MANDATORY ASSIGNMENT MAT 4800.

Hand in by November 7th.

Exercise 48. For each function $f : \Omega \to \mathbb{C}$ holomorphic on a *connected* open set $\Omega \subseteq \mathbb{C}$ prove the following statements.

(48.1) If $f'(z) = 0$ for every $z \in \Omega$, then f is constant.

(48.2) If there exists $c \in \mathbb{C}$ such that $f(z) = c \cdot \overline{f(z)}$ for every $z \in \Omega$, then f is constant.

(48.3) If $f(\Omega) \subseteq \mathbb{R}$, then f is constant.

(48.4) If $|f|$ is constant, then f is constant.

(48.5) If $g : \mathbb{C} \to \mathbb{C}$ is holomorphic and $g \circ f$ is constant, then f or g is constant.

(48.6) If $f_1, \dots, f_N$ are holomorphic on Ω, and if $|f_1|^2 + \dots + |f_n|^2$ is constant, then each f_j is constant.

Exercise 112. Prove that if $P : \mathbb{C} \to \mathbb{C}$ is a polynomial, if $f : \mathbb{C} \to \mathbb{C}$ is holomorphic on all of $\mathbb{C}$, and if there exists a real $C > 0$ such that $|f(z)| \leq C \cdot |P(z)|$ for every $z \in \mathbb{C}$, then $f = c \cdot P$ for some $c \in \mathbb{C}$. Is there an analogous statement with P replaced by an arbitrary holomorphic function on all of $\mathbb{C}$?

Exercise 220. Prove that if f and g are entire and $e^f + e^g = 1$, then f and g are constant.

Exercise 242. Show that for some compactly supported differentiable function φ, *none* of the solutions u of $\partial_{\bar{z}} u = \varphi$ has a compact support.

Exercise 279. With the notation as in the text, prove that for each open subset $\Omega \subseteq \mathbb{C}$ and for all compact subsets K and L of Ω such that

$$K \subset L^\circ \subset \Omega,$$

the following inclusion holds:

$$\widehat{K}_\Omega \subset \left(\widehat{L}_\Omega\right)^\circ .$$

Exercise 297. Prove that there is *no* proper holomorphic map from the open unit disc into the complex plane.