Exercise set 1: Basic properties of Brownian motion

Session 1: 2.3, 2.6, 2.7, 2.10, 2.14, 2.18, 2.19

Session 2:

Simple facts needed to solve these exercises:

- If $t \geq s$, then B_s and $B_t - B_s$ are independent.
- $\mathrm{E}[B_t^2] = t$ for all t.
- B_t is an $N(0,t)$ random variable.
- If X is $N(0,\sigma^2)$, then

$$\mathrm{E}[f(X)] = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} f(x) e^{-\frac{x^2}{2\sigma^2}}\, dx$$

Problem 1

Let $t \geq s$. Compute

$$\mathrm{E}[B_s^2 B_t - B_s^3].$$

Problem 2

Let $t \geq s$. Compute

$$\mathrm{E}[B_t B_s].$$

Hint: $B_t = B_t - B_s + B_s$.

Problem 3

Use $\mathrm{E}[B_r^2] = r$ for all r, to prove that

$$\mathrm{E}[(B_t - B_s)^2] = t - s.$$

Problem 4

Prove that

$$\mathrm{E}[B_r^4] = 3r^2$$

Hint: Use integration by parts to prove that $\mathrm{E}[B_r^4] = 3r\,\mathrm{E}[B_r^2]$.

Problem 5

Let $t \geq s$. Compute

$$\mathrm{E}[B_s^4 B_t^2 - 2B_t B_s^5 + B_s^6].$$

Problem 6

Let $0 \leq t_1 \leq t_2 \leq \cdots \leq t_N$. Prove that

$$\mathrm{E}\left[\left(\sum_{i=1}^{N} (B_{t_i+1} - B_{t_i})^2\right)^2\right] = \sum_{i=1}^{N} 2(\Delta t_i)^2 + t_N^2$$