

## a winding number

Let  $\gamma$  be a closed piecewise  $C^1$  curve in  $\mathbb{C}$ . Then for  $z \in \mathbb{C} \setminus \gamma$ ,

$$\text{Ind}(\gamma, z) = \frac{i}{2\pi i} \int_{\gamma} \frac{dz}{z - z}$$

is called the winding number of  $\gamma$  around  $z$ . Clearly  $\text{Ind}(\gamma, z) \in \Theta(C^1 \gamma)$ .

Lemma  $\text{Ind}(\gamma, z) \in \mathbb{Z}$ .

Pf: Assume  $\gamma$  is parametrized over  $[0, 1]$ , so  $\gamma(0) = \gamma(1)$ .

Then

$$\frac{d}{dt} \frac{\int_0^t \frac{\gamma'(s)}{\gamma(s)-z} ds}{\gamma(t)-z} = \frac{-e \cdot \frac{\gamma'(t)}{\gamma(t)-z} - (\gamma(t)-z) - e \cdot \gamma'(t)}{(\gamma(t)-z)^2} = 0.$$

Hence is constant, which must be  $\frac{i}{\gamma(0)-z}$ . Then

$$\int_0^1 \frac{\gamma'(s)}{\gamma(s)-z} ds = \frac{\gamma(0)-z}{\gamma(1)-z} = t, \text{ hence } \int_0^1 \frac{\gamma'(s) ds}{\gamma(s)-z} = 2\pi i \cdot n \quad n \in \mathbb{Z}.$$

- $\text{Ind}(\gamma, z)$  is constant in each connected comp. of  $\mathbb{C} \setminus \gamma$ , 0 in the unbounded
- $\Omega$  is simply connected if any closed curve is homotopic to a constant curve.

TFAE:

- (1)  $\Omega$  is simply connected
- (2) Any two curves between two points  $a$  and  $b$  are homotopic
- (3) For any closed curve  $\gamma \subset \Omega$  and  $z \notin \Omega$ ,  $\text{Ind}(\gamma, z) = 0$ .

Lemma Suppose  $g \in \Theta^*(\Omega)$ . TFAE

- (1)  $g$  has a holomorphic logarithm in  $\Omega$  ( $e^f = g$ )
- (2)  $g'/g$  has a holomorphic primitive
- (3)  $\int_{\gamma} g'/g \, dz = 0$  for all closed curves

Proof:

$$(1) \Rightarrow (2) \quad \text{If } e^f = g, \text{ then } g'/g = f'$$

$$(2) \Rightarrow (1) \quad \text{If } g'/g = f', \text{ let } h = e^{-f} g. \text{ Then } h' = e^{-f} (g' - f'g) = 0$$

Hence  $h \equiv c$ , so  $g = c e^f = e^{f+\alpha}$

Eq. of (2) and (3) well known from calculus class.

- If  $\Omega$  simply connected, then  $g$  has a holomorphic logarithm because (3) holds.
- Lemma. If  $z_0$  and  $z_1$  are in the same component of  $\mathbb{C} \setminus K$ , then  $g(z) = \frac{z-z_0}{z-z_1}$  has a hol. logarithm in a nbh of  $K$ . If  $z_0$  is in the unbounded component of  $\mathbb{C} \setminus K$ ,  $g(z) = z-z_0$  has a hol. logarithm.

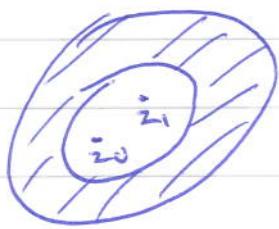
Pf: Pick a nbh  $\Omega$  of  $K$  such that  $z_0, z_1$  are in the same component of  $\mathbb{C} \setminus \Omega$ . Then

$$\frac{g'(z)}{g(z)} = \frac{1}{z-z_0} - \frac{1}{z-z_1}$$

Hence if  $\gamma \subset \Omega$  is a closed curve, then

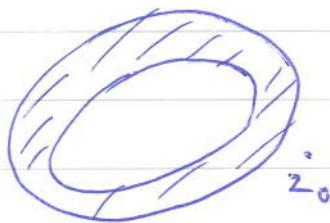
$$\int_{\gamma} \frac{g'(z)}{g(z)} \, dz = \int_{\gamma} \frac{dz}{z-z_0} - \int_{\gamma} \frac{dz}{z-z_1} = \text{Ind}(\gamma, z_0) - \text{Ind}(\gamma, z_1) = 0.$$

## Pushing zeroes



$$f(z) = \log \frac{z-z_0}{z-z_1} \in \Theta(K)$$

$\Rightarrow z - z_0 = e^{f(z)} (z - z_1)$ . Now, approximate  $f$  on  $K$  by  $\tilde{f}(z) \in \Theta(\mathbb{C} \setminus \{z_1\})$ , so  
 $z - z_0 \sim e^{\tilde{f}(z)} (z - z_1)$  on  $K$ .



$$f(z) = \log(z - z_0) \in \Theta(K)$$

$z - z_0 = e^{f(z)}$ . Approximate  $f$  on  $K$  by  $\tilde{f} \in \Theta(\mathbb{C})$   
 $\Rightarrow z - z_0 \sim e^{\tilde{f}(z)}$  on  $K$ . Thus we have

approximated  $z - z_0$  on  $K$  by a zero free entire function.

Theorem If  $K \subset \mathbb{C}$  is holomorphically convex, i.e.  $\hat{K}_{\alpha+1} = K$ .  
Then  $\Theta^*(\mathbb{C})|_K$  is dense in  $\Theta(K)$ .

Proof: Let  $f \in \Theta^*(K)$  and let  $\epsilon > 0$ ,  $\epsilon < \min \{|f(z)| ; z \in K\}$ .

Then there exists a rational function  $R(z) = \frac{P(z)}{Q(z)} \in \Theta(\mathbb{C})$

such that  $|f - R|_K < \frac{1}{2}\epsilon$ .  $P$  has no zeroes on  $K$ .  
Let  $a_1, \dots, a_k$  be the zeroes of  $P$  in the bounded components of  $\mathbb{C} \setminus K$ ,  $a_{k+1}, \dots, a_m$  the zeroes of  $P$  in the unbounded components of  $\mathbb{C} \setminus K$  and pick  $b_j, j=1, \dots, k$ ,  
 $b_j \in \mathbb{C}$ , in the same component as  $a_j$ . We may assume

$$P(z) = \prod_{j=1}^m (z - a_j)^{m_j}$$

$$\text{Then } g(z) = \sum_{j=1}^k m_j \log \left( \frac{z - a_j}{z - b_j} \right) + \sum_{j=k+1}^m m_j \log(z - a_j) \in \Theta(K)$$

$$\text{and } e^{g(z)} = \frac{P(z)}{\prod_{j=1}^k (z - b_j)^{m_j}} = \frac{P(z)}{P_0(z)}.$$

We have  $\min |Q(z)| = \delta > 0$ . Let  $M = \max_{z \in K} |P_0(z)|$ ,  
 $N = \max_{z \in K} |e^{g(z)}|$  and

let  $\mu > 0$  be given. If  $h \in \Theta(\Omega)$ ,  $|h - g|_K < \log(1 + \mu)$   
then  $|e^{h-g} - 1|_K < \mu$ . Hence for  $z \in K$ ,

$$\left| R(z) - \frac{P_0(z)e^{h(z)}}{Q(z)} \right| = \left| \frac{P_0(z)e^{g(z)}}{Q(z)} - \frac{P_0(z)e^{h(z)}}{Q(z)} \right| \leq \frac{M}{\delta} |e^{g(z)} - e^{h(z)}|$$

$$\leq \frac{M}{\delta} |e^{g(z)}| |1 - e^{h(z)-g(z)}| \leq \frac{MN}{\delta} \cdot \mu < \frac{1}{2}\epsilon \text{ when } \mu \text{ is suff.}$$

small. Therefore  $R_0(z) = \frac{P_0(z)}{Q(z)} e^{h(z)} \in \Theta^*(\Omega)$  is the required approximation.