

## MAT4701 Voluntary assignment 1

Can be handed in for correction on the groups March 12.

### Problem 1

Let  $t_i = \frac{i}{n}$ ,  $i = 0, 1, \dots, n$ , i.e.,  $\Delta t_i = \frac{1}{n}$ .

a) For  $n \in \mathbb{N}$  compute

$$F(n) = \mathbb{E} \left[ \sum_{i,j=0}^{n-1} B_{t_i} (B_{t_{i+1}} - B_{t_i}) B_{t_j} (B_{t_{j+1}} - B_{t_j}) \right]$$

b) Simplify the answer in a) using  $\sum_{k=1}^N k = \frac{N(N+1)}{2}$  and compute  $\lim_{n \rightarrow \infty} F(n)$ .

c) Explain the result in b) in light of the Itô isometry.

### Problem 2

a) Use the formula  $\mathbb{E}[e^{\alpha B_t}] = e^{\frac{1}{2}\alpha^2 t}$ , that is valid for all  $\alpha$  in  $\mathbb{C}$ , and standard trigonometric formulas to compute

$$\text{i) } \mathbb{E}[\cos(\alpha B_t)] \quad \text{ii) } \mathbb{E}[\sin(\alpha B_t)] \quad \text{iii) } \mathbb{E}[\cos^2(\alpha B_t)] \quad \text{iv) } \mathbb{E}[\sin^2(\alpha B_t)]$$

b) Assume that  $t \geq s$ . Use a) to compute  $\mathbb{E}[\cos(B_t) \cos(B_s)]$ . Hint: Use a trigonometric formula and independence of increments.

c) Use b) to compute  $\mathbb{E}[(\int_0^t \cos(B_s) ds)^2]$ .

Hint: For any reasonable function  $f$ ,  $(\int_0^t f(s) ds)^2 = 2 \int_0^t \int_0^u f(u) f(v) dv du$ .

### Problem 3

Compute  $\mathbb{E}[\int_0^t B_s dB_s \cdot \int_0^t B_s^2 dB_s]$

### Problem 4

Let  $t \geq s$  and compute  $\mathbb{E}[(B_s + \sin(B_s))e^{2B_t} | \mathcal{F}_s]$ .

### Problem 5

Let  $Z_t = e^{\int_0^t B_s dB_s - \frac{1}{2} \int_0^t B_s^2 ds}$ .

a) Prove that  $dZ_t = Z_t \cdot B_t dB_t$  and use this to compute  $\mathbb{E}[Z_t]$ .

b) Prove that if  $t \geq 0$ , then  $|Z_t|^2 \leq e^{B_t^2 - t}$  for all  $\omega$ .

c) Use the result in b) to prove that if  $0 \leq t < \frac{1}{2}$ , then  $\mathbb{E}[Z_t^2 B_t^2] < \infty$ . Hint:  $B_t$  is Gaussian.

d) Why do we need to prove an inequality like the one in c), and what can we conclude from that?