

$$ds_n^2$$

- If  $\rho dz d\bar{z}$  is metric on  $\Omega$  and  $f: U \rightarrow \Omega$  satisfies  $f'(z) \neq 0$  everywhere, then

$$f^*(ds_n^2) = |f'(z)|^2 \rho(f(z)) dz d\bar{z}$$

and

$$\mathcal{R}_{f^*(ds_n^2)}(z) = \mathcal{R}_{ds_n^2}(f(z)).$$

- The metric  $ds_a^2 = \frac{4\tilde{a}}{A} \frac{dz d\bar{z}}{(a - |z|^2)^2}$  on  $D_a = \{|z| < a\}$   

$$\left(\frac{\partial}{\partial z} \log \frac{1}{a - |z|^2}\right)^2 = \frac{2a^2}{(1 - \frac{|z|^2}{a})^2}$$

has curvature  $-A$ . Theorem 1.2. generalizes to

### Theorem 1.3 Ahlfors lemma.

If  $M$  is a Riemann surface with metric  $ds_M^2$  with curvature  $\leq -B$  ( $B > 0$ ) and  $f: D_a \rightarrow M$  is holomorphic,

then

$$f^*(ds_M^2) \leq \frac{A}{B} ds_a^2$$

Before proof, which  $M$  can have a metric with negative curvature?

1.  $\mathbb{C}$  does not have such a metric.

Pf: If  $ds_{\mathbb{C}}^2$  is such a metric, let  $f: D \rightarrow \mathbb{C}$  be defined by  $f(z) = az$ . Then

$$(f^* ds_{\mathbb{C}}^2)(0) = |a|^2 ds_{\mathbb{C}}^2(0), \text{ hence no such}$$

inequality can hold. The metric  $(1 + |z|^2) dz d\bar{z}$  has curvature  $\chi = -2/(1 + |z|^2)$  and is complete.

2.  $\mathbb{C} \setminus \{0\}$  does not have such a metric, since  $f(z) = e^z$  is a covering  $\mathbb{C} \rightarrow \mathbb{C}^*$ , hence if  $\mathbb{C}^*$  had a metric with negative curvature, so would  $\mathbb{C}$ .

The metric  $\frac{dz d\bar{z}}{\log(1 + |z|^2)}$  has curvature  $\chi = \frac{-2}{(1 + |z|^2)^2} \left( \frac{1 + |z|^2}{\log(1 + |z|^2)} - 1 \right) < 0$  and is complete.

3. The upper half plane  $\mathbb{C}^+$  has such a metric since it is biholomorphic to  $D$ . A biholomorphic map is

$$f(z) = \frac{z-i}{z+i} \quad f'(z) = \frac{2i}{(z+i)^2}$$

$$f^* \left( \frac{dz d\bar{z}}{(1-|z|^2)^2} \right) = \frac{|f'(z)|^2}{(1-|f(z)|^2)^2} dz d\bar{z} = \frac{4}{|z+i|^2 (1 - | \frac{z-i}{z+i} |^2)^2} dz d\bar{z}$$

$$= \frac{4}{(|z+i|^2 - |z-i|^2)^2} dz d\bar{z} = \frac{4 dz d\bar{z}}{((x^2 + (y+1)^2) - (x^2 + (y-1)^2))^2} = \frac{4 dz d\bar{z}}{(4y)^2}$$

$$= \frac{1}{4y^2} dz d\bar{z}$$

4. The punctured disc  $D^*$  has such a metric. We have a covering map  $p : \mathbb{C}^+ \rightarrow D^*$  given by  $p(z) = e^{iz}$ . This has local inverses  $p^{-1}(w) = \frac{1}{i} \log w$  and

$$(p^{-1})^* \left( \frac{dz d\bar{z}}{4y^2} \right) = \frac{|(p^{-1})'(w)|^2 dw d\bar{w}}{4 (\operatorname{Im} p^{-1}(w))^2}$$

$$= \frac{dw d\bar{w}}{4w^2 (\log|w|)^2} = \frac{dw d\bar{w}}{4w^2 (\log|w|^2)^2} =: \frac{d\omega_{D^*}}{4}$$

This metric is also complete. If  $0 < r < R < 1$ , then

$$\rho_{D^*}(r, R) = \int_r^R \frac{dt}{t(\log t^2)} = -\frac{1}{2} \int_r^R \frac{dt}{t \log t} = -\frac{1}{2} \log(-\log t) \Big|_r^R$$

$$= \frac{1}{2} (\log(\log \frac{1}{r}) - \log(\log \frac{1}{R})) \rightarrow \infty \text{ when } r \rightarrow 0 \text{ or } R \rightarrow 1.$$

The circle  $\gamma(t) = r e^{it}$  has length

$$l(\gamma) = \int_0^{2\pi} \frac{r dt}{r \log r^2} = \frac{\pi/2}{\log(1/r)} \rightarrow 0 \text{ when } r \rightarrow 0.$$

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We also get Ahlfors lemma for maps from  $D^*$ .  
 (We have put  $A=1$ ).

### Theorem 1.3 b - Ahlfors lemma for $D^*$

If  $M$  is a Riemann surface with metric  $d\omega_M^2$  with curvature  $\leq -B$  ( $B > 0$ ) and  $f: D^* \rightarrow M$  is holomorphic, then

$$f^*(d\omega_M^2) \leq \frac{1}{B} d\omega_{D^*}^2$$

Proof: We have  $d\omega_{D^*}^2 = (p^{-1})^* d\omega_D^2$ .  $f \circ p: D \rightarrow M$  is holomorphic, so by the Ahlfors lemma for  $D$  we have

$$(f \circ p)^*(d\omega_M^2) = p^*(f^*(d\omega_M^2)) \leq \frac{1}{B} d\omega_D^2$$

which gives

$$f^*(d\omega_M^2) = (p^{-1})^*(p^*(f^*(d\omega_M^2))) \leq (p^{-1})^*\left(\frac{1}{B} d\omega_D^2\right) = \frac{1}{B} d\omega_{D^*}^2$$

5. The doubly punctured plane  $C \setminus \{z_0, z_1\}$  has a complete metric  $h(z) dz d\bar{z}$  with curvature bounded above by a negative constant.

$h$  satisfies  $h(z) \geq \frac{C}{(1+|z|^2) \log(1+|z|^2)^2}$  for  $|z|$

sufficiently large.

The proof of this is quite technical and will be postponed. Notice the following consequence of the estimate above; Given  $R_0 > 0$ . Then there are constants  $c', c''$ ,  $c' > 0$  such that the following holds; if  $\gamma: [a, b] \rightarrow \mathbb{C} \setminus \{z_0, z_1\}$  is a curve with  $|\gamma(a)| \leq R_0$ ,  $|\gamma(b)| > R$ , then

$$\ell(\gamma) \geq c' \log(\log R) - c'' := g(R).$$

for  $R$  sufficiently large. In particular  $\lim_{R \rightarrow \infty} \ell(\gamma) = \infty$ .

This implies completeness at  $\infty$ .

Theorem 1.4. Schottky's Theorem Given  $R_0 > 0$  and  $n < 1$ , then there is a constant  $M (= M(R_0, n))$  such that if  $f: D \rightarrow \mathbb{C} \setminus \{z_0, z_1\}$  is holomorphic and  $|f(0)| \leq R_0$ , then  $|f(z)| \leq M$  for all  $z$  with  $|z| \leq n$ .

Proof: Let  $\gamma$  be the curve  $\gamma(t) = tz$ . Then  $\ell(\gamma) = \frac{1}{2} \log \frac{1+|z|}{1-|z|}$   
 $\leq \frac{1}{2} \log \frac{1+n}{1-n}$ . Hence if  $|f(z)| = R$ , we have  $g(R) \leq \ell(f \circ \gamma)$   
 $\leq \ell(\gamma) \leq \frac{1}{2} \log \frac{1+n}{1-n}$ . But then  $R$  must be bounded since  
 $g(R) \rightarrow \infty$  when  $R \rightarrow \infty$ . ■

The same proof can be used to prove bounds on maps  $f: D^* \rightarrow \mathbb{C} \setminus \{a, b\}$

on either annular regions or circles. Here is the wide version;

Theorem 1.4 b. Schottky's Theorem in  $D^*$  Given  $R_0 > 0$  and  $n < 1$  there is a constant  $M$  such that if  $f: D^* \rightarrow \mathbb{C} \setminus \{z_0, z_1\}$  is holomorphic and  $|f(z)| \leq R_0$  for some  $z$  with  $|z| \leq n$ , then  $|f(g)| \leq M$  for all  $g$  with  $|g| = |z|$ .

Pf: We simply use the curve  $g(t) = ze^{it}$ ,  $0 \leq t \leq 2\pi$  whose length is  $\frac{\pi}{2\log(1/z^n)} \leq \frac{\pi}{2\log(1/n)}$ .

and Ahlfors lemma for  $D^*$ .

Theorem 1.5 If  $\Omega$  has a metric  $ds_\omega^2$  with curvature  $\leq -B$ , then there is no nonconstant holomorphic map  $f: \mathbb{C} \rightarrow \Omega$ .

Proof: (A=1). Let  $ds_\omega^2 = h(z) dz d\bar{z}$ . Then the Ahlfors lemma gives

$$f^*(ds_\omega^2) = |f'(z)|^2 h(f(z)) dz d\bar{z} \leq \frac{1}{B} ds_n^2 = \frac{1}{B} \frac{4n^2}{(r^2 - |z|^2)^2} dz d\bar{z}$$

$\rightarrow 0$  when  $n \rightarrow \infty$

Hence  $f'(z) \equiv 0$  and  $f$  is constant.

Corollary Picard's Little Theorem An entire holomorphic map cannot omit more than 1 value.

- Can omit one value, of course,  $f(z) = e^z \neq 0$ .

We now turn to Picard's Big Theorem.

Lemma If  $f \in \Theta(D_n^*)$  has an essential singularity at 0, then  $f(D_n^*)$  is dense in  $\mathbb{C}$ .

Pf: If not, there exist  $a \in \mathbb{C}$  and  $\delta > 0$  such that  $|f(z) - a| \geq \delta$  for all  $z \in D_n^*$ . But then  $g(z) = \frac{1}{f(z) - a}$

satisfies  $|g(z)| \leq \frac{1}{\delta}$ , hence has a removable singularity at 0. But then  $f(z) = \frac{1}{g(z)} + a$  either has a pole or a removable singularity at 0.

Theorem 1.6. Picard's Big Theorem If  $f \in \Theta(D_n^*)$  has an essential singularity at 0, then  $f$  cannot omit more than one value.

Can assume  $n=1$

Proof: Assume  $f : D^* \rightarrow \mathbb{C} \setminus \{a, b\}$ . Since  $f(D_1^*)$  is dense in  $\mathbb{C}$  for all  $n > 0$ , there exist  $z_n \in D^*$ ,  $|z_n| \rightarrow 0$ ,  $|z_n| \leq \frac{1}{2} (= n)$  such that  $|f(z_n)| \leq 1 (= R_0)$  for all  $n$ . By Schottky's Theorem in  $D^*$  it follows that  $|f(g)| \leq M$  for all  $g$  with  $|g| = |z_m|$  for some  $m$ . It follows that  $|f(z)| \leq M$  for all  $z \in D_{|z_0|}^*$ , contradicting the lemma.

## Proof of Ahlfors lemma

Define  $u \geq 0$  on  $D_a$  by  $f^*(d\omega_M^2) = u d\omega_a^2 = u(z) \frac{4\tilde{a}^2 dz d\bar{z}}{A(\tilde{a} - |z|^2)^2}$

and for  $n \leq a$ ,  $u_n$  is defined by  $f^*(d\omega_M^2) = u_n d\omega_n^2$  on  $D_n$ .

so  $u = u_a$  and

$$u_n(z) = u(z) \frac{\tilde{a}(n - |z|^2)}{n(\tilde{a} - |z|^2)}$$

so  $u_n \rightarrow u$  when  $n \rightarrow a$ . It is therefore sufficient to prove that  $u_n(z) \leq \frac{A}{B}$  for  $z \in D_n$ .

By the formula above,  $u_n(z) = 0$  when  $|z| = n$ . If  $u_n(z) \equiv 0$ , we are done. Otherwise,  $u_n$  has a maximum at some  $z_0 \in D_n$ . Then  $f$  defines local coordinates around  $z_0$ , i.e. there is a nbh  $U$  of  $z_0$  with  $f'(z) \neq 0$  for  $z \in U$  and we can compute the curvature of  $d\omega_M^2$  by computing it in  $U$ .

We have

$$f^*(d\omega_M^2) = u_n d\omega_n^2 = u_n(z) \frac{4\tilde{n}^2 dz d\bar{z}}{A(n - |z|^2)^2} = h(z) dz d\bar{z}$$

so

$$\Re = -\frac{2}{h} \frac{\partial}{\partial z \partial \bar{z}} \log h = -\frac{2}{h} \frac{\partial}{\partial z \partial \bar{z}} \left( \log u_n + \log \frac{4\tilde{n}^2}{A(n - |z|^2)^2} \right)$$

$$= -\frac{2}{h} \left( \frac{\partial}{\partial z \partial \bar{z}} \log u_n + \frac{2n^2}{(n^2 - |z|^2)^2} \right)$$

$$= -\frac{2}{h} \frac{\partial}{\partial z \partial \bar{z}} \log u_n - \frac{A}{u_n} \leq -B$$

$$\frac{2\partial}{h} \log u_n \geq B - \frac{A}{u_n}$$

but  $\frac{\partial}{\partial z \partial \bar{z}} \log u_n = \frac{1}{4} \Delta \log u_n(z) \leq 0$   
since  $z_0$  is a maximum.

This gives  $u_n(z_0) \leq \frac{A}{B}$ .