

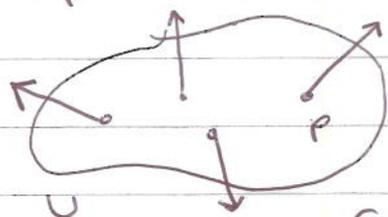
## Vector fields and forms on $U \subset \mathbb{C} = \mathbb{R}^2$

Def. A  $C^\infty$  vector field on  $U \subset \mathbb{R}^2$  is a  $C^\infty$  function  $X: U \rightarrow \mathbb{R}^2$ .

We often denote the value at  $p$  by  $X_p$  and visualize

$X_p$  as illustrated at  $p$ . The set of  $C^\infty$  vector fields is a real

vector space.



We have  $X_p = (\alpha(p), \beta(p)) =: \alpha \frac{\partial}{\partial x} + \beta \frac{\partial}{\partial y}$

$\alpha, \beta \in C^\infty(U) =$  set of  $C^\infty$  real functions

A vector field  $X$  acts on functions  $f \in C^\infty(U)$  by

taking the derivative of  $f$  in the  $X$  direction

$$X(f) = Jf \cdot X = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \alpha \frac{\partial f}{\partial x} + \beta \frac{\partial f}{\partial y}$$

Def.  $TU = U \times \mathbb{R}^2$  is called the tangent bundle of  $U$ . If  $X$  is a vector field, then the map  $U \rightarrow TU$  defined by  $p \rightarrow (p, X_p)$  is  $C^\infty$ .

Def. A  $C^\infty$  differential 1-form  $w$  on  $U$  is a map from  $C^\infty$  vector fields to  $C^\infty$  functions which is pointwise linear in the following sense:  $w(aX + bY) = aw(X) + bw(Y)$  for all  $a, b \in C^\infty(U)$  and vector fields  $X, Y$ .

Def.  $T^*U = U \times (\mathbb{R}^2)^*$  is called the cotangent bundle of  $U$ . If  $w$  is a 1-form, then the map  $U \rightarrow T^*U$  defined by  $p \rightarrow (p, w)$

Example. If  $f \in C^\infty(U)$ , the differential of  $f$  is the 1-form defined by

$$df(X) = X(f) = \alpha \frac{\partial f}{\partial x} + \beta \frac{\partial f}{\partial y}$$

It is clear that this is pointwise linear.

• In particular, if  $f = x$  or  $f = y$  we have

$$dx(X) = \alpha$$

$$dy(X) = \beta$$

• In general  $df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$

(2)

• At each point  $a \in U$ , the vector field

$$\frac{\partial}{\partial x}(a), \frac{\partial}{\partial y}(a)$$

is a basis for the tangent space at  $a$  and the dual basis is  $d_a x, d_a y$ .

• If  $\omega$  is a 1-form on  $U$  and  $\omega(\frac{\partial}{\partial x}) = \varphi$ ,  $\omega(\frac{\partial}{\partial y}) = \psi$ , then

$$\omega(X) = \omega(\alpha \frac{\partial}{\partial x} + \beta \frac{\partial}{\partial y}) = \alpha \omega(\frac{\partial}{\partial x}) + \beta \omega(\frac{\partial}{\partial y})$$

$$= \alpha \varphi + \beta \psi = \varphi dx(X) + \psi dy(X), \text{ i.e.}$$

$$\omega = \varphi dx + \psi dy$$

Hence all 1-forms is of this form for  $\varphi, \psi \in C^\infty(U)$ .

1-forms are also called cotangent vectors.

• We shall also allow forms with complex coefficients:

$$\mathcal{E}(U) = \{f: U \rightarrow \mathbb{C} \mid f \text{ is } C^\infty\}$$

$$\mathcal{E}^{(1)}(U) = \{\omega = \varphi dx + \psi dy \mid \varphi, \psi \in \mathcal{E}(U)\}.$$

If we split in  $\mathbb{C}$ -linear and  $\mathbb{C}$ -antilinear parts

$$\omega = \varphi dx + \psi dy$$

$$= \frac{1}{2} \underbrace{(\varphi - i\psi)}_f \underbrace{(dx + i dy)}_{dz} + \frac{1}{2} \underbrace{(\varphi + i\psi)}_g \underbrace{(dx - i dy)}_{d\bar{z}}$$

$$\stackrel{=:}{=} f dz + g d\bar{z}$$

If we regard  $X = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \in \mathbb{R}^2$  as a vector in  $\mathbb{C}$ , i.e.  $X = \alpha + i\beta$ ,

then  $dz(X) = X$ ,  $d\bar{z}(X) = \bar{X}$

(Hence  $dz d\bar{z}(X) = X \bar{X} = |X|^2$ )

- Forms of type (1,0)

$$\mathcal{E}^{1,0}(U) = \{ \omega = f dz \mid f \in \mathcal{E}(U) \}$$

Type (0,1)

$$\mathcal{E}^{0,1}(U) = \{ \omega = f d\bar{z} \mid f \in \mathcal{E}(U) \}$$

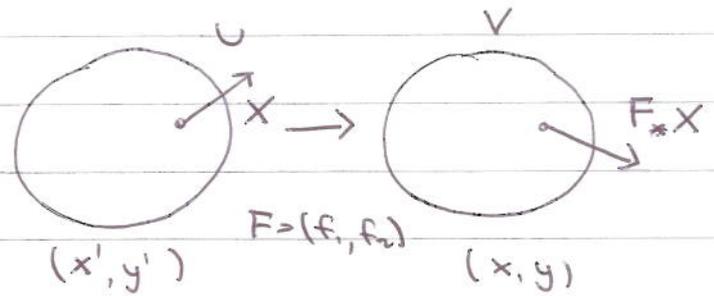
Holomorphic one-forms

$$\Omega(U) = \{ \omega = f dz \mid f \in \mathcal{O}(U) \}.$$

The corresponding sheaves are denoted  $\mathcal{E}^1$ ,  $\mathcal{E}^{1,0}$ ,  $\mathcal{E}^{0,1}$ ,  $\Omega$ .

Push forward and pullbacks of vector fields and forms.

If  $F: U \rightarrow V$  is a diffeomorphism and  $X$  a vector field on  $U$ , we can push it forward to  $V$ :



$$F_*X = JF \cdot X = \begin{pmatrix} \frac{\partial f_1}{\partial x'} & \frac{\partial f_1}{\partial y'} \\ \frac{\partial f_2}{\partial x'} & \frac{\partial f_2}{\partial y'} \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \dots$$

$$F_* \left( \frac{\partial}{\partial x'} \right) = \begin{pmatrix} \frac{\partial f_1}{\partial x'} \\ \frac{\partial f_2}{\partial x'} \end{pmatrix}$$

$$F_* \left( \frac{\partial}{\partial y'} \right) = \begin{pmatrix} \frac{\partial f_1}{\partial y'} \\ \frac{\partial f_2}{\partial y'} \end{pmatrix}$$

We can also pull back a form  $\omega$  on  $V$  to a form on  $U$  by

$$F^* \omega(X) = \omega(F_*X)$$

If  $\omega = \varphi dx + \gamma dy$  and  $X = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ , then

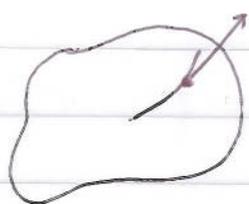
$$\begin{aligned} F^* \omega(X) &= \omega(F_* X) = \varphi \left( \frac{\partial f_1}{\partial x'} \alpha + \frac{\partial f_1}{\partial y'} \beta \right) + \gamma \left( \frac{\partial f_2}{\partial x'} \alpha + \frac{\partial f_2}{\partial y'} \beta \right) \\ &= \underbrace{\left( \varphi \frac{\partial f_1}{\partial x'} + \gamma \frac{\partial f_2}{\partial x'} \right)}_{\varphi'} \alpha + \underbrace{\left( \varphi \frac{\partial f_1}{\partial y'} + \gamma \frac{\partial f_2}{\partial y'} \right)}_{\gamma'} \beta = \varphi' dx' + \gamma' dy' \end{aligned}$$

$$\text{Hence } (\varphi' \ \gamma') = (\varphi \ \gamma) \begin{pmatrix} \frac{\partial f_1}{\partial x'} & \frac{\partial f_1}{\partial y'} \\ \frac{\partial f_2}{\partial x'} & \frac{\partial f_2}{\partial y'} \end{pmatrix} = (\varphi \ \gamma) JF.$$

We also have  $F^*(\varphi dx + \gamma dy) = \varphi df_1 + \gamma df_2$ .

• If  $\omega \in \Omega(V)$ ,  $\omega = f(z) dz$  and  $F: U \rightarrow V$  is holomorphic, then

$$F^* \omega = f(F(z)) F'(z) dz$$



• A vector  $X_p$  at  $p$  can also be thought of as

- 1) An equivalence class of curves
- 2) A derivation of  $C^\infty$  functions

- The  $d$ -operator  $d: \mathcal{F}(U) \rightarrow \mathcal{F}^{(1)}(U)$

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$$

$$= \frac{1}{2} \underbrace{\left( \frac{\partial f}{\partial x} - i \frac{\partial f}{\partial y} \right)}_{\frac{\partial f}{\partial z}} dz + \frac{1}{2} \underbrace{\left( \frac{\partial f}{\partial x} + i \frac{\partial f}{\partial y} \right)}_{\frac{\partial f}{\partial \bar{z}}} d\bar{z} = \frac{\partial f}{\partial z} dz + \frac{\partial f}{\partial \bar{z}} d\bar{z}$$

i.e.  $d = \partial + \bar{\partial}$  ,  $\partial: \mathcal{F}(U) \rightarrow \mathcal{F}^{(1,0)}(U)$

$\bar{\partial}: \mathcal{F}(U) \rightarrow \mathcal{F}^{(0,1)}(U)$ .

- If  $F: U \rightarrow V$  is a  $C^\infty$  map and  $f \in \mathcal{F}(V)$ , then  $df \in \mathcal{F}^{(1)}(V)$  and hence  $F^*(df) \in \mathcal{F}^{(1)}(U)$ . We have

$$\begin{aligned} F^*(df)(X) &= df(F_* X) = Jf \cdot (F_* X) = Jf \cdot (JF \cdot X) \\ &= (Jf \cdot JF) \cdot X = J(f \circ F) \cdot X = d(f \circ F)(X) \end{aligned}$$

Hence

$$F^*(df) = d(f \circ F)$$

"The  $d$  operator commutes with pullbacks."

- If  $F$  is also holomorphic,  $F^*$  preserves the linear and antilinear part, i.e.

$$F^*(\partial f) = \partial(f \circ F)$$

$$F^*(\bar{\partial} f) = \bar{\partial}(f \circ F)$$

## Vector fields and forms on a Riemann surface

- A vector field can be defined on a Riemann surface  $S$  like this: Given a complex atlas

$$U = \{ \varphi_i : U_i \rightarrow V_i \subset \mathbb{C} \}$$

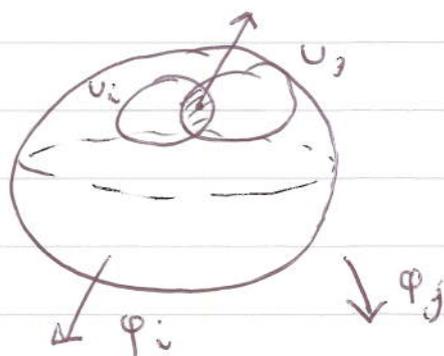
Assume for each  $i$  we are given a  $C^\infty$  vector field  $X_i$  on  $V_i$  which is independent of the local coordinates, i.e.

$$(\varphi_j \circ \varphi_i^{-1})_* (X_i) = X_j \quad (\text{when defined})$$

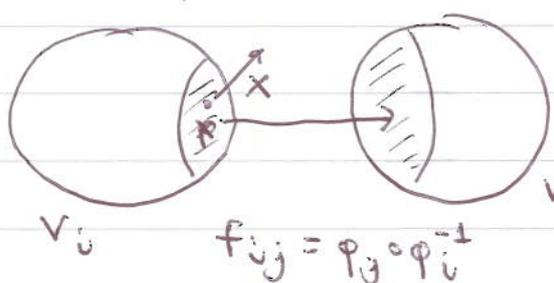
for all  $i$  and  $j$ . This defines a vector field  $X$  on  $S$ .

- It is possible to construct a tangent bundle  $TS$  of  $S$  by glueing together the tangent bundles

$$V_i \times \mathbb{R}^2 \text{ by identifying } (p, X) \in V_i \times \mathbb{R}^2 \text{ with } (f_{ij}(p), f_{ij*}(X)) \in V_j \times \mathbb{R}^2$$



We will not carry out the details of this.



We visualize the tangent vectors to  $S$  as attached to  $S$ .

There is a projection  $\pi : TS \rightarrow S$ .

A vector field may then be defined as a  $C^\infty$  map  $X$  from  $S$  to  $TS$  such that  $\pi \circ X(p) = p$  for all  $p \in S$ .

- Similarly we can define 1-forms on  $S$ , by a family of one-forms  $\omega_i$  on  $V_i$  such that

$$(\varphi_j \circ \varphi_i^{-1})^* (\omega_j) = \omega_i$$

The 1-forms act on tangent vectors to  $S$ . It is possible to construct an abstract "cotangent bundle"  $T^*S$ . The 1-forms are then maps from  $S$  to  $T^*S$ .

• The  $d$ -operator is defined on a RS by computing it local coordinates.

This is well defined. It does not depend on the local coordinates, since  $d$  commutes with pullbacks.

• The coordinate change maps  $\varphi_j \circ \varphi_i^{-1}$  are holomorphic. This implies that they preserve forms of type  $(1,0)$ ,  $(0,1)$  and holomorphic forms. Hence for any open set  $U \subset S$ , the spaces

$$\mathcal{E}(U), \mathcal{E}^{(1)}(U), \mathcal{E}^{1,0}(U), \mathcal{E}^{0,1}(U), \mathcal{Q}(U)$$

are well defined. The corresponding sheaves are denoted

$$\mathcal{E}, \mathcal{E}^1, \mathcal{E}^{1,0}, \mathcal{E}^{0,1}, \mathcal{Q}$$

• Finally, the  $\partial$  and  $\bar{\partial}$  operators are well defined.