

$$\text{Def. } \Theta^*(\Omega) = \{ f \in \Theta(\Omega); f(z) \neq 0 \quad \forall z \in \Omega \} \quad A$$

Review, part 2.

Th. $D = D(a, r)$ disc. If $f \in \Theta(D)$, then f has a holomorphic antiderivative, i.e. there is $F \in \Theta(D)$ such that $F' = f$.

If $f \in \Theta^*(D)$, then f has a holomorphic logarithm and m -th root of any order.

Pf: We know that $f = \sum_{n=0}^{\infty} c_n (z-a)^n$ in D .

$$\text{Let } F = \sum_{n=0}^{\infty} \frac{c_n}{n+1} (z-a)^{n+1}.$$

If $f \in \Theta^*(D)$, then $\frac{f'}{f} \in \Theta(D)$ and there is $F \in \Theta(D)$

such that $F' = \frac{f'}{f}$. Then $g = f e^{-F} \in \Theta^*(D)$ and
 $g' = f' e^{-F} + f \cdot e^{-F} \cdot \left(-\frac{f'}{f}\right) = 0$, hence $g = c \neq 0$, a constant.

Pick $\alpha \in \mathbb{C}$ such that $e^\alpha = c$. Then $f = e^{F+\alpha}$, so $G = F + \alpha$ is a holomorphic logarithm and $e^{\frac{m}{m}G}$ is a holomorphic m -th root for any $m \in \mathbb{N}$.

- Remark: This result is true in any simply connected domain Ω .

- Theorem 3.11 If Ω is a domain and $f \in \Theta(\Omega)$ is nonconstant, then $f(\Omega)$ is open.

Pf: Pick $a \in \Omega$. We have to show that $f(\Omega)$ contains a nbh of $f(a)$. We may assume $a = 0 = f(a)$.

Ω contains a disc $D = D(0, r)$ and f is not constant in D . If $f(D)$ does not contain a nbh of 0 , there exist $a_j \rightarrow 0$ such that $f(z) \neq a_j$ in D .

i.e. $g_j = \frac{1}{f - a_j} \in \Theta(D)$. If $r' < r$ is such that

$f(z) \neq 0$ for all z with $|z|=r$, then $|g_j|$ is uniformly bounded on this circle, but $|g_j(0)| = |g_j| \rightarrow \infty$ as $j \rightarrow \infty$. This contradicts the maximum principle on a disc.

Cor 3.12 (Maximum principle) If Ω is a domain, $f \in \Theta(\Omega)$ and $a \in \Omega$ such that $|f(z)| \leq |f(a)|$ for all $z \in \Omega$, then f is constant.

Pf: Follows from Open Mapping Th.

Prop 3.13 (Hurwitz theorem). If Ω is a domain, $f_j \in \Theta^*(\Omega)$ and $f_j \rightarrow f$ uniformly on compact then either $f \in \Theta^*(\Omega)$ or $f \equiv 0$ in Ω .

Pf: If $f(a) = 0$ and $f \neq 0$, pick $\epsilon > 0$ such that $f(z) \neq 0$ when $|z-a| = r$. Then $|f(z)| \geq \delta > 0$ when $|z-a| = r$. Since $|f_j(z)| \geq \frac{1}{2}\delta$ when $|z-a| = r$ for sufficiently large j . Therefore $g_j = \frac{1}{f_j} \in \Theta(\Omega)$ and $|g_j(z)| \leq \frac{2}{\delta}$ when $|z-a| = r$. But this is impossible, since $g_j(a) = \frac{1}{f_j(a)} \rightarrow \infty$ when $j \rightarrow \infty$. ■

* If $f \in \Theta(\Omega)$ has a zero of order k at a , then $f(z) = (z-a)^k g(z)$, where $g \in \Theta(\Omega)$ and $g(a) \neq 0$

$$\text{Then } \frac{f'}{f} = \frac{k}{z-a} + \frac{g'}{g} \text{ near } a$$

$$\text{and } \int_{|z-a|=r} \frac{f'}{f} dz = 2\pi i \cdot k \text{ for } r \text{ small.}$$

This gives :

Prop 3.14. If $\Omega \subset \mathbb{C}$ has piecewise smooth C^1 boundary, $f \in \Theta(\Omega) \cap C^1(\bar{\Omega})$ with $f(z) \neq 0$ for all $z \in b\Omega$, then

$$\int_{b\Omega} \frac{f'}{f} dz = 2\pi i \sum_{z \in \Omega} \text{ord}_z(f)$$

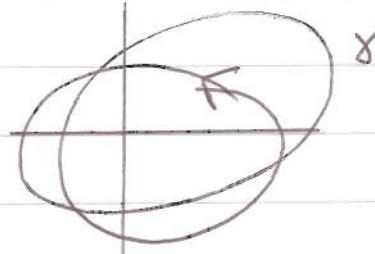
Pf: If a_1, \dots, a_m are the zeroes of f with orders k_1, \dots, k_m , D_1, \dots, D_m are small discs around the a_j 's, $\Omega' = \Omega \setminus \bigcup_{j=1}^m D_j$. Then Cauchy's theorem gives

$$0 = \int_{b\Omega'} \frac{f'}{f} dz = \int_{b\Omega} \frac{f'}{f} dz - \sum_{j=1}^m \int_{bD_j} \frac{f'}{f} dz = \int_{b\Omega} \frac{f'}{f} dz - 2\pi i \sum_{j=1}^m k_j.$$

Also called argument principle



$$\xrightarrow{f}$$



$$\int_{b\Omega} \frac{f'}{f} dz = \int_{\gamma} \frac{1}{z} dz = 2\pi i \cdot \text{winding number}.$$

- This is still true if f has poles in Ω .
- Theorem 3.15 (Rouche's theorem) $\Omega \subset \mathbb{C}$ as above, $f, g \in \Theta(\Omega) \cap C^1(\bar{\Omega})$ such that $|f(z) - g(z)| < |f(z)|$ for all $z \in b\Omega$. Then f and g have the same number of zeroes in Ω , i.e.

$$\sum_{z \in \Omega} \text{ord}_z f = \sum_{z \in \Omega} \text{ord}_z g.$$

Pf: Clearly f has no zeroes on $b\Omega$ and $\left|1 - \frac{g(z)}{f(z)}\right| < 1$ on $b\Omega$, so $F = \frac{g}{f}$ takes values in the disk $D(1, 1)$ on $b\Omega$ and therefore has a holomorphic logarithm near $b\Omega$. We have

$$(\log F)' = \frac{F'}{F} = -\frac{\frac{gf-f'g}{f^2}}{\frac{f}{g}} = \frac{g'}{g} - \frac{f'}{f}$$

Hence

$$0 = \int_{b \in \Omega} (\log F)' dz = \int_{b \in \Omega} \frac{g'}{g} - \int_{b \in \Omega} \frac{f'}{f} = \sum_{z \in \Omega} \text{ord}_z g - \sum_{z \in \Omega} \text{ord}_z f.$$

Prop 3.17 If Ω is a domain, $f_j \in \Theta(\Omega)$ are injective for all j and $f_j \rightarrow f$ uniformly on compact, then either f is injective or f is constant.

Proof: Assume $a, b \in \Omega$ and $f(b) = f(a)$. Let $g_j(z) = f_j(z) - f(a)$. Then $g_j \in \Theta^*(\Omega \setminus \{a\})$ and $g_j \rightarrow f - f(a)$ uniformly on compact. Then either $f - f(a)$ is constant, which must be zero, or $f \equiv f(a)$ or $f - f(a)$ is without zeroes, which contradicts the fact that $f(b) = f(a)$.

Prop 3.18+19 If $f \in \Theta(\Omega)$ is injective, then $f'(z) \neq 0$ for all $z \in \Omega$ and f has a holomorphic inverse $f^{-1} \in \Theta(f(\Omega))$.

Proof: We may assume $z=0$ and $f(z)=0$. We shall show that f has a zero of order 1 at 0. We have that $f(z) = z^k g(z)$ with $g \in \Theta(\Omega), g(0) \neq 0, k \in \mathbb{N}$.

In a disc D_n , g has a holomorphic k -th root, i.e.

there is $h \in \Theta(D_n)$ with $g(z) = h(z)^k$ and $h(0) \neq 0$.

We get $f(z) = (z h(z))^k$. The function $z h(z)$ is nonconstant, hence o.p.m. But then f takes values in a small disc at least k times in D_n . Hence $k=1$.

By the inverse mapping theorem f has a C^∞ smooth

inverse f^{-1} : $f(z) \rightarrow \Omega$. The derivative df^{-1} is the inverse of df , hence it is complex linear and f^{-1} is holomorphic.

$$\bullet A(r, s) = \{z \in \mathbb{C} \mid r < |z| < s\}, \quad 0 \leq r < s \leq \infty$$

Prop 3.19. (Laurent expansion) If $f \in \Theta(A(r, s))$ then f has a unique Laurent series expansion in $A(r, s)$

$$f(z) = \sum_{j=-\infty}^{\infty} c_j z^j$$

where $c_j = \frac{1}{2\pi i} \int_{|z|=p} \frac{f(z)}{z^{j+1}} dz$, any $p \in (r, s)$. The series

$\sum_{j \geq 0} c_j z^j$ converges for $|z| < s$ and the series $\sum_{j < 0} c_j z^j$

converges for $|z| > r$.

Proof: The Cauchy theorem gives that $\int_{|z|=p} \frac{f(z)}{z^{j+1}} dz$ is independent of $p \in (r, s)$. Let $g \in A(r, s)$ and pick r', s' such that

$$r < r' < |g| < s' < s.$$

By the Cauchy-Stokes formula, we have

$$f(g) = \frac{1}{2\pi i} \int_{|z|=s'} \frac{f(z)}{z-g} dz - \frac{1}{2\pi i} \int_{|z|=r'} \frac{f(z)}{z-g} dz$$

$$= \frac{1}{2\pi i} \int_{|z|=s'} \frac{f(z)}{z} \frac{1}{1-\frac{g}{z}} dz + \frac{1}{2\pi i} \int_{|z|=r'} \frac{f(z)}{z} \frac{1}{1-\frac{g}{z}} dz$$

$$= I + II.$$

$$\text{I} = \frac{1}{2\pi i} \int_{|z|=r} \frac{f(z)}{z} \cdot \sum_{j=0}^{\infty} \left(\frac{z}{r}\right)^j dz = \sum_{j=0}^{\infty} \left(\frac{1}{2\pi i} \int_{|z|=r} \frac{f(z)}{z^{j+1}} dz \right) r^j$$

$$\text{II} = \frac{1}{2\pi i} \int_{|z|=r} \frac{f(z)}{z} \cdot \sum_{j=0}^{\infty} \left(\frac{z}{r}\right)^j dz = \sum_{j=0}^{\infty} \left(\frac{1}{2\pi i} \int_{|z|=r} f(z) z^j dz \right) r^{-(j+1)}$$

$$= \sum_{j'=-k}^0 \left(\frac{1}{2\pi i} \int_{|z|=r} f(z) z^{-(j'+1)} dz \right) r^{j'}$$

- Exercise If $n=0$, $A(r, \delta)$ is the punctured disc $D_r^* = \{z \mid 0 < |z| < \delta\}$. f has a singularity at 0. There are three types

(1) Removable singularity: $a_n = 0$ for $n < 0 \Leftrightarrow f$ is bounded in D_0^*

(2) Pole of order k : $a_{-k} \neq 0$, $a_m = 0$ for $m < -k \Leftrightarrow |f| \rightarrow \infty$ when $z \rightarrow 0$.

(3) Essential singularity: $a_n \neq 0$ for infinitely many $n < 0$



$f(D_t^*)$ is dense in \mathbb{C} for all $0 < t \leq \delta$.

- Liouville's theorem If $f \in \mathcal{O}(\mathbb{C})$ is bounded, then f is constant.

Follows easily from Cauchy estimate of f' .