Exercise set 3: Itô's formula

Session 1: 4.1, 4.2, 4.3, 4.11, 4.13, 4.14

Session 2:

Simple facts needed to solve these exercises:

- $d(g(t,X_t)) = \frac{\partial g}{\partial t}(t,X_t)dt + \frac{\partial g}{\partial x}(t,X_t)dX_t + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(t,X_t)(dX_t)^2$
- $\mathrm{E}\left[\int_S^T f(s,\omega)dB_s\right] = 0$ for all $f \in \mathcal{V}$.

Problem 1

Let $g(t,x) = t\,x^2$, $X_t = B_t$. Let $Y_t = t\,B_t^2$, and use Itô's formula to compute dY_t. Write the answer on integral form.

Problem 2

Let $g(t,x) = e^{\alpha x}$, where α is a deterministic constant, and use Itô's formula to compute $de^{\alpha B_t}$.

Problem 3

Put $y(t) = \mathrm{E}[e^{\alpha B_t}]$, where α is a deterministic constant.

a) Prove that

$$y(t) = 1 + \frac{\alpha^2}{2}\int_0^t y(s)ds$$

Hint: Use the result in problem 2.

b) Show that

$$y' = \frac{\alpha^2}{2}y$$

and solve this differential equation to find a formula for $\mathrm{E}[e^{\alpha B_t}]$.

Problem 4

Let $X_t = (B_1(t), B_2(t), B_3(t))$. Compute

$$d(e^t B_1(t)B_2(t)B_3(t))$$

Problem 5

Let $X_t = (B_1(t), B_2(t))$. Compute

$$d(B_1^2 B_2^2)$$

Problem 6

Let $X_t = \begin{bmatrix} t \\ e^{2B_t} \end{bmatrix}$. Find u and v such that $dX_t = udt + vdB_t$.