

Some real analysis

σ and O -notation

Suppose f is defined in a nbhd of $0 \in \mathbb{R}^m$, $f: V \rightarrow \mathbb{R}^m$

$$f = o(|x|^k) \iff \lim_{x \rightarrow 0} \frac{|f(x)|}{|x|^k} = 0 \quad (k=0 \text{ is called } o(1))$$

$$f = O(|x|^k) \iff \exists C > 0 \text{ s.t. } |f(x)| \leq C|x|^k, \text{ } x \text{ small}$$

Def. f is differentiable at a if there is a linear map $L: \mathbb{R}^m \rightarrow \mathbb{R}^m$ such that

$$\lim_{x \rightarrow 0} \frac{|f(a+x) - f(a) - L(x)|}{|x|} = 0$$

\iff

$$f(a+x) = f(a) + L(x) + o(|x|)$$

- L is called the derivative of f at a and denoted df_a
- If f is differentiable at a , then the partial derivatives

$\frac{\partial f_j}{\partial x_i}(a)$ exist and

$$df_a(v) = \sum_{j=1}^m \left(\sum_{i=1}^n \frac{\partial f_j}{\partial x_i}(a) v_i \right) e_j$$

$$df = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \dots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1} & & \frac{\partial f_m}{\partial x_n} \end{pmatrix}$$

Jacobian matrix.

- If the partial derivatives $\frac{\partial f_j}{\partial x_i}$ exist in a nbhd of a and are continuous at a , then f is differentiable at a .

$$C(\Omega) = \{f: \Omega \rightarrow \mathbb{C}; f \text{ is continuous}\}$$

$$C^1(\Omega) = \{f: \Omega \rightarrow \mathbb{C}; \frac{\partial f}{\partial x_i} \in C(\Omega), i=1, \dots, m\}$$

$$C^k(\Omega) = \{f: \Omega \rightarrow \mathbb{C}; \text{all partial derivatives of order} \leq k \text{ are cont.}\}$$

Order does not matter.

$$\alpha = (\alpha_1, \dots, \alpha_m) \in \mathbb{N}^m \text{ multiindex}$$

$$|\alpha| = \alpha_1 + \dots + \alpha_m, \quad \text{order the multiindex}$$

$$D^\alpha f = \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \dots \partial x_m^{\alpha_m}}$$

$$C^\infty(\Omega) = \bigcap_k C^k(\Omega)$$

Complex function of a complex variable, $\Omega \subset \mathbb{C}$

$$f: \Omega \rightarrow \mathbb{C}, \quad z = x + iy, \quad f = u + iv$$

$$f(z) = f(x, y) = u(x, y) + i v(x, y)$$

As a real function $f: \underset{\mathbb{R}^2}{\Omega} \rightarrow \mathbb{R}^2, f = (u, v)$

Let $\lambda = \alpha + i\beta \in \mathbb{C} \cong \mathbb{R}^2$. What is $df(\lambda)$?

$$df(\lambda) = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \frac{\partial u}{\partial x} \alpha + \frac{\partial u}{\partial y} \beta \\ \frac{\partial v}{\partial x} \alpha + \frac{\partial v}{\partial y} \beta \end{pmatrix} = \left(\frac{\partial u}{\partial x} \alpha + \frac{\partial u}{\partial y} \beta \right) + i \left(\frac{\partial v}{\partial x} \alpha + \frac{\partial v}{\partial y} \beta \right) = \alpha \cdot \frac{\partial f}{\partial x} + \beta \cdot \frac{\partial f}{\partial y}$$

Want to express in terms of λ

$$\alpha = \text{Re } \lambda = \frac{1}{2}(\lambda + \bar{\lambda}) \quad \beta = \text{Im } \lambda = \frac{1}{2i}(\lambda - \bar{\lambda})$$

$$df(\lambda) = \frac{1}{2}(\lambda + \bar{\lambda}) \frac{\partial f}{\partial x} + \frac{1}{2i}(\lambda - \bar{\lambda}) \frac{\partial f}{\partial y}$$

$$= \frac{1}{2} \left(\frac{\partial f}{\partial x} - i \frac{\partial f}{\partial y} \right) \lambda + \frac{1}{2} \left(\frac{\partial f}{\partial x} + i \frac{\partial f}{\partial y} \right) \bar{\lambda} =: \frac{\partial f}{\partial z} \lambda + \frac{\partial f}{\partial \bar{z}} \bar{\lambda}$$

\mathbb{R} complex linear

$$L(c\lambda) = cL(\lambda)$$

Complex antilinear

$$L(c\lambda) = \bar{c}L(\lambda)$$

$$df \text{ is } \mathbb{C}\text{-linear} \Leftrightarrow \frac{\partial f}{\partial \bar{z}} = 0.$$

$-\frac{\partial f}{\partial \bar{z}} = 0$ is called the Cauchy-Riemann equations, i.e. $\frac{\partial f}{\partial x} = -i \frac{\partial f}{\partial y}$

Real form

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Exercise

a) Show that $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \bar{z}}$ satisfy Leibniz rule!

b) Suppose $L: \mathbb{C}^m \rightarrow \mathbb{C}^m$ is \mathbb{R} -linear. Show that

$$L \text{ is } \mathbb{C}\text{-linear} \Leftrightarrow L(i v) = i L(v) \quad \forall v \in \mathbb{C}^m.$$

$$L \text{ is } \mathbb{C}\text{-antilinear} \Leftrightarrow L(i v) = -i L(v) \quad \text{--- " ---}$$

c) Show that every \mathbb{R} linear $L: \mathbb{C}^m \rightarrow \mathbb{C}^m$ splits uniquely in a \mathbb{C} -linear and \mathbb{C} -antilinear part

$$L = L_{\mathbb{C}} + L_{\bar{\mathbb{C}}}$$

$$L_{\mathbb{C}}(v) = \frac{1}{2}(L(v) - i L(i v)), \quad L_{\bar{\mathbb{C}}}(v) = \frac{1}{2}(L(v) + i L(i v)).$$

Def. $f: \Omega \rightarrow \mathbb{C}$ is called \mathbb{C} -differentiable at a if

$$\lim_{\lambda \rightarrow 0} \frac{f(a+\lambda) - f(a)}{\lambda}$$

exists. This is denoted by $f'(a)$.

\Leftrightarrow

$$f(a+\lambda) = f(a) + f'(a)\lambda + o(|\lambda|)$$

f is \mathbb{C} -diff. at $a \Leftrightarrow f$ is differentiable at a and df_a is \mathbb{C} -linear

Def. Let Ω be an open subset of \mathbb{C} . We say that a complex function $f(z)$ defined in Ω is holomorphic if $f \in C^1(\Omega)$ and f is complex differentiable at all points in Ω , i.e. f satisfies the C-R equations.

- The set of holomorphic functions is denoted by $\mathcal{O}(\Omega)$
- It is not necessary to assume $f \in C^1(\Omega)$. (this follows automatically when f is \mathbb{C} -differentiable), but it makes things easier, because we can use Green's theorem in the plane.

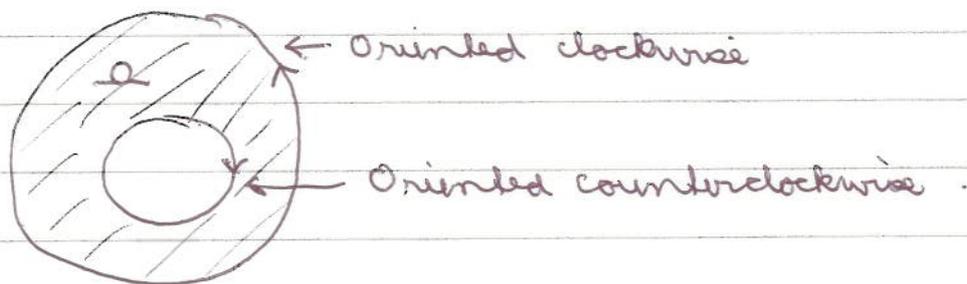
- Green's theorem in the plane

If $\Omega \subset \mathbb{R}^2$ is an open set with piecewise smooth boundary $\partial\Omega$ and M, N are two C^1 functions in $\bar{\Omega} = \Omega \cup \partial\Omega$, then

$$\int_{\partial\Omega} M dx + N dy = \iint_{\Omega} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy$$

Remarks:

1. $\partial\Omega$ is oriented such that Ω lies to the left of $\partial\Omega$.



2. It does not matter if M and N are real or complex valued.

3. $\int_{\partial\Omega} M dx + N dy$ is computed by parametrizing $\partial\Omega$

$(x(t), y(t))$, $a \leq t \leq b$. Then -

$$\int_{\partial \Omega} M dx + N dy = \int_a^b M(x(t), y(t)) x'(t) + N(x(t), y(t)) y'(t) dt$$

i.e. $dx = x'(t)dt$, $dy = y'(t)dt$.

- If $\gamma \subset \mathbb{C}$ is a curve parametrized by $z(t) = x(t) + iy(t)$, $a \leq t \leq b$

and f is a complex function on γ , then the complex line integral is defined by

$$\begin{aligned} \int_{\gamma} f(z) dz &= \int_a^b f(x(t) + iy(t)) z'(t) dt = \int_a^b f(x(t) + iy(t)) (x'(t) + iy'(t)) dt \\ &= \int_{\gamma} f dx + i f dy. \end{aligned}$$

If $\gamma = \partial \Omega$ as in Green's theorem, we get

$$\int_{\partial \Omega} f dz = \iint_{\Omega} \left(i \frac{\partial f}{\partial x} - \frac{\partial f}{\partial y} \right) dx dy = 2i \iint_{\Omega} \frac{\partial f}{\partial \bar{z}} dx dy$$

(Complex form of Green's theorem)

Remarks:

1. If f is holomorphic, we get Cauchy's theorem

$$\int_{\partial \Omega} f dz = 0$$

2. If γ is the circle $z = \zeta + re^{i\theta}$, then

$dz = ire^{i\theta} d\theta$ and

$$\int_{\gamma} \frac{f(z)}{z - \zeta} dz = \int_0^{2\pi} \frac{f(\zeta + re^{i\theta})}{re^{i\theta}} \cdot ire^{i\theta} d\theta = \int_0^{2\pi} i f(\zeta + re^{i\theta}) d\theta$$

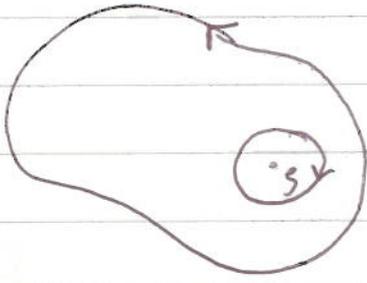
Cauchy - Stokes formula

Assume f is C^1 in $\bar{\Omega}$, as in Green's theorem and let $z \in \Omega$. For small r , let

$$\Omega_r = \Omega \setminus \bar{D}(z, r). \text{ Then } \partial\Omega_r = \partial\Omega \cup \partial D(z, r)$$

where $\partial D(z, r)$ is oriented counterclockwise.

Applying the complex form of Green's theorem to $\frac{f(z)}{z-z}$ in Ω_r , we get



$$\int_{\partial\Omega_r} \frac{f(z)}{z-z} dz - i \int_0^{2\pi} f(z + re^{i\theta}) d\theta = 2i \iint_{\Omega_r} \frac{\partial f / \partial \bar{z}}{z-z} dx dy$$

$$\downarrow n \rightarrow 0 \qquad \qquad \qquad \downarrow n \rightarrow 0$$

$$2\pi i f(z) \qquad \qquad \qquad 2i \iint_{\Omega} \frac{\partial f / \partial \bar{z}}{z-z} dx dy$$

(In the limit to the right, we have used the fact that $\frac{1}{z-z}$ has a finite integral over Ω , i.e. is integrable, see Lemma 2 on page 99 of Norvaišaitis). This proves:

Theorem If f is C^1 in $\bar{\Omega}$ and $z \in \Omega$ then

$$f(z) = \frac{1}{2\pi i} \int_{\partial\Omega} \frac{f(z)}{z-z} dz - \frac{1}{\pi} \iint_{\Omega} \frac{\partial f / \partial \bar{z}}{z-z} dx dy$$

In particular, if f is holomorphic, we get Cauchy's formula

$$f(z) = \frac{1}{2\pi i} \int_{\partial\Omega} \frac{f(z)}{z-z} dz$$

Another particular case is if $f \in C^1(\mathbb{C})$ has compact support, then

$$f(z) = -\frac{1}{\pi} \iint_{\mathbb{C}} \frac{\partial f / \partial \bar{z}}{z-z} dx dy \quad \text{for all } z \in \mathbb{C}.$$

Resumé of one complex variable

- Power series expansion.

If $f \in \mathcal{O}(\Omega)$ and $D(a, r) \subset \Omega$, then f is given by its power series expansion in $D(a, r)$

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (z-a)^n$$

- Conversely, every power series $f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$ converges in a disc $D(a, r)$, $0 \leq r \leq \infty$, uniformly on compacts and the sum is holomorphic. We may def. formulae, $a_n = \frac{f^{(n)}(a)}{n!}$

- (Identity th.) If Ω is connected, $f \in \mathcal{O}(\Omega)$ and $f \equiv 0$ on an open set V (or more generally a set E with an accumulation point in Ω), then $f \equiv 0$ in Ω .

- If Ω is connected, $f \in \mathcal{O}(\Omega)$, $f \not\equiv 0$, then $Z(f)$ is discrete.

- Every nonconstant holomorphic function is open

Maximum principle If $f \in A(\Omega) = \mathcal{O}(\Omega) \cap C(\bar{\Omega})$

and $M = \max_{z \in \partial \Omega} |f(z)|$, then

$$|f(z)| \leq M$$

for all $z \in \Omega$. If equality holds at one point in Ω , then f is constant

- Cauchy formula for derivatives If $f \in \mathcal{O}(\Omega)$, $\bar{D}(a, r) \subset \Omega$

then

$$f^{(n)}(a) = \frac{n!}{2\pi i} \int_{|z-a|=r} \frac{f(z) dz}{(z-a)^{n+1}}$$

Cauchy estimates If $|f(z)| \leq M$ in $\bar{D}(a, r)$, then

$$f^{(n)}(a) \leq \frac{n! M}{r^n}$$

(Supnorm estimates on f give supnorm estimates on all derivatives).

- Liouville's theorem If $f \in \mathcal{O}(\mathbb{C})$ is bounded, then f is constant.

- Weierstrass theorem If $f_n \in \mathcal{O}(\Omega)$ and $f_n \rightarrow f$ uniformly on compact, then $f \in \mathcal{O}(\Omega)$ and all derivatives $f_n^{(k)} \rightarrow f^{(k)}$ uniformly on compact.

- Montel's theorem If \mathcal{F} is a family of holomorphic functions that is uniformly bounded on compact ($\sup \{|f(z)|; f \in \mathcal{F}, z \in K\} < \infty$ for all compact), then every sequence $f_n \in \mathcal{F}$ contains a subsequence that converges uniformly on compact.

- Laurent series If $f \in \mathcal{O}(A(a, r_1, r_2))$ then f is given by

$$f = \sum_{n=-\infty}^{\infty} a_n (z-a)^n$$

- Particular case $r_1 = 0, A = D^*(a, r_2)$ punctured disc.

f has a singularity at a . Three types

- Removable sing. $a_n = 0 \quad \forall n < 0, \Leftrightarrow f$ bd in D

- Pole of order $k \quad a_{-k} \neq 0, a_n = 0 \quad \forall n < -k \Leftrightarrow |f| \rightarrow \infty \text{ as } z \rightarrow a$

- Essential singularity. $a_n \neq 0$ for infinitely many $n < 0 \Leftrightarrow f(D^*(a, r))$ dense in \mathbb{C} for all r .

- Meromorphic functions Ω open, $E \subset \Omega$ discrete, f is called meromorphic in Ω if

a) $f \in O(\Omega \setminus E)$

b) $\forall a \in E \exists D(a, r)$ and $g, h \in O(D)$, $h \neq 0$ such that

$$hf = g \text{ in } D \setminus E.$$

We write $f \in M(\Omega)$

Nonremovable singularities are poles.

- The residue theorem If $f \in M(\Omega)'$ and $\gamma \subset \Omega'$ is a ^{f has no poles on γ .} simple closed curve such that $\text{inside}(\gamma) \subset \Omega$, then

$$\frac{1}{2\pi i} \int_{\gamma} f(z) dz = \sum_{a \in \text{inside}(\gamma)} \text{res}_f(a)$$

The argument principle

If f has no zeros or poles on γ , then

$$\frac{1}{2\pi i} \int_{\gamma} \frac{f'(z)}{f(z)} dz = \text{number of zeros inside } \gamma - \text{number of poles inside } \gamma$$

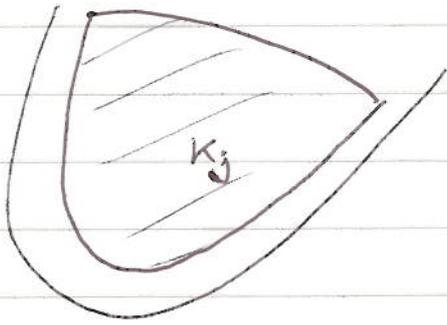
(zeros and poles counted with multiplicity).

- If $f: \Omega \rightarrow \Omega'$ is holomorphic and bijective, then f has a holomorphic inverse, i.e. f is biholomorphic.

Partitions of unity

- If $U \subset \mathbb{R}^m$ is open, then there exist an exhaustion $\{K_j\}_{j=1}^\infty$ of U by compact such that $K_j \subset K_{j+1}^\circ$, $\bigcup_j K_j = U$

Proof: If $U = \mathbb{R}^m$ this is trivial. If not, let $K_j = \{z \in U; d(z, \mathbb{R}^m \setminus U) \geq \frac{1}{j}\} \cap \bar{B}(j)$



- We say that a family \mathcal{F} of subsets of \mathbb{R}^m is locally finite if every $a \in \mathbb{R}^m$ has a nbhd. $B(a, r)$ such that $B(a, r) \cap E \neq \emptyset$ for only a finite number of sets $E \in \mathcal{F}$.

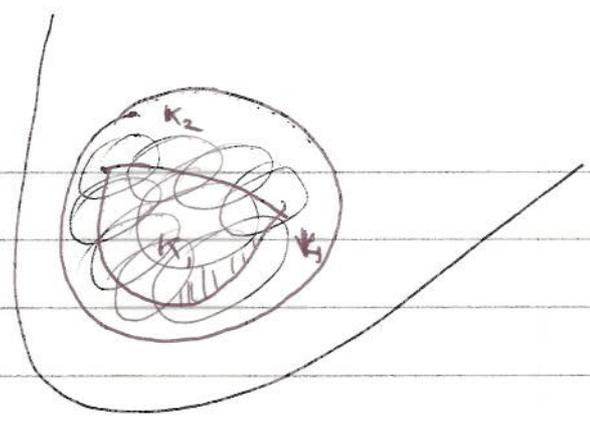
This is equivalent to $K \cap E \neq \emptyset$ for only a finite number of sets $E \in \mathcal{F}$ for any compact K .

- Let $\mathcal{U} = \{U_i\}_{i \in I}$ be a collection of open sets. We say that $\mathcal{V} = \{V_j\}_{j \in J}$ is a refinement of \mathcal{U} if for each V_j there is a U_i with $V_j \subset U_i$ and $\bigcup_{j \in J} V_j = \bigcup_{i \in I} U_i$.

- If $\mathcal{U} = \{U_i\}$ is an open covering of U (i.e. $U = \bigcup U_i$) then there is a locally finite refinement $\mathcal{V} = \{V_j\}$ of \mathcal{U} and compact $K_j \subset V_j$ such that $\bigcup_{j \in J} K_j = U$.

Proof:

Set $K'_1 = K_1$. There are finitely many $U_1, \dots, U_{k_1} \in \mathcal{U}$ such that $K'_1 \subset \bigcup_{d=1}^{k_1} U_d$.



If $L'_j = K'_1 \setminus \bigcup_{d \neq j} U_d$, then

$L'_j \subset U_j$ and $L'_l \cap U_j = \emptyset$ for all $l \neq j$.

Then there are disjoint open sets $U'_j \subset U_j$ such that $L'_j \subset U'_j$. Let $L_j = K'_1 \setminus \bigcup_{l \neq j} U'_l$. Then $L_j \subset U_j$ and $\bigcup L_j = K'_1$.

Now, let $K'_2 = K_2 \setminus \bigcup_{l=1}^{k_1} U_l$. Then $\{U_j \setminus K'_1\}_{j \in I}$ covers K'_2 and we can continue.

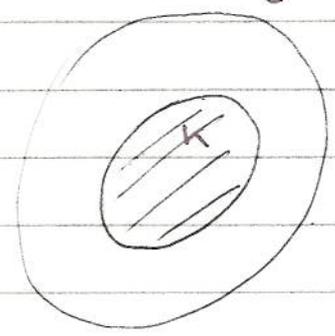
None of the open sets to follow will intersect K_1 .

This proves that the V'_α constructed is locally finite.

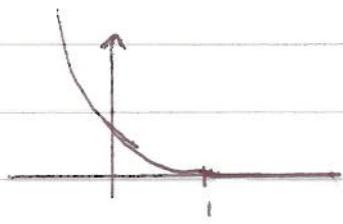
- ϕ defined on U , $\text{supp } \phi = \{x; \phi(x) \neq 0\} \subset U$.
- $C_0^\infty(U) = \{ \phi \in C^\infty(U); \phi \text{ real and } \text{supp } \phi \text{ is a compact subset of } U \}$.

- If $\mathcal{U} = \{U_i; i \in I\}$ is an open cover of U , then a partition of unity relative to \mathcal{U} is a family $\phi_i \in C^\infty(U)$ such that
 - $\phi_i \geq 0$, $S_i = \text{supp } \phi_i \subset U_i$
 - S_i of ϕ_i is locally finite
 - $\sum \phi_i \equiv 1$ in U .

Lemma 1. If U is open, $K \subset U$ is compact, then there is $\phi \in C_0^\infty(U)$ such that $\phi(x) > 0$ for $x \in K$.



Proof: The function $\psi(t) = \begin{cases} e^{-1/(1-t)} & t \leq 1 \\ 0 & t \geq 1 \end{cases}$ is in $C^\infty(\mathbb{R})$



There exist $\delta > 0$ such that $\text{dist}(K, \mathbb{R}^m \setminus U) \geq 2\delta$

There are a finite number of points $a_1, \dots, a_N \in K$ such that $K \subset \bigcup_{i=1}^N B(a_i, \delta)$. Let

$$\phi(x) = \sum_{i=1}^n \psi\left(\frac{|x-a_i|^2}{\delta^2}\right)$$

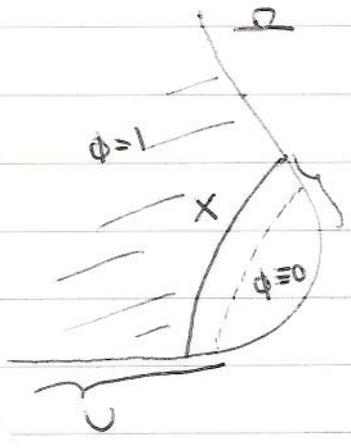
Theorem 1. If $\mathcal{U} = \{U_i\}_{i \in I}$ is an open cover of U , then there is a partition of unity relative to U .

Proof: Let $\mathcal{V} = \{V_j\}_{j \in J}$ be a locally finite refinement of \mathcal{U} and $K_j \subset V_j$ compact which cover U . Then there are $\psi_j \in C_0^\infty(V_j) \subset C^\infty(U)$ such that $\psi_j > 0$ in K_j .

Let $\psi = \sum_j \psi_j$. This sum is locally finite, hence $\psi \in C^\infty(U)$ and $\psi > 0$ in U . If we let $\chi_j = \psi_j / \psi$, then χ_j is a partition of unity relative to V_j . For each $j \in J$ pick $\tau(j) \in I$ such that $V_j \subset U_{\tau(j)}$ and for each $i \in I$ define $\phi_i = \sum_{j \in \tau^{-1}(i)} \chi_j \in C^\infty(U)$. Clearly, $\{\text{supp } \phi_i\}$ is locally finite.

If $x \in U \setminus U_i$ there is a nbhd V of x such that $V \cap \text{supp } \chi_j \neq \emptyset$ for only finitely many j . If $j \in \tau^{-1}(i)$, then $\text{supp } \chi_j$ is a compact subset of U_i , hence $\phi_i \equiv 0$ in $V \setminus \bigcup_{j \in \tau^{-1}(i)} \text{supp } \chi_j$ and $x \notin \text{supp } \phi_i$. This proves that $\text{supp } \phi_i \subset U_i$.

Theorem 2 (Separation of closed sets) If $\Omega \subset \mathbb{R}^n$ is open, $X \subset \Omega$ closed (relatively), $X \subset U$ open, then there exist $\phi \in C^\infty(\Omega)$, $0 \leq \phi \leq 1$, $\phi|_X = 1$, $\phi|_{\Omega \setminus U} = 0$



Proof: Let ϕ_U, ϕ_V be a partition of unity relative to the covering $\{U, \Omega \setminus X\}$.
 Must have $\phi_V|_X = 0 \Rightarrow \phi_U = 1$ on X
 Also $\phi_U = 0$ in $\Omega \setminus U$.

Patching C^∞ functions on disjoint closed sets

Theorem 3 If $\Omega \subset \mathbb{R}^n$ is open, $X_1, X_2 \subset \Omega$ two disjoint closed sets and $\phi_1, \phi_2 \in C^\infty(\Omega)$. Then there exist $\phi \in C^\infty(\Omega)$ such that $\phi|_{X_1} = \phi_1$, $\phi|_{X_2} = \phi_2$

Proof: Pick $\alpha \in C^\infty(\Omega)$, $0 \leq \alpha \leq 1$, $\alpha|_{X_1} = 1$, $\alpha|_{X_2} = 0$ and let $\phi = \alpha\phi_1 + (1-\alpha)\phi_2$.

The $\bar{\partial}$ -equation, $\frac{\partial u}{\partial \bar{z}} = \phi$.

Recall Cauchy-Stokes formula in $\Omega \subset \mathbb{C}$ ($z = x + iy, \zeta = \xi + i\eta$)

• $f \in C^1(\bar{\Omega}), z \in \Omega \Rightarrow f(z) = \frac{1}{2\pi i} \int_{\partial\Omega} \frac{f(\zeta)}{\zeta - z} d\zeta - \frac{1}{\pi} \iint_{\Omega} \frac{\partial f / \partial \bar{\zeta}}{\zeta - z} d\zeta d\eta$

• f also holomorphic in Ω : $f(z) = \frac{1}{2\pi i} \int_{\partial\Omega} \frac{f(\zeta)}{\zeta - z} d\zeta$

• $f \in C_0^1(\mathbb{C}), z \in \mathbb{C} \Rightarrow f(z) = -\frac{1}{\pi} \iint_{\mathbb{C}} \frac{\partial f / \partial \bar{\zeta}}{\zeta - z} d\zeta d\eta$

Given $\phi \in C_0^1(\mathbb{C})$, we want to find f such that

$$\frac{\partial f}{\partial \bar{z}} = \phi$$

It is natural to try

$$f(z) = -\frac{1}{\pi} \iint_{\mathbb{C}} \frac{\phi(\zeta)}{\zeta - z} d\zeta d\eta = -\frac{1}{\pi} \iint_{\mathbb{C}} \frac{\phi(\zeta + z)}{\zeta} d\zeta d\eta$$

If we can diff. under sign of integration

$$\frac{\partial f}{\partial \bar{z}}(z) = -\frac{1}{\pi} \iint_{\mathbb{C}} \frac{\partial \phi / \partial \bar{z}(\zeta + z)}{\zeta} d\zeta d\eta = \phi(z)$$

Differentiation is allowed. Dg. with respect to x , let $h \in \mathbb{R}$

$$\frac{f(z+h) - f(z)}{h} = -\frac{i}{\pi} \iint_C \frac{\frac{1}{h} [\phi(s+z+h) - \phi(s+z)]}{s} d\xi d\eta \xrightarrow{\text{dom. conv.}} -\frac{i}{\pi} \iint_C \frac{\partial \phi / \partial x}{s} d\xi d\eta$$

$\frac{1}{s} \in L^1_{loc}(\mathbb{R}^2)$

Can do the same in y direction.

Hence we have proved

Theorem 2 If $\phi \in C_c^\infty(\mathbb{C})$ and

$$f(z) = -\frac{i}{\pi} \iint_C \frac{\phi(s)}{s-z} d\xi d\eta$$

then $f \in C^\infty(\mathbb{C})$ and $\frac{\partial f}{\partial \bar{z}} = \phi$

- Notice that in general f does not have compact support since for large R

$$0 = \int_{|z|=R} f \partial \bar{z} = 2i \iint_{|z| \leq R} \frac{\partial f}{\partial \bar{z}} dx dy = 2i \iint_{|z| \leq R} \phi dx dy \Rightarrow \int_C \phi dx dy = 0$$

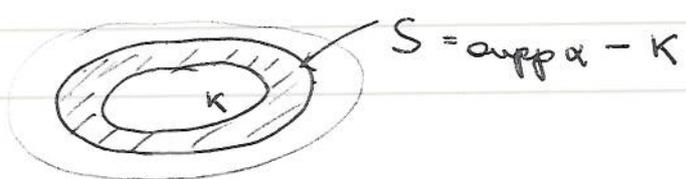
Theorem 3 (Smeared out Cauchy integral formula)

If $K \subset \Omega$ is compact, $f \in \mathcal{O}(\Omega)$ and $\alpha \in C_c^\infty(\Omega)$ is $\equiv 1$ on K , then for $z \in K$

$$f(z) = -\frac{i}{\pi} \iint_\Omega f(s) \frac{\partial \alpha}{\partial \bar{s}} \frac{1}{s-z} d\xi d\eta$$

In particular $\iint_\Omega f(s) \frac{\partial \alpha}{\partial \bar{s}} d\xi d\eta = 0$

Proof: Apply Cauchy-Stokes to $\phi = \alpha f$.



Theorem 1. (Runge) $\Omega \subset \mathbb{C}$ open, $K \subset \Omega$ compact.

The following are equivalent:

- (1) $\mathcal{O}(\Omega)|_K$ is dense in $\mathcal{O}(K)$
- (2) No connected component of $\Omega \setminus K$ is relatively compact in Ω
- (3) $\forall a \in \mathbb{C} \setminus K$ there is $f \in \mathcal{O}(\Omega)$ such that $|f(a)| > |f|_K$

Proof:

(1) \Rightarrow (2) If U is a connected component of $\Omega \setminus K$ which is relatively compact in Ω , then $\partial U \subset K$, because otherwise we could attach a disc to $z \in \partial U \setminus K$ to obtain a bigger connected set. If $z_0 \in U$ and

$f(z) = \frac{1}{z-z_0} \in \mathcal{O}(K)$, then f cannot be approximated by

$f_n \in \mathcal{O}(\Omega)$, because if $\frac{1}{z-z_0} - f_n \rightarrow 0$ on K , then

$g_n = 1 - (z-z_0)f_n \rightarrow 0$ on K , but $g_n(z_0) = 1$, so this violates the maximum modulus theorem since $\partial U \subset K$.

(2) \Rightarrow (1) We must prove that every $f \in \mathcal{O}(K)$ can be approximated uniformly on K by $f_n \in \mathcal{O}(\Omega)$.

Pick $f \in \mathcal{O}(W)$ for some open neighbourhood W of K .

Step 1. Approximation of f by rational functions with poles outside K .

Pick $\alpha \in C_0^\infty(W)$ such that $\alpha = 1$ in a nbhd W_0 of K .

For $z \in K$ we have by Cauchy - Stokes formula

$$f(z) = \frac{1}{\pi} \iint_{\mathbb{C}} f(\zeta) \frac{\partial \alpha}{\partial \bar{\zeta}} \frac{1}{z-\zeta} d\zeta d\bar{\zeta} = \frac{1}{\pi} \iint_{L = \text{supp } \alpha \setminus W_0} f(\zeta) \frac{\partial \alpha}{\partial \bar{\zeta}} \frac{1}{z-\zeta} d\zeta d\bar{\zeta}$$

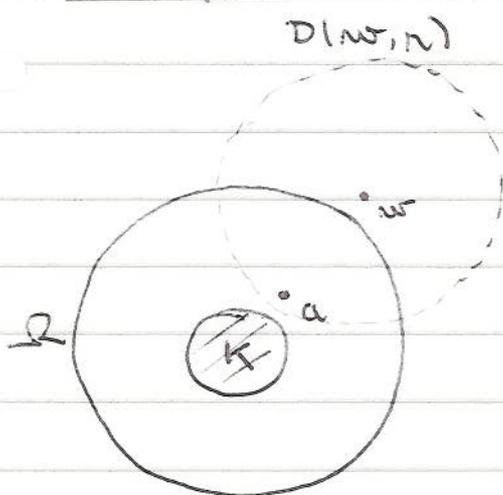
If we subdivide \mathbb{C} by small squares and form the corresponding Riemann sum for the integral,

$$\frac{1}{\pi} \sum_{\nu} f(z_{\nu}) \frac{\partial \alpha}{\partial \bar{z}}(z_{\nu}) \frac{1}{z - z_{\nu}}$$

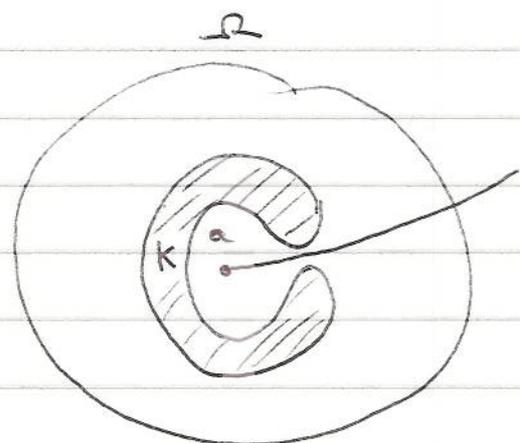
then these Riemann sums will approximate the integrals, uniformly on K , since the integrand is compactly supported, hence uniformly continuous in \mathbb{C} . The z_{ν} 's will be close to $L = \text{supp} \alpha \cap W_0$, hence in $\Omega \cap K$. It follows that f can be approximated on K by a finite sum $\sum_{\nu} c_{\nu} \frac{1}{z - z_{\nu}}$ with $z_{\nu} \in \Omega \cap K$.

Step 2. We now look at terms of the form $\frac{1}{z-a}$ with $a \in \Omega \cap K$. We shall approximate these by functions which are holomorphic in Ω by "pushing the poles out of Ω ".

Example



$\frac{1}{z-a}$ is holomorphic outside $D(w, r)$ and is given there by a power series in $\frac{1}{z-w}$.



The pole a can be gradually pushed out of Ω .

Therefore, let $a \in \mathbb{C} \setminus K$ and consider

$$U_a = \left\{ w \in \mathbb{C} \setminus K; \frac{1}{z-a} \text{ can be approximated on } K \text{ by polynomials in } \frac{1}{z-w} \right\}.$$

We will show that U_a is a connected component of $\mathbb{C} \setminus K$.

U_a is open: Suppose $w \in U_a$ and $D(w, r) \cap K = \emptyset$.

If P_ϵ is a polynomial in $\frac{1}{z-w}$ which approximates f on K and $w' \in D(w, r/2)$, then $P_\epsilon(\frac{1}{z-w})$ is holomorphic outside $\bar{D}(w', r/2)$ and can therefore be developed in a power series in $1/z-w'$ there.

A finite sum of this power series will approximate P_ϵ on the compact $K \subset \mathbb{C} \setminus \bar{D}(w', r/2)$.

U_a is closed in $\mathbb{C} \setminus K$: Assume $w_n \in U_a$ and $w_n \rightarrow w \in \mathbb{C} \setminus K$. Then there is a disc $\bar{D}(w, r) \subset \mathbb{C} \setminus K$ and a $w_n \in \bar{D}(w, r)$. $\frac{1}{z-a}$ can be approximated on K by polynomials in $\frac{1}{z-w_n}$. These are holomorphic outside $\bar{D}(w, r)$ and the same argument as above gives that $w \in U_a$.

This proves the claim.

We now prove that $\frac{1}{z-a}$ can be approximated on K by a function which is holomorphic in Ω .

If U_a is bounded, then we claim that $U_a \cap (\mathbb{C} \setminus \Omega) \neq \emptyset$. Otherwise, $U_a \subset \Omega$ and U_a is a connected component

of $\Omega \setminus K$. But $\partial U_a = K$, hence U_a would be relatively compact in Ω , which is impossible. Hence there is some $w \in U_a \setminus \Omega$ and by definition $\frac{1}{z-a}$ can be approximated by a polynomial in $\frac{1}{z-w}$, which is holomorphic in Ω .

If U_a is unbounded, then there is $w \in U_a$ with $|w| > \sup\{|z|, z \in K\}$. Let $r = |w|$. In this case a polynomial in $\frac{1}{z-w}$ is holomorphic in the disc $D(0, r)$, hence is given by a power series there, and can be approximated by a polynomial on K .

(3) \Rightarrow (2) is analogous with (1) \Rightarrow (2): If $U \subset \subset \Omega$ is a connected component of $\Omega \setminus K$, then $\partial U = K$ and for all $a \in U$ we have by the max. modulus princ.

$$|f(a)| \leq |f|_{\partial U} \leq |f|_K.$$

which contradicts (3).

(2) \Rightarrow (3). If $a \in \Omega \setminus K$, then $L = K \cup \{a\}$ has the same property and by the implication (2) \Rightarrow (1), $\mathcal{O}(\Omega)|_L$ is dense in $\mathcal{O}(L)$. If U and V are disjoint open sets, $K \subset U$, $a \in V$ and ϕ is defined by $\phi = 0$ in U , $\phi = 1$ in V , then $\phi \in \mathcal{O}(L)$, hence there exist $f \in \mathcal{O}(\Omega)$ such that $|f - \phi|_L < \frac{1}{2}$. But then

$$|f|_K < \frac{1}{2} < |f(a)|$$

This completes the proof of the theorem.

- Remark: From the implication (2) \Rightarrow (1) we that if
 - No connected component of $\Omega \setminus K$ is rel.comp. in Ω
 - $A \subset \mathbb{C}$ is a set which contains at least one point in every bounded component of $\mathbb{C} \setminus \Omega$.
 - $f \in \mathcal{O}(K)$

then f can be approximated uniformly on K by rational functions with poles in A .

- The polynomials are dense in $\mathcal{O}(\mathbb{C})$. Hence if we let $\Omega = \mathbb{C}$ in Runge's theorem, we get:

Corollary For a compact set $K \subset \mathbb{C}$ the following are equivalent:

- (1) Every $f \in \mathcal{O}(K)$ can be approximated by polynomials
- (2) $\mathbb{C} \setminus K$ is connected (i.e. K has no holes)
- (3) For any $z \notin K$ there is a polynomial P such that $|P(z)| > |P|_K$.

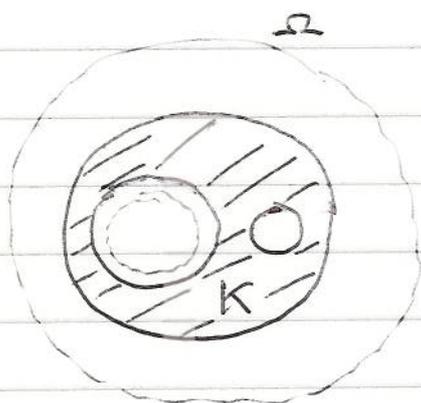
Such K are called polynomially convex.

- Def. Let $K \subset \Omega$ be compact. The holomorphically convex hull of K in Ω is defined by

$$\hat{K}_\Omega = \{z \in \Omega; |f(z)| \leq |f|_K \text{ for all } f \in \mathcal{O}(\Omega)\}.$$

(3) in Runge's theorem states that $\hat{K}_\Omega = K$, in which case we call K holomorphically convex in Ω .

We have $\hat{\hat{K}}_\Omega = \hat{K}_\Omega$. We shall see that \hat{K}_Ω fills in the holes in K which do not contain holes in Ω .

Example.

\hat{K}_Ω fills in the hole to the right, not the left.

Exercise: - \hat{K}_Ω does not get closer to $\partial\Omega$, i.e. $d(\hat{K}_\Omega, \partial\Omega) = d(K, \partial\Omega)$.

- \hat{K}_Ω is compact

Theorem \hat{K}_Ω is the union of K and all relatively compact components of $\Omega \setminus K$.

Proof: If U is such a component, then $\partial U \subset K$ and therefore $U \subset \hat{K}_\Omega$ by the maximum modulus theorem.

This shows that

$$K_1 := K \cup \left(\bigcup_{U \subset \subset \Omega} U \right) \subset \hat{K}_\Omega.$$

Also $\Omega \setminus K_1 = \bigcup_{U \subset \subset \Omega} U$ is open, hence K_1 is closed in Ω

and therefore compact. Also, no components of $\Omega \setminus K_1$ are relatively compact. Runge's theorem gives that any $z \notin K_1$ can be separated from K_1 (and hence K) by a holomorphic function in Ω . This proves that $z \notin \hat{K}_\Omega$, i.e. $\hat{K}_\Omega \subset K_1$.

Lemma If $\Omega \subset \mathbb{C}$ is open, then

$$K_m = \left\{ z \in \Omega; d(z, \mathbb{C} \setminus \Omega) \geq \frac{1}{m}, |z| \leq m \right\}$$

is a holomorphically convex exhaustion of Ω .

Theorem 3 (Classical Runge theorem)

If $\Omega \subset \mathbb{C}$ is open, $A \subset \mathbb{C}$ is a set which contains one point from each bounded component of $\mathbb{C} \setminus \Omega$, then every $f \in \mathcal{O}(\Omega)$ can be approximated uniformly on compact by rational functions with poles in A .

Proof: Pick $f \in \mathcal{O}(\Omega)$ and a compact set $K \subset \Omega$. Replacing K by \hat{K}_Ω , we may assume that K is holomorphically convex in Ω . The result follows from the remark to Runge's theorem.

Mittag-Leffler theorem

Def. $\mathbb{C}_a^* = \mathbb{C} \setminus \{a\}$, \mathbb{C}_0^* is denoted \mathbb{C}^*

If f is holomorphic in a punctured disc around a , we have

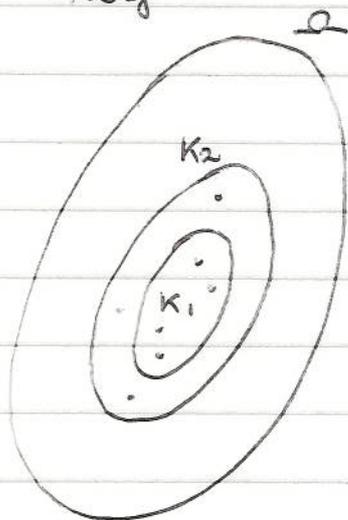
$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-a)^n$$

The negative powers $p_a = \sum_{n=-\infty}^{-1} c_n (z-a)^n$ is called the principal part of f at a . We have $p_a \in \mathcal{O}(\mathbb{C}_a^*)$.

Theorem 1 (Mittag-Leffler) Prescribing principal parts

If $E \subset \Omega$ is discrete and for every $a \in E$ there is given a principal part $p_a \in \mathcal{O}(\mathbb{C}_a^*)$, then there is $f \in \mathcal{O}(\Omega \setminus E)$ such that $f - p_a$ is holomorphic in a neighbourhood of a for all $a \in E$.

Proof:



Let $\{K_n\}$ be a holomorphically convex exhaustion of Ω and put $K_0 = \emptyset$.

Let $E_n = E \cap \{K_n \setminus K_{n-1}\}$. E_n is finite. Put

$$g_n = \sum_{a \in E_n} p_a \in \mathcal{O}(\mathbb{C} \setminus E_n) \supset \mathcal{O}(K_{n-1})$$

Let $f_1 = g_1$. Then $f_1 - p_a$ is holomorphic in a for all $a \in E_1$ and is holomorphic outside K_1 . We would like to add g_2 ,

but the problem is convergence. However, since $g_2 \in \mathcal{O}(K_1)$ and K_1 is holomorphically convex, we can find $h_2 \in \mathcal{O}(\Omega)$ such that $|g_2 - h_2|_{K_1} < 2^{-2}$.

If we let $f_2 = g_1 + (g_2 - h_2)$, then $f_2 - p_a$ is holomorphic at all $a \in E_1 \cup E_2$. We proceed inductively to find $h_n \in \mathcal{O}(\Omega)$ such that $|g_n - h_n|_{K_{n-1}} < 2^{-n}$. It follows that

$$f = \lim f_n = g_1 + \sum_{n=2}^{\infty} (g_n - h_n) \text{ solves the problem} \quad \blacksquare$$

- If every $p_a \in \mathcal{M}(\mathbb{C})$, i.e. only has a pole at a , then $f \in \mathcal{M}(\Omega)$
- Enough to assume $p_a \in \mathcal{O}(D^*(a, \nu))$ for some $\nu > 0$.

- Equivalent formulation:

Theorem 1' If $E \subset \Omega$ is discrete, $\Omega = \bigcup_{j \in J} U_j$ and $g_j \in \mathcal{O}(U_j \setminus E)$ such that $g_j - g_k \in \mathcal{O}(U_j \cap U_k)$ for all j, k , then there is $g \in \mathcal{O}(\Omega \setminus E)$ such that $g - g_j \in \mathcal{O}(U_j) \forall j$.

(1') \Rightarrow (1) Put $E = \{z_j\}$, $U_j = (\Omega \setminus E) \cup \{z_j\}$ and $g_j = p_{z_j}$

(1) \Rightarrow (1') For $a \in E$ pick $j(a)$ such that $a \in U_{j(a)}$ and let $p_a =$ the principal part of $g_{j(a)}$ at a . This is

independent of the choice of $j(a)$. If $g \in \mathcal{O}(\Omega, E)$ such that $g - p_a$ is holomorphic at a for all $a \in E$, then $g - g_j \in \mathcal{O}(U_j)$.

In theorem 1', suppose we can find the "holomorphic correction term", $f_j = g - g_j \in \mathcal{O}(U_j)$ directly. How can we be sure that they patch together to a global g ?

We must have

$$f_i + g_i = f_j + g_j \text{ in } (U_i \cap U_j) \setminus E$$
$$f_i - f_j = g_j - g_i \text{ in } U_i \cap U_j$$

Let $f_{ij} = g_j - g_i \in \mathcal{O}(U_i \cap U_j)$. The existence of f_i follows from:

Theorem 4 If $\{U_j\}_{j=1}^{\infty}$ is an open covering of Ω and $f_{ij} \in \mathcal{O}(U_i \cap U_j)$ satisfy the cocycle condition
$$f_{ij} + f_{jk} + f_{ki} = 0 \text{ in } U_i \cap U_j \cap U_k$$

for all indices i, j, k . Then there exist $f_j \in \mathcal{O}(U_j)$ such that $f_{ij} = f_i - f_j$ in $U_i \cap U_j$ for all i, j .

- Notice that the cocycle condition implies that $f_{ii} = 0$ and $f_{ji} = -f_{ij}$ for all i, j .
- The argument above shows that Theorem 4 \Rightarrow Theorem 1'
- We shall now prove Theorem 4. We first prove a ~~solution theorem for the $\bar{\partial}$ -equation~~.

Proof of theorem 1.

Step 1. We first prove that there are smooth solutions to the problem, i.e. there are $\phi_i \in C^\infty(U_i)$ such that $f_{ij} = \phi_i - \phi_j$ in $U_i \cap U_j$. For this, it is sufficient that $f_{ij} \in C^\infty(U_i \cap U_j)$.

Proof: Let α_j be a partition of unity relative to $U = \{U_i\}$ and define in U_i :

$$\phi_i = \sum_j \alpha_j f_{ij}$$

This is in $C^\infty(U_i)$, since $\text{supp } \alpha_j \subset U_j$ and the sum is locally finite. In $U_i \cap U_j$ we have

$$\phi_i - \phi_j = \sum_k \alpha_k (f_{ik} - f_{jk}) = \sum_k \alpha_k f_{ij} = f_{ij}.$$

Step 2. We now correct the ϕ_i to make a holomorphic solution. Notice that since $\phi_i - \phi_j$ differ by a holomorphic function on $U_i \cap U_j$, the function

$$\psi(z) = \frac{\partial \phi_i}{\partial \bar{z}} \quad \text{for } z \in U_i$$

is globally defined in Ω . If we can find $u \in C^\infty(\Omega)$ such that

$$\frac{\partial u}{\partial \bar{z}} = \psi$$

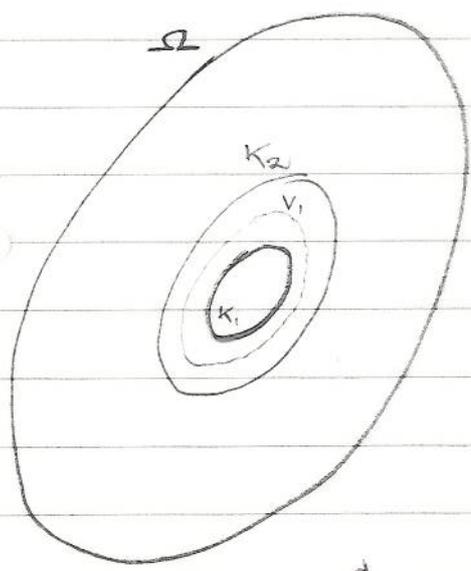
then $f_u = \phi_i - u \in \mathcal{O}(U_i)$ and solves the problem.

Hence Theorem 4 follows from the following result:

Theorem 2 (Solution of $\bar{\partial}$ -equation) If $\psi \in C^\infty(\Omega)$, then there exist $u \in C^\infty(\Omega)$ such that $\frac{\partial u}{\partial \bar{z}} = \psi$.

Proof: Notice that we can solve the equation in a nbh of any compact set $K \subset \Omega$. Just chop off ψ with a smooth function. The solution is in $C^\infty(\mathbb{C})$.

We shall now build the solution inductively as in Runge's theorem. Let $\{K_n\}_{n=1}^\infty$ be a holomorphically convex exhaustion of Ω .



First, solve

$$\frac{\partial u_1}{\partial \bar{z}} = \psi \text{ in an open nbh } V_1 \text{ of } K_1,$$

$u_1 \in C^\infty(\mathbb{C})$. We now want to correct u_1 so the equation is satisfied in an open nbh V_2 of K_2 .

$$\text{Let } \phi = \psi - \frac{\partial u_1}{\partial \bar{z}}. \text{ Then } \phi \in C^\infty(\Omega) \text{ and}$$

$$\phi = 0 \text{ in } V_1. \text{ Now solve } \frac{\partial v_2}{\partial \bar{z}} = \phi \text{ in } V_2,$$

$v_2 \in C^\infty(\mathbb{C}) \cap \mathcal{O}(V_1)$. $u_1 + v_2$ solves the problem in V_2 .
but we want the process to converge, so we pick $f_2 \in \mathcal{O}(\Omega)$ such that $|v_2 - f_2|_{K_1} < 2^{-2}$ and let $u_2 = v_2 - f_2$.

Now, proceed to find $u_3, \dots, u_n \in C^\infty(\mathbb{C})$ and open nbhs V_j of K_j , $j = 3, \dots, n$, such that

$$\bullet u_j \in \mathcal{O}(V_{j-1}), |u_j|_{K_{j-1}} < 2^{-j}$$

$$\bullet \frac{\partial u_2}{\partial \bar{z}} + \dots + \frac{\partial u_n}{\partial \bar{z}} = \psi \text{ in } V_n$$

Then $u = \sum_{n=1}^\infty u_n$ is the required solution. ■

Weierstrass theorem Shall prove result on prescription of zeros and poles. For this we need to study infinite products.

Let $\{a_n\} \in \mathbb{C}$. We say that $\prod_{n=1}^{\infty} a_n$ is convergent if $p_N = \prod_{n=1}^N a_n$ is a convergent sequence, and we set $\prod_{n=1}^{\infty} a_n = \lim_{N \rightarrow \infty} p_N$

If this limit is nonzero, it is clearly necessary that $\lim_{n \rightarrow \infty} a_n = 1$. We shall consider products

$$\prod_{n=1}^{\infty} (1 + u_n) \text{ with } u_n \rightarrow 0.$$

Stoopy calculation

$$\log \prod_{n=1}^N (1 + u_n) = \sum_{n=1}^N \log(1 + u_n) \approx \sum_{n=1}^N u_n$$

Hence it follows that the convergence of $\prod (1 + u_n)$ is related to the convergence of the series $\sum u_n$. Correct calc!

$$\begin{aligned} |p_N| &\leq \prod_{n=1}^N (1 + |u_n|) \\ \log |p_N| &\leq \sum_{n=1}^N \log(1 + |u_n|) \leq \sum_{n=1}^N |u_n| \quad (\log(1+x) \leq x) \\ |p_N| &\leq e^{\sum |u_n|} \end{aligned}$$

Hence $\{p_N\}$ is bounded if $\sum_{n=1}^{\infty} |u_n| < \infty$

$p_N - 1$ is a polynomial in u_1, \dots, u_N , without constant term.

This gives

$$|p_N - 1| \leq \prod_{n=1}^N (1 + |u_n|) - 1 \leq e^{\sum |u_n|} - 1$$

Lemma 1 If $\{u_n(z)\}$ are bounded functions on a set E such that $\sum |u_n(z)|$ converges uniformly on E , then

$$f(z) = \prod_{n=1}^{\infty} (1 + u_n(z))$$

converges uniformly on E and $f(z_0) = 0$ iff $u_n(z_0) = -1$ for some n .

Proof: It follows that from $|p_N(z)| \leq e^{\sum_{n=1}^N |u_n(z)|}$ that $\{p_N(z)\}$ is uniformly bounded on E , i.e. $|p_N(z)| \leq C \forall z \in E$.

For $M > N$ we have

$$\begin{aligned} |p_M(z) - p_N(z)| &= |p_N(z)| \left| \prod_{n=N+1}^M (1 + u_n(z)) - 1 \right| \\ &\leq C \left(e^{\sum_{n=N+1}^M |u_n(z)|} - 1 \right) \xrightarrow{N, M \rightarrow \infty} 0 \end{aligned}$$

which proves that $\{p_N(z)\}$ converges uniformly on E .

The inequality also shows that

$$|p_M(z)| \geq |p_N(z)| (1 - \epsilon) \text{ for } N \text{ suff. large, } M > N$$

Hence, the infinite product has a zero ^{at z_0} iff some finite p_N does.

Theorem If Ω is connected, $f_n \in \mathcal{O}(\Omega)$, no f_n is identically equal to zero and $\sum |1 - f_n(z)|$ converges u.o.c. in Ω ,

then $f(z) = \prod_{n=1}^{\infty} f_n(z)$ converges u.o.c. and

$$\text{ord}_a(f) = \sum_{n=1}^{\infty} \text{ord}_a(f_n)$$

To prove Weierstrass we need the following lemma

* Comment on winding number / Simple connectedness.

Lemma 2 Suppose $g \in \mathcal{O}(\Omega)$, $g(z) \neq 0$ for all z . TFAE:

- (1) g has a holomorphic logarithm in Ω . ($e^f = g$)
- (2) g'/g has a hol. primitive
- (3) $\int_{\gamma} g'/g dz = 0$ for all closed curves in Ω .

(1) \Rightarrow (2)Proof: If $g = e^f$, then $g'/g = f'$ (2) \Rightarrow (1) If $g'/g = f'$, let $h = e^{-f} g$. Then $h' = e^{-f} (g' - f'g) = 0$,
hence $h \equiv C$, so $g = Ce^f = e^{f+\alpha}$ ■COROLLARY: If Ω is simply connected, then g has a holomorphic logarithm.Lemma 2 If z_0 and z_1 are in the same component of $\mathbb{C} \setminus K$,
then $g(z) = \frac{z-z_0}{z-z_1}$ has a holomorphic logarithm in a nbhv of K .
If z_0 is in the unbounded comp of $\mathbb{C} \setminus K$, then $g(z) = (z-z_0)$ has a hol. log.Proof: K has a nbhv Ω such that z_0, z_1 are in same comp.
of $\mathbb{C} \setminus \Omega$. Then

$$\frac{g'(z)}{g} = \frac{1}{z-z_0} - \frac{1}{z-z_1}$$

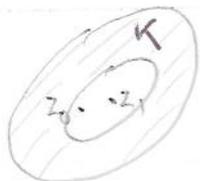
hence for any closed curve in Ω

$$\int_{\gamma} \frac{g'}{g} dz = \int_{\gamma} \left(\frac{1}{z-z_0} - \frac{1}{z-z_1} \right) dz = 2\pi i (\text{Ind}(\gamma, z_0) - \text{Ind}(\gamma, z_1)) = 0$$

* See below

Lemma 3 If K is holomorph. convex in Ω and V is abounded component of $\mathbb{C} \setminus K$, then V contains a $U \cap \{\Omega \neq \emptyset\}$ Proof: If not, then $\partial V \subset K$ and $\bar{U} = U \cup \partial U \subset \Omega$, so U is a
relatively compact component of $\Omega \setminus K$.Theorem 2 WeierstrassIf $E \subset \Omega$ is discrete and for every $a \in E$ there is given
an integer $k_a \in \mathbb{Z}$, then there exists a ~~meromorphic~~ ^{holomorphic} function
in $\Omega \setminus E$ such that $f(z) \neq 0$ for all $z \in \Omega \setminus E$ and $(z-a)^{k_a} f(z)$
is holomorphic and nonzero in a nbhv. of a for all $a \in E$.

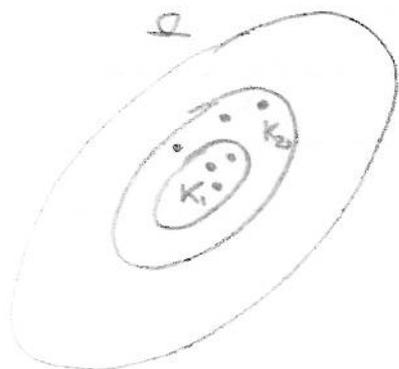
*



$$f(z) = \log \frac{z-z_0}{z-z_1} \in \mathcal{O}(K)$$

 $\Rightarrow z-z_0 = e^{f(z)} (z-z_1)$, Approximate $f(z)$ on K by
 $\tilde{f} \in \mathcal{O}(\mathbb{C} \setminus \{z_1\})$, so $z-z_0 \sim e^{\tilde{f}(z)} (z-z_1)$ on K .

Proof:



Let $\{K_n\}$ be a holomorph. convex exhaustion of Ω and let $E_n = E \cap (K_n \setminus K_{n-1})$, $K_0 = \emptyset$.
 Let $g_n = \prod_{a \in E_n} (z-a)^{k_a}$ and put $f_1 = g_1$, so

f_1 has the required property for $a \in E_1$. We

now would like to multiply by g_2 , but the problem is convergence. We therefore want

to correct g_2 by multiplying with $h_2 \in \mathcal{O}(\Omega)$ such that $h_2 \neq 0$ in all of Ω and

$$|g_2 h_2 - 1|_{K_1} < 2^{-2}$$

Let a_1, \dots, a_l be the points in E_2 contained in bounded components of $\mathbb{C} \setminus K_1$, a_{l+1}, \dots, a_m the points in the unbounded component and k_1, \dots, k_m the corresponding multiplicities. Let $b_j \notin \Omega$ be a point in the same component of $\mathbb{C} \setminus K_1$ as a_j for $j=1, \dots, l$. Then, by Lemma 2, $\log \left(\frac{z-a_j}{z-b_j} \right)$, $j=1, \dots, l$ and $\log(z-a_j)$, $j=l+1, \dots, m$ are holomorphic in a nbhd of K . Hence

$$g := \sum_{j=1}^l k_j \log \left(\frac{z-a_j}{z-b_j} \right) + \sum_{j=l+1}^m k_j \log(z-a_j) \in \mathcal{O}(K)$$

and we may pick $h \in \mathcal{O}(\Omega)$ such that

$$|g - h|_{K_1} < \log(1+2^{-2})$$

which implies

$$|e^{g-h} - 1|_{K_1} < 2^{-2}$$

But $e^{g-h} = \prod_{j=1}^l \left(\frac{z-a_j}{z-b_j} \right)^{k_j} \prod_{j=l+1}^m (z-a_j)^{k_j} \cdot e^{-h} = g_2 h_2$ with

$$h_2 = \frac{e^{-h}}{\prod_{j=1}^l (z-b_j)^{k_j}} \in \mathcal{O}(\Omega) \text{ and has no zeroes.}$$

Inductively, we now construct $h_n \in \mathcal{O}(\Omega)$, h_n has no zeros and $|g_n h_n - 1|_{K_{n-1}} < 2^{-n}$. This implies that

$$f = g_1 \cdot \prod_{n=2}^{\infty} g_n h_n \text{ has the required properties.}$$

Exercise: Formulate analogous versions of Theorem 1' and Theorem 4 for Weierstrass theorem. Theorem 4' is then called the multiplicative Cousin problem (Cousin II) (see page 238-241 in Forster).

Theorem 4 (Interpolation in a discrete set).

If $E \subset \Omega$ is discrete and for every $a \in E$ there is given $\phi_a \in \mathcal{O}(D^*(a, r_a))$ and $k_a \geq 0$, then there is $f \in \mathcal{O}(\Omega \setminus E)$ such that $f - \phi_a$ is holomorphic at a and $\text{ord}_a(f - \phi_a) > k_a$ for all $a \in E$.

Proof: By Weierstrass there is $g \in \mathcal{O}(\Omega)$ such that $Z(g) = E$ and $\text{ord}_a g = k_a + 1$ for all $a \in E$. We must find f such that

$$\frac{f}{g} - \frac{\phi_a}{g} \text{ is holomorphic near } a$$

By Mittag-Leffler, there is $h \in \mathcal{O}(\Omega \setminus E)$ such that $\psi_a = h - \frac{\phi_a}{g}$ is holomorphic near every a . Put $f = g \cdot h$.

Then $f - \phi_a = g \cdot \psi_a$. This proves the theorem. \square

If ϕ_a is meromorphic, then we can find such f without any other zeroes:

Theorem 5. If $E \subset \Omega$ is discrete and for every $a \in E$ there is given $\phi_a \in \mathcal{M}(D^*(a, r_a))$, then there $\exists f \in \mathcal{M}(\Omega)$ which is holomorphic and nonzero outside E such that $\text{ord}_a(f - \phi_a) > k_a$ for all $a \in E$.

Proof:

- $E_0 = \{a; \phi_a \neq 0\}$
- $m_a = \text{ord}_a \phi_a, a \in E_0.$

• By Weierstrass, pick $g \in \mathcal{K}(\Omega)$ s.t.

$$\text{ord}_a g = \begin{cases} m_a & a \in E_0 \\ > k_b & b \in E \setminus E_0. \end{cases}$$

$g \neq 0$ in $\Omega \setminus E.$

• If we multiply g by $e^{h(z)}$, then everything holds except possibly $\text{ord}_a (f - \phi_a) > k_a$ for $a \in E_0$. How can we achieve this?

$$\text{ord}_a (g e^h - \phi_a) = \text{ord}_a g \left(e^h - \frac{\phi_a}{g} \right) = \text{ord}_a g (e^h - e^{h_a})$$

Hol + nonzer near a

$$= \text{ord}_a g e^{h_a} (e^{h-h_a} - 1) = m_a + \text{ord}_a (h-h_a).$$

By Theorem 4 there is $h \in \mathcal{O}(\Omega)$ such that $\text{ord}_a (h-h_a) > |m_a| + k_a$

This completes the proof.

Automorphism of the disc

Def: An automorphism of an open set $\Omega \subset \mathbb{C}$ is a biholomorphic map of Ω onto itself, i.e. a holomorphic map $f: \Omega \rightarrow \Omega$ which has a holomorphic inverse. Denoted by $\text{Aut}(\Omega)$. This is a group.

$D = D(0, 1) = \{ |z| < 1 \}$ the unit disc; $T = \{ \lambda; |\lambda| = 1 \}$

Theorem 1. Schwarz lemma.

If $f \in \mathcal{O}(D)$, $|f(z)| \leq 1$ for all $z \in D$ and $f(0) = 0$, then

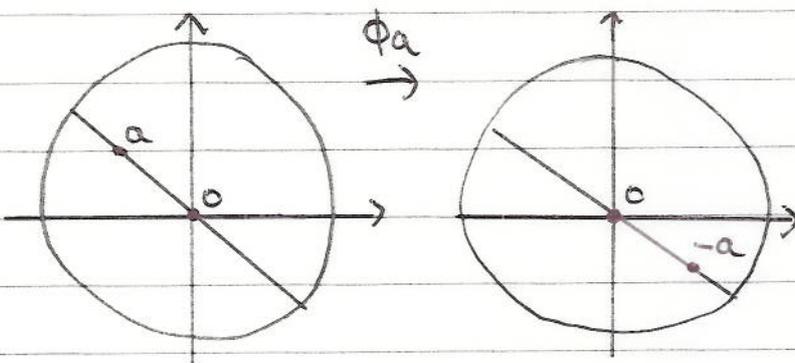
$$|f'(0)| \leq 1 \quad \text{and} \quad |f(z)| \leq |z|$$

Equality holds for some $z \in D \Leftrightarrow f(z) = \lambda z$, $|\lambda| = 1$.

Proof: Let $g(z) = \frac{f(z)}{z}$, $g(0) = f'(0)$. Then $g \in \mathcal{O}(D)$ and $\lim_{z \rightarrow \zeta \in T} |g(z)| \leq 1$, hence the maximum modulus theorem implies that either $|g(z)| < 1$ for all $z \in D$ or $g(z) \equiv \lambda \in T$.

In the first case $|f(z)| < |z|$ and $|f'(0)| < 1$, in the second case $f(z) = \lambda z$.

For $a \in D$, let $\phi_a(z) = \frac{z-a}{1-\bar{a}z}$ $\phi_a(a) = 0$, $\phi_a(0) = -a$



If $|z| = 1$, then

$$\begin{aligned} |\phi_a(z)| &= \left| \frac{z-a}{(1-\bar{a}z)\bar{z}} \right| \\ &= \left| \frac{z-a}{z-\bar{a}} \right| = 1. \end{aligned}$$

Hence $\phi_a: D \rightarrow D$. Easy to see that $\phi_a^{-1} = \phi_{-\bar{a}}$.

ϕ_a is an automorphism

Theorem 2 Every automorphism of D is of the form $\psi(z) = \lambda \phi_a(z)$ for some $\lambda \in T$.

Proof: If $\psi(0) = 0$, then $(\psi^{-1})'(0) \cdot \psi'(0) = 1$. Since $\psi, \psi^{-1} \in \text{Aut}(D)$ and are 0 at 0, their derivatives at zero must be ≤ 1 in absolute value. $<$ is impossible, so $|\psi'(0)| = 1$ and $\psi = \lambda z$ by the Schwarz lemma.

In general, if $\psi(a) = 0$, consider $\phi = \psi \circ \phi_{-a}$. Then $\phi \in \text{Aut}(D)$, $\phi(0) = 0$, so $\phi(z) = \lambda z$ hence $\psi(z) = \lambda \phi_a(z)$.

Hurwitz theorem If Ω is connected $f_n \in \mathcal{O}(\Omega)$ without zeros and $f_n \rightarrow f \neq 0$ uniformly on compact, then f is without zeros.

Proof: Let $a \in \Omega$ and pick $r > 0$ such that f has no zeros on $\gamma = \{z; |z-a|=r\}$. Then $f_n'/f_n \rightarrow f'/f$ uniformly on γ , so

$$\# \text{ zeros of } f \text{ in } D(a,r) = \frac{1}{2\pi i} \int_{\gamma} \frac{f'}{f} dz = \lim \frac{1}{2\pi i} \int_{\gamma} \frac{f_n'}{f_n} dz = 0$$

Corollary If Ω is connected, $f_n \in \mathcal{O}(\Omega)$ are injective and $f_n \rightarrow f \neq c$ uniformly on compact, then f is injective.

Proof: If $f(a) = f(b) = w$ and $D(a,r) \cap D(b,r) = \emptyset$, then by Hurwitz theorem $f_n(z) - w$ must have a zero in both $D(a,r)$ and $D(b,r)$ for sufficiently large n , hence f_n is not injective.

Riemann mapping theorem

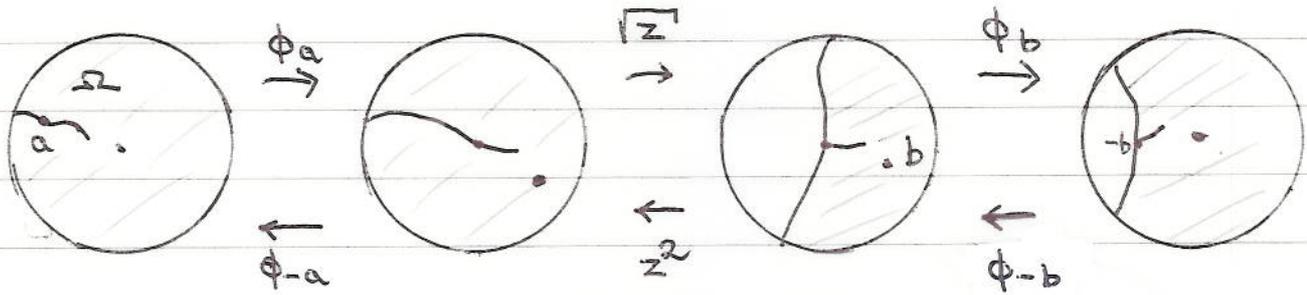
Theorem 1. If $\Omega \neq \mathbb{C}$ is simply connected (and connected), then Ω is biholomorphic to D .

- We shall see that this follows from the fact that every $f \in \mathcal{O}(\Omega)$, f without zeros, has a holomorphic square root. This is true in a simply connected domain since f has a holomorphic logarithm. If $g = e^{\frac{1}{2} \log f}$, then $g^2 = f$.
- $f: \Omega \rightarrow \mathbb{C}$ is biholomorphic onto its image $\Leftrightarrow f$ is injective
- The square root property is invariant under biholomorphism.
- If $f: \Omega \rightarrow \Omega'$ is biholomorphic and has a holomorphic square root, then \sqrt{f} is also biholomorphic. Also; if $w \in \text{Im}(\sqrt{f})$, then $-w \notin \text{Im}(\sqrt{f})$.

Proposition (Koebe) If $0 \in \Omega \subset D$, $\Omega \neq D$ is connected and has the square root property, then there is a $\chi \in \mathcal{O}(\Omega)$ such that

- (i) $\chi(0) = 0, \chi(\Omega) \subset D$
- (ii) χ is injective
- (iii) $|\chi(z)| > |z|$ for all $z \in D, z \neq 0$.

Proof: Pick $a \in D \setminus \Omega$



Let $\mathcal{H} = \phi_b \circ \sqrt{z} \circ \phi_a$. Then (i) and (ii) holds.

\mathcal{H}^{-1} is defined in all of D and is 2-1 (except at $-b$), therefore $|\mathcal{H}^{-1}(w)| < |w|$ for all $w \neq 0$, so $|\mathcal{H}(z)| > |z|$ for all $z \neq 0$. ■

Proof Theorem 1. We know that Ω has the square root property.

Step 1. To map Ω biholomorphically onto a bounded domain

Pick $a \in \mathbb{C} \setminus \Omega$ and $g \in \mathcal{O}(\Omega)$ such that $g^2(z) = z - a$.

If $D(r, \nu) \subset g(\Omega)$ (which is open), then $D(-r, \nu) \cap g(\Omega) = \emptyset$

and

$$\psi(z) = \frac{1}{g(z) + r}$$
 is biholomorphic in Ω and

$$|\psi(\nu)| < \frac{1}{\nu}.$$

For small ϵ , $h(z) = \epsilon(\psi(z) - \psi(z_0))$, is biholomorphic onto $0 \in \Omega_0 \subset D$. Ω_0 has the square root property.

Step 2. We shall produce a biholomorphic map $\Omega_0 \rightarrow D$ which is "maximal". Let

$$F = \{ f: \Omega_0 \rightarrow D; f \text{ is hol, injective and } f(0) = 0 \}$$

Let $z_0 \in \Omega_0$, $z_0 \neq 0$ and put

$$\alpha = \sup_{f \in \mathcal{F}} |f(z_0)| \in (0, 1].$$

and pick $f_n \in \mathcal{F}$ such that $\lim_{n \rightarrow \infty} |f_n(z_0)| = \alpha$. By Montel's

theorem there is a convergent subsequence, i.e. we

may assume $f_n \rightarrow f$ u.c.c. Since $f(0) = 0$ and $|f(z_0)| = \alpha > 0$,

f is not constant. By corollary of Hurwitz theorem,

f is injective, so f is a biholomorphism $f: \Omega_0 \rightarrow \Omega_1 = f(\Omega_0) \subset \mathbb{D}$

We cannot have $\Omega_1 = \mathbb{D}$, because by Koebe's theorem

there is a $\chi: \Omega_1 \rightarrow \mathbb{D}$ injective such that

$|\chi(f(z_0))| > |f(z_0)| = \alpha$, contradicting the definition of α . \blacksquare

It is instructive to read Theorem 1 of section 7.3.

Schwarz-Pick and Ahlfors lemma.

$$\varphi_a(z) = \frac{z-a}{1-\bar{a}z}$$

$$\varphi_a'(z) = \frac{1-|a|^2}{(1-\bar{a}z)^2}$$

$$\varphi_a'(0) = 1-|a|^2$$

$$\varphi_a'(a) = \frac{1}{1-|a|^2}$$

If $f: D \rightarrow D$ is holomorphic, $z \in D$, let

$$g = \varphi_{f(z)} \circ f \circ \varphi_{-z}$$

Then $g(0) = 0$ and

$$g'(0) = \varphi_{f(z)}'(f(z)) \cdot f'(z) \cdot \varphi_{-z}'(0)$$

$$= \frac{1}{1-|f(z)|^2} \cdot f'(z) \cdot (1-|z|^2)$$

We get

Theorem 1.1. If $f: D \rightarrow D$ is holomorphic, then

$$\frac{|f'(z)|}{1-|f(z)|^2} \leq \frac{1}{1-|z|^2}$$

Equality at one point implies that f is an automorphism.

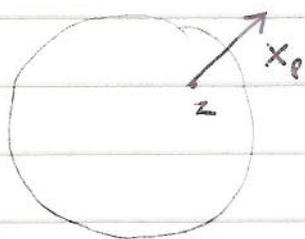
Pf: The last statement follows from $g(w) = \lambda w$, so

$$f(w) = \varphi_{-f(z)}(\lambda \varphi_z(w)) \Rightarrow f = \varphi_{-f(z)} \circ (\lambda \varphi_z)$$

This formulation is equivalent to the Schwarz lemma.
 Plick gave an invariant definition of this:
 Consider the (Kähler) metric

$$ds_{\mathbb{H}}^2 = \frac{dz d\bar{z}}{(1-|z|^2)^2}$$

on \mathbb{D} , i.e. for a tangent vector $X \in T_p \mathbb{D}$, $p \in \mathbb{D}$



$$ds_{\mathbb{H}}^2(X) = \frac{|X|^2}{(1-|z|^2)^2}$$

Then

$$f^*(ds_{\mathbb{H}}^2) = \frac{|f'(z)|^2}{(1-|f(z)|^2)^2} dz d\bar{z} \leq \frac{dz d\bar{z}}{(1-|z|^2)^2} = ds_{\mathbb{H}}^2 \quad \text{i.e.}$$

$$f^*(ds_{\mathbb{H}}^2) \leq ds_{\mathbb{H}}^2$$

with equality at one point iff f is an automorphism.

- We can define length of curves $\gamma: [a, b] \rightarrow \mathbb{D}$ using the metric $ds_{\mathbb{H}}$:

$$L(\gamma) = \int_a^b ds_{\mathbb{H}}(\gamma(t), \gamma'(t)) dt$$

It follows that holomorphic functions decrease the length of curves:

$$L(f \circ \gamma) \leq L(\gamma)$$

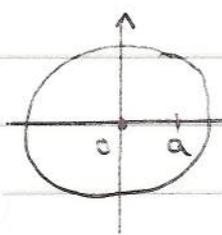
and automorphisms preserve length.

- This defines a distance on D by

$$\rho_N(z_1, z_2) = \inf L(\gamma) \quad \gamma \text{ curve from } z_1 \text{ to } z_2.$$

Holomorphic functions are distance decreasing and automorphisms preserve distances. It follows that

$$\rho_N(z_1, z_2) = \rho_N(0, |\varphi_{z_1}(z_2)|)$$



$$\rho_N(0, a) = \int_0^a \frac{dt}{1-t^2} = \frac{1}{2} \log \frac{1+a}{1-a}$$

$$\rho_N(z_1, z_2) = \frac{1}{2} \log \frac{1+|\varphi_{z_1}(z_2)|}{1-|\varphi_{z_1}(z_2)|}$$

Theorem 1.2. If $f: D \rightarrow D$ is holomorphic, then

$$1. f^*(d\rho_N) \leq d\rho_N$$

$$2. \rho_N(f(z), f(w)) \leq \rho_N(z, w)$$

Equality in one point in 1 or one pair $z \neq w$ in 2 implies f is auto.
 $d\rho_N$ is called the Poincaré metric

ρ_N

Poincaré distance.

- The curvature of a metric $h dz d\bar{z}$ is defined by

$$\mathcal{K} = -\frac{2}{h} \frac{\partial^2}{\partial z \partial \bar{z}} \log h$$

$$\text{For } h = \frac{1}{(1-|z|^2)^2} \text{ we get}$$

$$\mathcal{K} = -2(1-|z|^2)^2 \frac{\partial}{\partial z \partial \bar{z}} \log (1-|z|^2)^{-2}$$

$$= 4(1-|z|^2)^2 \frac{\partial}{\partial z \partial \bar{z}} \log (1-z\bar{z}) = 4(1-|z|^2)^2 \frac{\partial}{\partial z} \frac{-z}{1-z\bar{z}}$$

$$= 4(1-|z|^2)^2 \cdot \frac{-1(1-z\bar{z}) - (-z) \cdot (-\bar{z})}{(1-z\bar{z})^2} = 4(1-|z|^2)^2 \cdot \frac{-1}{(1-z\bar{z})^2} = -4$$

$do_N =$

- If $R dz d\bar{z}$ is metric on Ω and $f: U \rightarrow \Omega$ satisfies $f'(z) \neq 0$ everywhere, then

$$f^*(do_N) = |f'(z)|^2 R(f(z)) dz d\bar{z}$$

and

$$\chi_{f^*(do_N)}(z) = \chi_{do_N}(f(z)).$$

- The metric $do_a^2 = \frac{4\tilde{a}^2}{A} \frac{dz d\bar{z}}{(a^2 - |z|^2)^2}$ on $D_a = \{|z| < a\}$

has curvature $-A$. Theorem 1.2. generalizes to

Theorem 1.3. Ahlfors lemma.

If M is a Riemann surface with metric do_M^2 with curvature $\leq -B$ ($B > 0$) and $f: D_a \rightarrow M$ is holomorphic,

then $f^*(do_M^2) \leq \frac{A}{B} do_a^2$

Before proof, which M can have ~~an~~ metric with negative curvature?

1. \mathbb{C} does not have such a metric.

Pf: If $do_{\mathbb{C}}^2$ is such a metric, let $f: D \rightarrow \mathbb{C}$ be defined by $f(z) = az$. Then

$$(f^* do_{\mathbb{C}}^2)(0) = |a|^2 do_{\mathbb{C}}^2(0), \text{ hence no such}$$

inequality can hold.

2. $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ does not have such a metric, since

$f(z) = z^2$ is a covering $\mathbb{C} \rightarrow \mathbb{C}^*$, hence if \mathbb{C}^* had a metric with negative curvature, so would \mathbb{C} .

3. The upper half plane \mathbb{C}^+ has such a metric since it is biholomorphic to \mathbb{D} . A biholomorphic map is

$$f(z) = \frac{z-i}{z+i} \quad f'(z) = \frac{2i}{(z+i)^2}$$

$$f^* \left(\frac{dzd\bar{z}}{(1-|z|^2)^2} \right) = \frac{|f'(z)|^2}{(1-|f(z)|^2)^2} dzd\bar{z} = \frac{4}{|z+i|^2 \left(1 - \left|\frac{z-i}{z+i}\right|^2\right)^2} dzd\bar{z}$$

$$= \frac{4}{(|z+i|^2 - |z-i|^2)^2} dzd\bar{z} = \frac{4 dzd\bar{z}}{((x^2+(y+1)^2) - (x^2+(y-1)^2))^2} = \frac{4 dzd\bar{z}}{(4y)^2}$$

$$= \frac{1}{4y^2} dzd\bar{z}$$

4. $\mathbb{C} \setminus \{0, 1\}$ has a metric with constant negative curvature. We shall prove this later using the modular function $\lambda: \mathbb{C}^+ \rightarrow \mathbb{C} \setminus \{0, 1\}$.