

# SOLUTIONS TO MANDATORY ASSIGNMENT

①

48. This problem can be done entirely by elementary means. Part (3) and (4) of course also follow from the open mapping theorem. We use the identity

$$\frac{\partial}{\partial \bar{z}} \bar{f}(z) = \overline{\frac{\partial f}{\partial z}(z)}$$

which is valid for all  $C^1$  functions.

(1) Pick  $z_0 \in \Omega$ . If  $z_1 \in \Omega$ , there is a smooth path  $\gamma : [0, 1] \rightarrow \Omega$  from  $z_0$  to  $z_1$ , since  $\Omega$  is connected.  
If  $f'(z) \equiv 0$ , we have

$$\begin{aligned} 0 &= \int_{\gamma} f'(z) dz = \int_0^1 f'(\gamma(t)) \gamma'(t) dt = \int_0^1 \frac{d}{dt} (f(\gamma(t))) dt \\ &= f(\gamma(t)) \Big|_0^1 = f(z_1) - f(z_0) \end{aligned}$$

by the fundamental theorem of calculus.

Hence  $f(z_1) = f(z_0)$  and  $f$  is constant.

(2) We differentiate with respect to  $\bar{z}$ :

$$0 = \frac{\partial f}{\partial \bar{z}} = c \cdot \frac{\partial \bar{f}}{\partial \bar{z}} = c \overline{\frac{\partial f}{\partial z}}$$

Hence either  $c = 0$ , in which case  $f = 0$ , or

$$\frac{\partial f}{\partial z} = 0, \text{ so } f \text{ is constant by (1).}$$

(3) If  $f$  is real,  $f(z) = \bar{f}(\bar{z})$ , i.e. (2) holds with  $c = 1$ .

(4) is just a special case of (6).

(5) If  $f$  is not constant, then  $f'(z) \neq 0$  in some nonempty open set  $V$  and since  $g'(f(z)) \cdot f'(z) = 0$ ,  $g' = 0$  in the open set  $f(V)$ , hence  $g$  is constant.

$$(6) 0 = \frac{\partial^2}{\partial z \partial \bar{z}} \left( |f_1|^2 + \dots + |f_n|^2 \right) = \frac{\partial}{\partial z} \left( \sum_{j=1}^n f_j \bar{f}_j \right) = \frac{\partial}{\partial z} \left( \sum_{j=1}^n f_j \frac{\partial \bar{f}_j}{\partial \bar{z}} \right) \\ = \frac{\partial}{\partial z} \left( \sum_{j=1}^n f_j \bar{f}_j \frac{\partial}{\partial z} \right) = \sum_{j=1}^n \frac{\partial f_j}{\partial z} \frac{\partial \bar{f}_j}{\partial z} = \sum_{j=1}^n |f'_j(z)|^2, \approx$$

$f'_j(z) = 0$  for all  $j$  and  $z$  and  $f_j$  is constant.

112  $h(z) = \frac{f(z)}{P(z)}$  is holomorphic away from the zeroes of  $P$

and  $|h(z)| \leq C$ , hence the singularities are removable.  $h$  is therefore a bounded entire function. Hence constant by Liouville's theorem. Therefore  $f(z) = cP(z)$ .

This argument would work if  $P$  was an arbitrary entire function, since the zeroes of  $P$  are discrete in  $\mathbb{C}$ .

220. If  $e^f + e^g = 1$ , then  $e^f$  and  $e^g$  would both avoid the two values 0 and 1. By Picard's theorem, they are constant. But then  $f$  and  $g$  must be constant.

242 If  $\frac{\partial u}{\partial \bar{z}} = \varphi$  and  $u$  has compact support, then for large  $R$  we get

$$0 = \int u dz = 2i \iint \frac{\partial u}{\partial \bar{z}} = 2i \iint \varphi dx dy = 2i \iint \varphi dx dy$$

Hence  $\iint_{\mathbb{C}} q dx dy = 0$ , which is not always true. Pick

for instance a non-negative, nonconstant  $q$ .

279. We have  $\hat{K}_e \subset \hat{L}_e$ . Also

$$\hat{K}_e = K \cup \bigcup U_\alpha$$

$U_\alpha$  connected component of  $\partial^1 K$ ,  
 $U_\alpha \subset \mathbb{C}$

This shows that  $\partial \hat{K}_e \subset \partial K$ . But  $\partial K \subset K \subset L^0 \subset (\hat{L}_e)^\circ$ .

297. If  $f: D \rightarrow \mathbb{C}$  is proper, then  $f^{-1}(0)$  is compact and discrete, hence a finite number of points  $a_1, \dots, a_n$ . If  $f$  has a zero of order  $m_j$  at  $a_j$ , then if  $P(z) = \prod_{j=1}^n (z - a_j)^{m_j}$ ,

$$h(z) = \frac{f(z)}{P(z)}$$

is holomorphic and nonzero in  $D$ . It is still proper, since  $P$  is bounded in  $D$ . Then  $1/h(z)$  is holomorphic and  $M_r = \sup \{ |1/h(z)|; |z| = r \}$  will tend to zero when  $r \rightarrow 1$ . This contradicts the maximum modulus theorem.