MANDATORY ASSIGNMENT — 2015

1. Exercise on top of page 12 in the course notes.

2. Prove that if $\Omega \subseteq \mathbb{C}$ and $f \in \mathcal{H}(\Omega)$, then $f^{-1}(\mathbb{R})$ is *not* a non-empty compact subset of Ω.

3. Prove that if a power series $\sum_{n=0}^{\infty} a_n z^n$ has a finite positive radius of convergence $R \in]0, \infty[$, and if $a_n \geq 0$ for every index n, then the holomorphic function f defined by $f(z) := \sum_{n=0}^{\infty} a_n z^n$ has a "singularity" at $z := R$, in the sense that f cannot be extended holomorphically to $D(0, R) \cup D(R, \delta)$ for any $\delta > 0$.

4. Let $\Omega = \{z : |z| < 1 \text{ and } |2z - 1| > 1\}$, and suppose $f \in H(\Omega)$.
   (*a*) Must there exist a sequence of polynomials P_n such that $P_n \to f$ uniformly on compact subsets of Ω?
   (*b*) Must there exist such a sequence which converges to f uniformly in Ω?
   (*c*) Is the answer to (*b*) changed if we require more of f, namely, that f be holomorphic in some open set which contains the closure of Ω?

5. Is there a sequence of polynomials P_n such that

$$\lim_{n\to\infty} P_n(z) = \begin{cases} 1 & \text{if } \operatorname{Im} z > 0, \\ 0 & \text{if } z \text{ is real,} \\ -1 & \text{if } \operatorname{Im} z < 0? \end{cases}$$

6. Suppose Ω is a region, $f_n \in H(\Omega)$ for $n = 1, 2, 3, \ldots$, $f_n \to f$ uniformly on compact subsets of Ω, and f is one-to-one in Ω. Does it follow that to each compact $K \subset \Omega$ there corresponds an integer $N(K)$ such that f_n is one-to-one on K for all $n > N(K)$? Give proof or counterexample.

7. If $\alpha_n \in D = \{|z| < 1\}$, $\alpha_n \neq 0$ and $\sum_1^{\infty} (1 - |\alpha_n|) < \infty$, $k \in \mathbb{N}$ and

$$B(z) = z^k \prod_{n=1}^{\infty} \frac{\alpha_n - z}{1 - \bar{\alpha}_n z} \frac{|\alpha_n|}{\alpha_n}$$

then B is a bounded holomorphic function in D with zeroes precisely at α_n and 0.

8. Problems 1.4 and 1.5 in Forster.