pq}$ er en lineær isomorfi av vektorrommet V_p på V_q.

Oppgave 7

La V og W være vektorrom av endelig dimensjon. La B_1 og B_2 være to basiser for V, la C_1 og C_2 være to basiser for W. La $B_1^{B_2}$ og $C_1^{C_2}$ være matrisene for basisskifte fra B_1 til B_2 og fra C_1 til C_2. La F være en lineær avbildning fra V til W, og la $F_{B_i}^{C_i}$ være matrisen for F m.h.t. basis B_i i V og basis C_i i W. Hva blir matrisen $F_{B_2}^{C_2}$ uttrykt ved $F_{B_1}^{C_1}$?

Oppgave 8

La U være et åpent område i $\mathbf{R}^m$ og la $f \in C^1(U, \mathbf{R}^m)$. La $p \in U$ og $q = f(p)$. La V_p være vektorrommet av lokale derivasjoner i p og W_q vektorrommet av lokale derivasjoner i q. Vi definerer en avbildning $f_*(p)$ på V_p ved $f_*(p)(x)(g) = x(g \circ f)$ for $x \in V_p$ og $g \in C^\infty(\mathbf{R}^m, \mathbf{R})$.

a) Vis at f_* er en lineær avbildning fra V_p til W_q (sml. også oppgave 6).

b) La $(x^1, \cdots, x^m)$ og $(u^1, \cdots, u^n)$ være koordinatene på U og $\mathbf{R}^n$. Vi bruker de naturlige basiser $B_1 = \left(\frac{\partial}{\partial x^1}, \cdots, \frac{\partial}{\partial x^m}\right)$ og $C_1 = \left(\frac{\partial}{\partial u^1}, \cdots, \frac{\partial}{\partial u^n}\right)$ for V_p og W_q. Hva blir matrisen for $f_*(p)$? Sammenlign med matrisen for $Df(p)$.

c) La $(y^1, \cdots, y^n)$ være et kart på U og la $(v^1, \cdots, v^n)$ være et kart på en åpen omegn om q i $\mathbf{R}^n$. Vis at de lokale derivasjoner $\left(\frac{\partial}{\partial y^1}, \cdots, \frac{\partial}{\partial y^m}\right)$ danner en basis B_2 for V_p (og tilsvarende: $\left(\frac{\partial}{\partial v^1}, \cdots, \frac{\partial}{\partial v^n}\right)$ danner en basis C_2 for W_q). Hva blir matrisene $B_1^{B_2}$ og $C_1^{C_2}$?

d) Finn $f_*\left(\frac{\partial}{\partial y^i}\right)$ uttrykt ved $\left(\frac{\partial}{\partial v^1}, \cdots, \frac{\partial}{\partial v^n}\right)$ og derav matrisen $f_*(p)_{B_2}^{C_2}$. Verifiser resultatet i oppgave 7.

e) La T_pU være vektorrommet av vektorer med fotpunkt i p. Gi en fortolkning av den basis som svarer til B_2 under den naturlige identifikasjon $T_pU \cong V_p$.

Oppgave 9

La S^2 være kuleflaten med radius 1 i $\mathbf{R}^3$. La $u(v)$ være stereografisk projeksjon fra nordpol (henholdsvis sydpol). Finn de inverse avbildninger fra $\mathbf{R}^2$ til S^2. Bruk dette til å finne koordinatovergangsformlene vu^{-1} på $\mathbf{R}^2$ ved skifte av koordinatsystem.

Oppgave 10

La $f \in C^1(\mathbf{R}^2, \mathbf{R})$. Vis at hvis f er injektiv, må $\frac{\partial f}{\partial x^1}$ være forskjellige fra 0 i minst ett punkt. Definer $g(x^1, x^2) = (f(x^1, x^2), x^2)$. Finn Jacobideterminanten til g, og vis så at f ikke kan være injektiv.

Oppgave 11. (Det implisitte funksjonsteorem)

La U være et åpent område i $\mathbf{R}^m \times \mathbf{R}^n$ og anta $f \in C^1(U, \mathbf{R}^n)$. La (a, b) være et punkt i U slik at $f(a, b) = 0$. La M være $(n \times n)$-matrisen $\left(\frac{\partial f^i}{\partial x^{j+m}}(a,b)\right) 1 \leq i, j \leq n$, og anta at determinanten til M er forskjellig fra null. Definer $F : U \to \mathbf{R}^m \times \mathbf{R}n$ ved $F(x, y) = (x, f(x, y))$. Vis at determinanten til den deriverte avbildningen i (a, b); $DF(a, b)$, er forskjellig fra null. Herav følger at det finnes en åpen omegn V om (a, b) i $\mathbf{R}^m \times \mathbf{R}^n$ og en åpen omegn W om $F(a, b) = (a, 0)$ i $\mathbf{R}^m \times \mathbf{R}^n$ slik at $F : V \to W$ har en invers avbildning $h : W \to V$ som også er en C^1-avbildning. (Hvorfor?). Vis at h er av formen $h(x, y) = (x, k(x, y))$ der $k \in C^1(W, \mathbf{R}^n)$. Vis at funksjonen $g(x) = k(x, 0)$ definerer en C^1-funksjon fra en åpen omegn om a i $\mathbf{R}^m$ slik at $f(x, g(x)) = 0$ (dvs. $f(x, y) = 0$ definerer y implisitt som en C^1-funksjon av x i en omegn om $x = a$) Anta at $n = 1$. Finn $\frac{\partial g}{\partial x^j}(a)$.

Oppgave 12

La $A \subseteq \mathbf{R}^m$ være et åpent rektangel og la $f : A \to \mathbf{R}^m$ være en C^1-avbildning slik at $|\frac{\partial f^i}{\partial x^j}| \leq M$ på A for $1 \leq i, j \leq m$. Vis at da er $|f(x) - f(y)| \leq m^2 M |x - y|$ for alle $x, y \in A$.

Oppgave 13

a) La S^2 være kuleflaten i $\mathbf{R}^3$ med radius 1 og la U være den del av S^2 som ligger i første oktant, dvs. $U = \{(x^1, x^2, x^3) \in S^2 \mid x^1, x^2, x^3 > 0\}$. La kartet $\Phi_1 = (\phi, \theta)$ på U være definert ved sfæriske koordinater, dvs. $x^1 = \sin\phi\cos\theta, x^2 = \sin\phi\sin\theta, x^3 = \cos\phi$. La T_pU være tangentrommet til S^2 i p (betraktet som et rom av vektorer i $\mathbf{R}^3$ med fotpunkt i p). Finn den basis B_1 for

T_pU som defineres av kartet Φ_1 (dvs. av koordinatkurvene i Φ_1). Basisvektorene angis som vektorer i $\mathbf{R}^3$. La så kartet Φ_2 på U være definert ved koordinatene (x^1, x^2). Finn den basis B_2 for T_pU som defineres av Φ_2, og matrisen $B_1^{B_2}$ for basisskiftet i T_pU.

b) La U_p være vektorrommet av lokale derivasjoner i p. La B og B^1 være basisene $\left(\frac{\partial}{\partial\phi}, \frac{\partial}{\partial\theta}\right)$ og $\left(\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}\right)$ i U_p. Verifiser at matrisen for basisskiftet i U_p : B^{B^1} blir den samme som $B_1^{B_2}$ (og altså er lik Jacobimatrisen for koordinatovergangen fra (ϕ, θ) til (x^1, x^2)).

c) La V være den delen av den åpne enhetsdisken i (x^1, x^2)-planet som ligger i første kvadrant, dvs.

$$V = \{(x^1, x^2) | (x^1)^2 + (x^2)^2 < 1; x^1, x^2 > 0\}.$$

La kartet $\Psi_1 = (x^1, x^2)$ på V være gitt ved de vanlige kartesiske koordinater, og la C_1 være den tilhørende basis for T_qV. La kartet $\Psi_2 = (r, \alpha)$ være gitt ved polarkoordinater, dvs. $x^1 = r\cos\alpha, x^2 = r\sin\alpha$ og la C_2 være den tilhørende basis for T_qV. Finn matrisen $C_1^{C_2}$ for basisskiftet i T_qV.

d) Vi definerer en avbildning f fra U til V ved å ta projeksjonen av et punkt i (x^1, x^2)-planet, dvs. $f((x^1, x^2, x^3)) = (x^1, x^2)$. M.h.t. kartene Φ_i og ψ_i får f lokale uttrykk. $f_i : \Phi_i(U) \to \psi_i(V), i = 1, 2$. La $f_*(p)_{B_i}^{C_i}$ være matrisen for $Df_j(\Phi_i(p))$. Vis at $f_*(p)_{B_2}^{C_2} = C_1^{C_2} \circ f_*(p)_{B_1}^{C_1} \circ (B_1^{B_2})^{-1}$.

e) Hva blir $f_*(p)_{B_2}^{C_2}$?

Oppgave 14

Vis at funksjonene

$$g_1(x) = \begin{matrix} e^{\frac{1}{x}} & \text{for} & x > 0 \\ 0 & \text{for} & x \leq 0 \end{matrix}$$

$$g_2(x) = \frac{g_1(x)}{g_1(x) + g_1(1-x)}$$

og

$$g_3(x) = g_2(x+2)g_2(2-x)$$

tilhører $C^\infty(\mathbf{R}; \mathbf{R})$. La M være en C^∞-mangfoldighet, la $p \in M$ og la U være en åpen omegn om p. Vis at det finnes omegner V_1 og V_2 om p og en funksjon $f \in C^\infty(M, \mathbf{R})$ slik at $p \in V_1 \subseteq \bar{V}_1 \subseteq V_2 \subseteq \bar{V}_2 \subseteq U$ og $f = 1$ på $V_1, f = 0$ utenfor V_2. Vis at $C^\infty(M, \mathbf{R})$ har uendelig dimensjon.

Oppgave 15

a) La $f \in C^k(\mathbf{R}^m, \mathbf{R}^n)$, $1 \le k \le \infty$. Anta at rangen av $Df(p)$ er k for et punkt p i $\mathbf{R}^m$. La (U, u) være et kart rundt p (der $u = (u^1, \cdots, u^m)$ og la (V, y) være et kart rundt $q = f(p), y = (y^1, \cdots, y^n)$). Vis at ved evt. å permutere koordinatene u^i og y^i kan vi anta at $\left|\frac{\partial(y^i \circ f)}{\partial u^j}(p)\right| \neq 0$ for $i, j = 1, \cdots, k$. Definer funksjonene $x^i = y^i \circ f$ for $i = 1, \cdots, k$ og $x^r = u^r$ for $r = k+1, \cdots, n$ på U. Vis at funksjonene definerer et C^k-kart i en omegn om p. Vis at f's lokale uttrykk rundt p m.h.p. kartene $x = (x^1, \ldots, x^n)$ og $y = (y^1, \ldots, y^n)$ er gitt ved $(a^1, \ldots, a^m) \to (a^1, \ldots, a^k, \psi^{k+1}(a^1, \ldots, a^m), \ldots, \psi^n(a^1, \ldots, a^m))$.

b) (Rangteoremet). Anta at Df har konstant rang k i en omegn om p. La $x = (x^1, \ldots, x^m)$ og $v = (v^1, \ldots, v^n)$ være kart som i a), dvs. slik at f's lokale uttrykk i en omegn om p er:

$$(a^1, \ldots, a^m) \to (a^1, \ldots, a^k, \psi^{k+1}(a^1, \ldots, a^m), \ldots, \psi^n(a^1, \ldots, a^m)).$$

Betrakt Jacobimatrisen

$$\frac{\partial(v^i \circ f)}{\partial x^j} = \begin{pmatrix} I & 0 \\ * & (\frac{\partial \psi^i}{\partial x^j}, k+1 \le i, j \le n) \end{pmatrix}$$

og vis at matrisen $(\frac{\partial \psi^i}{\partial x^j}, k+1 \le i, j \le n)$ er identisk null i en omegn om p. Derfor avhenger funksjonene $\psi^{k+1}, \ldots, \psi^n$ bare av $(x^1, \ldots, x^k)$, dvs. vi har $\psi^r(a) = \bar{\psi}^r(a^1, \ldots, a^k)$, $r = k+1, \ldots, m$. Definer

$$y^i = v^i \quad \text{for} \quad i = 1, \ldots, k,$$
$$y^r = v^r - \bar{\psi}^r \circ (v^1, \ldots, v^k) \quad \text{for} \quad r = k+1, \ldots, n.$$

Vis at $y = (y^1, \ldots, y^n)$ definerer et kart rundt $f(p)$. Vis at det lokale uttrykk til f rundt p m.h.p. kartene $x = (x^1, \ldots, x^m)$ og $y = (y^1, \ldots, y^n)$ blir: $(a^1, \ldots, a^m) \to (a^1, \ldots, a^k, 0, 0, \ldots, 0)$.

c) Vis at det inverse funksjonsteorem er et spesialtilfelle av rangteoremet.

Oppgave 16

La $A \subseteq \mathbf{R}^m$. A har mål 0 hvis det for ethvert $\varepsilon > 0$ eksisterer en følge $B_1, \ldots, B_n, \ldots$ av rektangler som overdekker A og slik at $\sum_{n=1}^{\infty} v(B_n) < \varepsilon$, der $v(B_n)$ er volumet av B_n. Vis at en tellbar union av mengder av mål 0 har mål 0. Vis at hvis $f \in C^1(\mathbf{R}^n, \mathbf{R}^n)$ og $A \subseteq \mathbf{R}^n$ har mål 0, så har $f(A)$ mål 0. (Bruk oppgave 12.)

Oppgave 17

La $\mathfrak{gl}(\mathbf{R}^n)$ være vektorrommet av $n \times n$-matriser. La $A \in \mathfrak{gl}(\mathbf{R}^n)$. Vis at rekken $I + A + \frac{1}{2!}A^2 + \cdots + \frac{1}{n!}A^n + \cdots$ konvergerer mot en matrise som vi kaller $\exp(A)$. Bruk binominialformelen for $(A+B)^k$ til å vise at $\exp(A+B) = \exp A \exp B$ hvis A og B kommuterer, og slutt herav at determinanten til $\exp A$ alltid er forskjellig fra 0.

La $A \in \mathfrak{gl}(\mathbf{R}^n)$. Da er $\exp((t_1+t_2)A) = (\exp t_1 A)(\exp t_2 A)$. $\{(\exp tA) | t \in R\}$ kalles en én-parameter gruppe av matriser, og blir en C^∞-kurve i $\mathfrak{gl}(\mathbf{R}^n)$ med fotpunkt i ℓ (= identitetsmatrisen). Finn den deriverte avbildning $\exp_*(0)$.

Vis at $\exp^{-1}$ definerer et kart i en omegn om I i $GL(\mathbf{R}^n)$. La $T \in GL(\mathbf{R}^n)$, og la L_T betegne venstre-translasjon med T i $GL(\mathbf{R}^n)$,dvs. $L_T(B) = T \circ B$. Vis at $L_T \in C^\infty(GL(\mathbf{R}^n), GL(\mathbf{R}^n))$.

Vis at $\exp^{-1} \cdot L_T^{-1}$ definerer et kart rundt T i $GL(\mathbf{R}^n)$.

Oppgave 18

a) Hvis M er en kompakt, differensiabel mangfoldighet, må et atlas for M alltid ha mer enn ett kart.

b) Hvis M og N er diffeomorfe sammenhengende differensiable mangfoldigheter, må de ha samme dimensjon.

c) Hvis M er en sammenhengende, differensiabel mangfoldighet og p, q er to punkter i M, da eksisterer en differensiabel enkel kurve $c : \langle -1, +1 \rangle \to M$ slik at $c(0) = p$. $c\left(\frac{1}{2}\right) = q$.

Oppgave 19

a) La V være et endeligdimensjonalt vektorrom og la V^* være det duale vektorrom til V. La B_1 og B_2 være to basiser for V og la B_1^* og B_2^* være de tilsvarende duale basiser for V^*. Vis at matrisen for basis-skiftet $(B_1^*)^{B_2^*}$ blir den inverse av den transponerte til matrisen $B_1^{B_2}$.

b) La $f : V \to W$ være en lineær avbildning fra V inn i et endeligdimensjonalt vektorrom W. Den adjungerte avbildning $f^* : W^* \to V^*$ er definert ved $\langle v, f^* w^* \rangle = \langle fv, w^* \rangle$ når $v \in V$, $w^* \in W^*$. La B_1 og C_1 være basiser for V og W og la B_1^* og C_1^* være de tilsvarende duale basiser for V^* og W^*. Hva blir matrisen $(f^*)_{C_1^*}^{B_1^*}$ uttrykt ved matrisen $f_{B_1}^{C_1}$?

Oppgave 20

a) Vis at enhver vektorbunt har et tversnitt (med tversnitt menes *differensiabelt* tversnitt hvis intet annet er presisert).

b) La E være en vektorbunt over en differensiabel mangfoldighet M med fiber $\mathbf{R}^K$. E kalles triviell hvis det finnes en buntisomorfi fra E til den trivielle vektorbunten $M \times \mathbf{R}^K$ som induserer identitetsavbildningen på M. Vis at E er triviell hvis og bare hvis det finnes K seksjoner $s_1, \dots, s_K$ i E slik at $s_1(p), \dots, s_K(p)$ danner en basis for fiberen E_p over hvert punkt $p \in M$.

c) M kalles parallelliserbar hvis tangentbunten TM er triviell. Vis at sirkelen og torusen er parallelliserbaremangfoldigheter.

d) La $M = S^2 \subseteq \mathbf{R}^3$ og anta $p \in S^2$. Da kan tangentrommet til M i p identifiseres med et to-dimensjonalt underrom av $T_p(\mathbf{R}^3)$ (vektorrommet av vektorer med fotpunkt i p). La N_p være det ortogonale komplement til M_p i $T_p(\mathbf{R}^3)$ (dvs. N_p er det vektorrom som utspennes av normalen til flaten S^2 i punktet p). La $N = \bigcup_{p \in S} N_p$. Vis at N blir en vektorbunt over $M = S^2$. (Normalbunten til S^2 i $\mathbf{R}^3$). Vis at dette blir en triviell vektorbunt.

Oppgave 21

La P^n være mengden av linjer gjennom origo i $\mathbf{R}^{n+1}$ (dvs. P^n er det projektive n-rom). La H_i være hyperplanet $\{(x^1, \dots, x^{n+1}) | x^i = 0\}$ i $\mathbf{R}^{n+1}$ og la U_i være mengden av linjer gjennom origo i $\mathbf{R}^{n+1}$ som ikke ligger i H_i. Vis at $\{U_i, i = 1, \dots, n+1\}$ danner en overdekning av P^n. Innfør koordinater på U_i som følger: la p være et punkt i U_i, dvs. p er en linje som genereres av en vektor $(x^1, \dots, x^{n+1})$ der $x^i \neq 0$. Siden $x^i \neq 0$ kan vi normalisere koordinatene ved å multiplisere med skalaren $\frac{1}{x^i}$, og setter:

$$x_i^j(p) = \frac{x^j}{x^i} \quad \text{for} \quad j = 1, \dots, i-1$$

$$x_i^j(p) = \frac{x^j}{x^i} \qquad \text{for} \quad j = i, \dots, n$$

Da definerer $(U_i, x_i^1, \dots, x_i^n)$ et kart på P^n. Finn koordinatovergangsfunksjonene på $U_i \cap U_j$ og vis at kartene definerer et C^∞-atlas på P^n.

Oppgave 22

La M være en sammenhengende differensiabel mangfoldighet av dimensjon m. La $A \subseteq M$ og la (U_α, x_α) være et atlas for M. A sies å ha mål null hvis $x_\alpha(A \cap U_\alpha)$ har mål null i $\mathbf{R}^m$ for alle α. Vis at dette blir uavhengig av hvilket atlas som benyttes (bruk oppgave 16). Vis at hvis A har mål null, er komplementet til A tett i M. La N være en sammenhengende differensiabel mangfoldighet av dimensjon $n \geq m$ og la $f \in C^1(M, N)$. Vis at hvis Λ har mål null i M, så har $f(A)$ mål null i N (sml. oppgave 16).

Oppgave 23

La i være inklusjonsavbildingen fra sfæren S^2 inn i $\mathbf{R}^3$, og la (x^1, x^2, x^3) være de Euklidske koordinater på $\mathbf{R}^3$. Vis at $i^*(dx^1)$ og $i^*(dx^2)$ danner en basis for kotangentrommet til S^2 for hvert punkt på S^2 i det øvre halvplan $(x^3 > 0)$. Finn $i^*(dx^3)$ uttrykt i denne basisen.

Oppgave 24

La det være determinantavbildingen fra $GL(\mathbf{R}^m)$ inn i $\mathbf{R}$. Vis at $\det_*(I) : \mathfrak{gl}(\mathbf{R}^m) \to \mathbf{R}$ er den lineære avbildingen gitt ved $A \to trA = a_1^1 + \cdots + a_m^m$, der I er identitetsmatrisen og $A = (a_j^i)$. La $B \in GL(\mathbf{R}^m)$, da er det $(B \cdot C) = \det B \cdot \det C$. Vis at den lineære avbildning $\det_*(B)$ har rang 1 for enhver $B \in GL(\mathbf{R}^m)$. La $SL(\mathbf{R}^m) = \{A \in GL(\mathbf{R}^m) | \det A = 1\}$. Forklar hvorfor $SL(\mathbf{R}^m)$ blir en undermangfoldighet av $GL(\mathbf{R}^m)$.

Oppgave 25

La M være en differensiabel mangfoldighet og la $f \in C^\infty(M, \mathbf{R})$. Vis at den deriverte av f i et punkt p, $f_*(p)$ er det samme som differensialet av f i p, df_p, når vi identifiserer kotangentrommet til M i p med det duale til tangentrommet i p.

Oppgave 26

La V være et vektorrom av dimensjon m. La $B(V)$ være mengden av basiser i V. Vis at ethvert element B i $B(V)$ (dvs. B er en basis i V) definerer en bijeksjon $X_B : C \to C^B$ fra $B(V)$ på $GL(\mathbf{R}^m)$. Vis at koordinatovergangsformelen $X_{B'} \cdot (X_B)^{-1}$ er gitt ved venstremultiplikasjon med matrisen $B^{B'}$ på $GL(\mathbf{R}^m)$, og at $B(V)$ blir en m^2-dimensjonal differensiabel mangfoldighet med $\{B(V), X_B\}_{B \in B(V)}$ som et atlas.

Oppgave 27

a) La M være en differensiabel mangfoldighet, og la $B(M) = \bigcup_{p \in M} B(M_p)$. La projeksjonen $\pi : B(M) \to M$ være definert ved at $\pi(B(M_p)) = p$. Vis at et kart (U, X) på M definerer en bijeksjon ϕ_U fra $\pi^{-1}(U)$ til $X(U) \times GL(\mathbf{R}^m)$. (Vink: Kartet definerer en basis $\left(\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^m}\right)$ i hver fiber M_p, dvs. et element i $B(M_p)$. Bruk så oppgave 26). La (V, y) være et annet kart, og la $p \in U \cap V$. Finn koordinatovergangsformelene $\phi_v \cdot \phi_u^{-1}$ fra $X(U \cap V) \times GL(\mathbf{R}^m)$ til $y(U \cap V) \times GL(\mathbf{R}^m)$. Vis at $B(M)$ blir en differensiabel mangfoldighet og $\pi : B(M) \to M$ en differensiabel avbildning. $B(M)$ er en "fiberbunt" over M med totalrom $B(M)$ og fiber $GL(\mathbf{R}^m)$. (Fiberen er altså her ikke et vektorrom). $B(M)$ kalles basisbunten til TM.

b) På fiberen $GL(\mathbf{R}^m)$ ovenfor er koordinatovergangene gitt ved venstremultiplikasjon med Jacobi-matrisen $\left(\frac{\partial y^i}{\partial x^j}\right)$. Det var disse matrisene som ga koordinatovergangene i tangentbunten også, men da skulle de virke som lineære operatorer på vektorrommet $\mathbf{R}^m$. $B(M)$ kalles prinsipalbunten assosiert til tangentbunten TM. Vis at en differensiabel seksjon i $B(M)$ definerer m differensiable seksjoner i TM. Vis at TM er triviell hvis og bare hvis det finnes en differensiabel seksjon i den assosierte prinsipalbunt $B(M)$. (Sml. oppgave 20).

Oppgave 28

La E være en orienterbar vektorbunt over en differensiabel mangfoldighet M. Vis at da blir den duale vektorbunten E^* også orienterbar.

Oppgave 29

La M være en differensiabel mangfoldighet. En (global) derivasjon av $C^\infty(M)$ er en lineær avbildning X fra $C^\infty(M)$ til $C^\infty(M)$ slik at $X(f \cdot g) = X(f)g + f \cdot X(g)$. Hvis X er en global derivasjon og $p \in M$, kan vi definere $X_p(f) = X(f)(p)$. Vis at X_p blir en lokal derivasjon av $C^\infty(M)$ i p. Vis at X på denne måten bestemmer et (differensiabelt) vektorfelt på M, og at omvendt ethvert differensiabelt vektorfelt på M bestemmer en derivasjon av $C^\infty(M)$.

Oppgave 30. (Sards teorem, lokal versjon)

La K være et lukket rektangel i $\mathbf{R}^m$ med sider av lengde 1. La V være en åpen omegn om K, og la $f \in C^1(V, \mathbf{R}^m)$. La C være mengden av kritiske punkt for f i K. Vis at $f(C)$ (de kritiske verdier for f) har mål null. Vink: La $f = (f^1, \ldots, f^m)$ og definer den affine avbildningen $f_x = (f_x^1, \cdots, f_x^m)$ ved $f_x(y) = f(x) + Df(x)(y - x)$ for $x \in K$. Bruk mellomverdisatsen til å estimere $f^i(y) - f_x^i(y)$. Bruk den uniforme kontinuiteten av de partielt deriverte $\frac{\partial f^i}{\partial x^j}$ til å vise at det til ethvert $\varepsilon > 0$ finnes et N slik at $\|f(y) - f(x)\| < \varepsilon\|x - y\|$ hvis x og y ligger i samme rektangel S, der S er ett av de N^m rektangler som fremkommer ved en partisjon av K i rektangler med side av lengde $\frac{1}{N}$. Vis at hvis $x \in C$, så er $f_x(K)$ inneholdt i et hyperplan i $\mathbf{R}^m$ gjennom $f(x)$. Vis at det finnes en konstant M uavhengig av S, slik at hvis S er et rektangel som ovenfor som inneholder et kritisk punkt for f, så er $f(S)$ inneholdt i en sylinder med volum $\leq M(\frac{1}{N})^m\varepsilon$. (Bruk oppgave 12).

Oppgave 31

La M være en differensiabel mangfoldighet og $\{U_i\}_{i\in I}$ et C^∞-atlas for M. Anta at det er gitt differensiable avbildninger $\phi_{ji} : U_i \cap U_j \to GL(\mathbf{R}^k)$ slik at

$$\phi_{kj}(x)\phi_{ji}(x) = \phi_{ki}(x) \qquad \text{for} \quad x \in U_i \cap U_j \cap U_k.$$

(Herav følger: $\phi_{ii}(x)=$ identitetsmatrisen, $\phi_{ij}(x) = \phi_{ji}(x)^{-1}$). La T være den disjunkte union av $U_i \times \mathbf{R}^k$, $i \in I$. Innfør en relasjon i T som følger: Hvis $(x, s) \in U_i \times \mathbf{R}^k$ og $(y, t) \in U_j \times \mathbf{R}^k$ så er $(x, s) \sim (y, t)$ hvis $x = y$ og $t = \phi_{ji}(x)s$. Vis at dette blir en ekvivalensrelasjon. La E være kvotientrommet $T/\sim$. (dvs. mengden av ekvivalensklasser). Vis at projeksjonene $\pi_i : U_i \times \mathbf{R}^k \to U_i$ definerer en avbildning $\pi : E \to M$. Vis at E blir en vektorbunt over M med π som projeksjon.

Oppgave 32

La V være et endelig-dimensjonalt vektorrom. Vis at det til enhver bilineær avbildning $f : V^* \times V^* \to \mathbf{R}$ svarer en entydig lineær avbildning $g : T^2(V) = V^* \otimes V^* \to \mathbf{R}$ slik at $f = g \cdot h$ der h er definert ved $h(v^*, w^*) = v^* \times w^*$.

Oppgave 33

Anta at dim $V > 1$. Vis at ikke alle element i $V^* \otimes V^*$ kan skrives som $v^* \otimes w^*$ der v^* og w^* er element i V^*.

Oppgave 34. (Sards teorem)

La M og N være sammenhengende differensiable mangfoldigheter og la $f \in C^1(M, N)$. Anta at $\dim N = \dim M$. Vis at da danner de kritiske verdier for f en mengde av mål null i N. (Du har lov til å bruke at en sammenhengende differensiabel mangfoldighet kan overdekkes med et tellbart antall kartomegne Sml. oppg. 30.)

Oppgave 35

a) Gi et eksempel på vektorfelt X, Y slik at $X_p = 0$, men $[X, Y]_p \neq 0$.

b) La X, Y være vektorfelt på en differensiable mangfoldighet M og la $f, g \in C^\infty(M)$. Vis at $[fX, gY] = fg[X, Y] - f(Xg)Y + g(Yf)X$.

Oppgave 36

a) La $S^1 = \{X \pmod 1; X \in \mathbf{R}\}$ være sirkelen. Vis Kroneckers teorem: $\{nb \pmod 1; n \in \mathbf{N}\}$ er tett i S^1 hvis og bare hvis b er irrasjonal.

(Vink: Vis at for ethvert $\varepsilon > 0$, finnes tall n, m $(n \neq m)$ slik at $|(nb - mb) \pmod 1| < \varepsilon$).

b) Vi kan oppfatte torusen T^2 som R^2/Z^2. Avbildningen $t \to f(t) = (t, bt)$ definerer da en differensiabel avbildning $\bar{f}$ fra R inn i T^2. Vis at $\bar{f}$ blir en immersjon. Vis at bildet til $\bar{f}$ blir tett i T^2 hvis og bare hvis b er irrasjonal.

Oppgave 37

En vektor ω i $\Omega^k(V)$ er *dekomponerbar* hvis $\omega = \phi_1, \cdots, \phi_k$ der $\phi_i \in V^* = \Omega^1(V)$.

a) Anta $\dim V \leq 3$. Vis at enhver vektor i $\Omega^2(V)$ er dekomponerbar.

b) Anta $\phi_i, i = 1, 2, 3, 4$ er lineært uavhengige i V^*. Vis at $\omega = \phi_1 \wedge \phi_2 + \phi_3 \wedge \phi_4$ ikke er dekomponerbar. (Vink: betrakt $\omega \wedge \omega$).

Oppgave 38

Anta $\dim V = m$. La $T \in L(V, V)$. Den induserte lineære avbildning $\wedge^m(t) : \wedge^m(V) \to \wedge^m(V)$ er definert ved: $\wedge(T)(\phi_1 \wedge \cdots \wedge \phi_m) = T^*(\phi_1) \wedge \cdots \wedge T^*(\phi_m) \cdots (\phi_1, \cdots, \phi_m \in V^*)$. Innfør en basis i V, og finn sammenhengen mellom $\wedge^m(T)$ og $\det T$.

Oppgave 39

La M være en differensiabel mangfoldighet av dimensjon m. Vis at m funksjoner $f^1, \ldots, f^m$ i $C^\infty(M)$ definerer et kart i en omegn om p hvis og bare hvis $df_p^1 \wedge \cdots \wedge df_p^m \neq 0$.

Oppgave 40

Vis at de lineært uavhengige mengder av vektorer $\{v_1, \cdots, v_r\}$ og $\{w_1, \cdots, w_r\}$ er basiser for det samme r-dimensjonale underrom av V hvis og bare hvis $v_1 \wedge \cdots \wedge v_r = cw_1 \wedge \cdots \wedge w_r$ der $c \neq 0$.

Oppgave 41

La $\omega = \frac{1}{2}\sum_{i,j} a_{ij} dx^i \wedge dx^j$ der $a_{ij} + a_{ji} = 0$. Vis at $d\omega = 0$ hvis og bare hvis $\frac{\partial a_{ij}}{\partial x^k} + \frac{\partial a_{jk}}{\partial x^i} + \frac{\partial a_{ki}}{\partial x^j} = 0$ for alle i, j, k.

Oppgave 42

La M være en differensiabel mangfoldighet og ω en 1-form på M. Vis at 2-formen $d\omega$ tilfredsstiller:

$$d\omega(X,Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X,Y])$$

der X og Y er to vilkårlige vektorfelt på M.

Vink: Det er tilstrekkelig å vise dette i lokale koordinater, dvs.

$$\omega = \sum a_i dx^i, \quad X = \sum b^j \frac{\partial}{\partial x^j}, \quad Y = \sum c^k \frac{\partial}{\partial x^k}$$

Oppgave 43

La ω være en k-form på en differensiabel mangfoldighet M og la X være et vektorfelt på M. Vis at $L_X\omega$ (den Lie-deriverte av ω) er en k-form.

Oppgave 44

Bruk resultatet: $d^2 = 0$ og Poincaré's lemma til å vise de klassiske resultater fra vektoranalysen i $\mathbf{R}^3$:

1) curl grad $f = 0$ 2) div curl $X = 0$.