Exercise set 5: Diffusions/Markov property

Session 1: 7.1, 7.2, 7.5, 7.9, 7.15

Session 2:

Problem 1

Consider the diffusion X_t given by

$$dX_t = 0dt + 1dB_t \qquad X_0 = x$$

i.e. $X_t = x + B_t$ (Brownian motion started at x).

a) Let $f(x) = x^2$ and compute for all x the value of the function $g(x) = \mathrm{E}^x[f(X_t)]$.

b) Let $h \geq 0$ be any given number, and compute for all x the conditional expectation

$$Y(\omega) = \mathrm{E}^x[f(X_{t+h})|\mathcal{F}_t]$$

c) Verify that $Y(\omega) = g(X_h)$ (which is the Markov property).

Problem 2

Consider the stochastic process $X_t = x\, e^{3t+2B_t}$. In this problem we will need to use the formula $\mathrm{E}[e^{\alpha B_t}] = e^{\frac{1}{2}\alpha^2 t}$ several times.

a) Use Itô's formula to prove that X_t is a diffusion.

b) Let $f(x) = x^r$ (where $r > 0$ is a constant) and compute for all x the value of the function $g(x) = \mathrm{E}^x[f(X_t)]$.

c) Let $h \geq 0$ be any given number, and compute for all x the conditional expectation

$$Y(\omega) = \mathrm{E}^x[f(X_{t+h})|\mathcal{F}_t]$$

d) Verify that $Y(\omega) = g(X_h)$ (which is the Markov property).

Problem 3

Consider the diffusion $X_t = x + B_t$ and the stopping time $\tau = \inf\{t > 0 | X_t = 0\}$.

a) Write down the generator A of X_t.

b) Let $f(x) = x$, and find the value of $\mathrm{E}^x[X_\tau]$.

c) Let $f(x) = x$, and find the value of $x + \mathrm{E}[\int_0^t Af(X_s)ds]$.

d) Compare the results in b) and c) with the result in Dynkin's formula. What can you conclude about $\mathrm{E}[\tau]$?