

## Exercise set 2: Itô isometry

Session 1: 3.1, 3.2, 3.3, 3.4, 3.5, 3.8, 3.9, 3.18

Session 2:

Simple facts needed to solve these exercises:

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$$\mathbb{E} \left[ \left( \int_S^T f(t, \omega) dB_t(\omega) \right)^2 \right] = \int_S^T \mathbb{E}[f^2(t, \omega)] ds$$

- $\mathbb{E}[(\Delta B_i)^2] = \Delta t_i$  (Problem 6)
- $\mathbb{E}[(\Delta B_i)^4] = 3(\Delta t_i)^2$  (Problem 6)

### Problem 1

Compute

$$\mathbb{E} \left[ \left( \int_0^t (1 + B_s) dB_s \right)^2 \right].$$

### Problem 2

Compute

$$\mathbb{E} \left[ \left( \int_0^t B_s^2 dB_s \right)^2 \right].$$

### Problem 3

Compute

$$\mathbb{E} \left[ \left( \int_0^t \sin(B_s) dB_s \right)^2 + \left( \int_0^t \cos(B_s) dB_s \right)^2 \right].$$

### Problem 4

Compute

$$\mathbb{E} \left[ \int_0^t 1 dB_s \cdot \int_0^t B_s^2 dB_s \right].$$

Hint:  $x \cdot y = \frac{1}{4}((x+y)^2 - (x-y)^2)$

### Problem 5

Use the technique from Problem 4 to prove that if  $f, g \in \mathcal{V}[0, T]$ , then

$$\mathbb{E} \left[ \int_S^T f(t, \omega) dB_t(\omega) \cdot \int_S^T g(t, \omega) dB_t \right] = \int_S^T \mathbb{E}[f(t, \omega) \cdot g(t, \omega)] ds$$

**Problem 6\***

Assume that  $0 = t_1 \leq t_2 \leq \dots \leq t_N = T$ , and let  $\Delta B_j = B_{t_{j+1}} - B_{t_j}$ . Prove that

$$\mathbb{E} \left[ \left( t - \sum_{i=1}^N (\Delta B_i)^2 \right)^2 \right] = 2 \sum_{i=1}^N (\Delta t_i)^2$$

Hint: Use that  $\left( \sum_{i=1}^N a_i \right)^2 = \sum_{i,j=1}^N a_i a_j$ .