SOLUTIONS

1 (1) If f is bounded in some punctured disc D_t^*, then for $n \geq 1$ we have

$$|a_{-n}| = \left|\frac{1}{2\pi i}\int_{|z|=\rho} f(z)\cdot z^{n-1}\,dz\right| = \left|\frac{1}{2\pi}\int_0^{2\pi} f(\rho e^{it})\rho^n e^{int}\,dt\right|$$

$$\leq \frac{1}{2\pi}\cdot 2\pi M\rho^n \to 0 \quad \text{as } \rho \to 0. \text{ Hence } a_{-n} = 0$$

and the singularity is removable.

(2) + (3) It is sufficient to prove $\Rightarrow$ in both statements.

(2) If $a_n = 0$ for $n < -k$, $a_{-k} \neq 0$.

$$f(z) = \sum_{n=-k}^{\infty} a_n z^n = \frac{1}{z^k}\sum_{n=0}^{\infty} a_{n-k} z^n = \frac{1}{z^k}\, g(z),$$

where $g \in \mathcal{O}(D_t)$ and $g(0) = a_{-k} \neq 0$. Hence $|f(z)| \to \infty$ $z \to 0$.

(3) If $f(D_t^*)$ is not dense in $\mathbb{C}$ for some t, then there is some $c \in \mathbb{C}$ such that $|f(z) - c| > \delta > 0$ for all $z \in D_t^*$. Hence

$$g(z) = \frac{1}{f(z) - c}$$

is bounded in D_t^*, hence has a removable singularity at 0. Then $g(z) = z^k h(z)$ for some $k \geq 0$ and $h(0) \neq 0$. and $f(z) = c + \frac{1}{z^k}\frac{1}{h(z)}$ will have a pole of order k at 0. (Removable singularity if $k = 0$).

2. If $f^{-1}(\mathbb{R}) = K \neq \emptyset$ is compact, then

$f(\Omega) \cap \mathbb{R} = f(K)$ is a compact subset of $\mathbb{R}$.

contradicting the fact that $f(\Omega)$ is open.
(If f is constant, then either $f^{-1}(\mathbb{R}) = \Omega$ or $f^{-1}(\mathbb{R}) = \emptyset$).

3. We shall use the fact that if $\sum_{n=0}^{\infty} c_n z^n$ converges in some disc $D(0,r)$ and $|d_n| \le |c_n|$ for all n, then $\sum_0^{\infty} d_n z^n$ also converges in $D(0,r)$.

If f is holomorphic in $D(0,R) \cup D(R,\delta)$, then for all $\varepsilon > 0$ f is holomorphic in $D(R-\varepsilon, \delta-\varepsilon)$ and is given by its power series there

$$f(z) = \sum_{n=0}^{\infty} c_n (z-(R-\varepsilon))^n$$

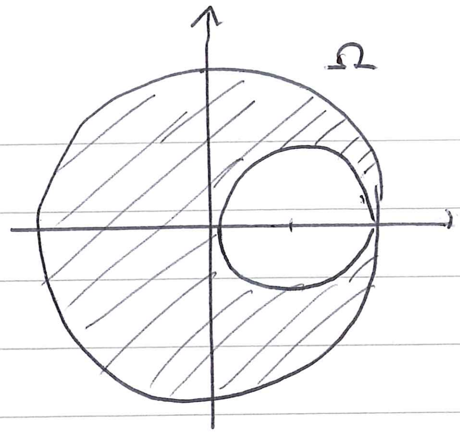
where $$c_n = \frac{f^{(n)}(R-\varepsilon)}{n!} = \sum_{k=0}^{\infty} \binom{n+k}{k} a_{n+k} (R-\varepsilon)^k \ge 0.$$

If we look at the power series at $(R-\varepsilon)e^{i\theta}$, its coefficients are

$$d_n = \sum_{k=0}^{\infty} \binom{n+k}{k} a_{n+k} (R-\varepsilon)^k \cdot e^{ik\theta}$$

so $|d_n| \le c_n$ since $a_n \ge 0$ for all n. Therefore the power series converges in $D((R-\varepsilon)e^{i\theta}, \delta-\varepsilon)$ for all $\varepsilon > 0$. It follows that f extends holomorphically to $D(Re^{i\theta}, \delta)$ for all θ, hence to $D(0, R+\delta)$. This contradicts the fact that R is the radius of convergence for f.

4.



$f \in \mathcal{O}(\Omega)$

(a) Yes, Ω is simply connected, so this follows from Runge's theorem.

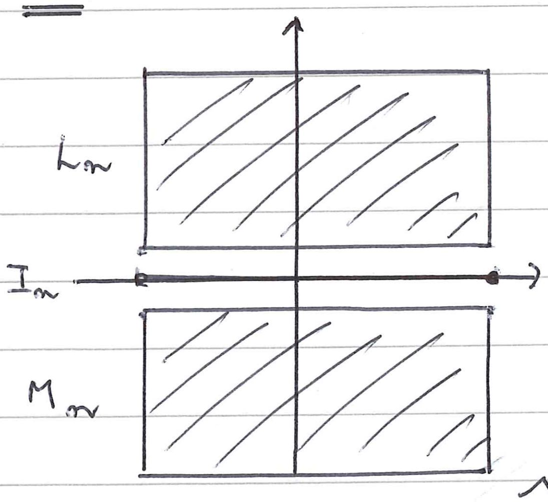
(b) + (c). No. The function $f(z) = \dfrac{1}{z-\frac{1}{2}}$ is holomorphic in a neighbourhood of $\overline{\Omega}$, but cannot be approximated uniformly by polynomials in all of Ω. Such a sequence would converge uniformly to f on (for instance) the circle $\gamma = \{|z - \frac{1}{3}| = \frac{2}{3}\}$, but

$$\int_\gamma P_n \, dz = 0$$

and

$$\int_\gamma \frac{1}{z-\frac{1}{2}} \, dz = 2\pi i \cdot \mathrm{Ind}_\gamma\left(\tfrac{1}{2}\right) = \underline{2\pi i}$$

5.



Yes. Let

$$L_n = [-n, n] \times [\tfrac{1}{n}, n]$$

$$I_n = [-n, n] \times \{0\}$$

$$M_n = [-n, n] \times [-\tfrac{1}{n}, -n]$$

and $K_n = L_n \cup I_n \cup M_n$. Then $\mathbb{C} \setminus K_n$ is connected, so K_n is polynomially convex, i.e. $\widehat{(K_n)}_{\mathbb{C}} = K_n$.

Let $f(z) = \begin{cases} 1 & z \in L_n \\ 0 & z \in I_n \\ -1 & z \in M_n \end{cases}$. Then $f \in \mathcal{O}(K)$ hence there is polynomial P_n with $|f - P_n| < \frac{1}{n}$ on K_n. It is clear that P_n have the required properties.

6. This is true. If not, there is a compact set $K \subset \Omega$, a subsequence of f_n (also denoted by f_n) and points $z_n, w_n \in K$ with $f_n(z_n) = f_n(w_n)$ and $z_n \neq w_n$. By compactness of K we may, after passing to subsequences, assume that $z_n \to a$ and $w_n \to b$. If $a \neq b$ we have

$$f(a) = \lim f_n(z_n) = \lim f_n(w_n) = f(b)$$

contradicting the fact that f is one-to-one.

Hence $a = b$. For any $\rho > 0$ such that $\overline{D}(a,\rho) \subset \Omega$ it follows that $f(z) \neq f(a)$ for all z with $|z-a| = \rho$. This implies that $f_n(z) \neq f_n(z_n)$ for n sufficiently large. Hence

$\#$ solutions of $f(z) = f(a)$ in $D(a,\rho)$

$$= \frac{1}{2\pi i} \int_{|z-a|=\rho} \frac{f'(z)}{f(z)-f(a)} dz = \lim_{n \to \infty} \frac{1}{2\pi i} \int_{|z-a|=\rho} \frac{f_n'(z)}{f_n(z)-f_n(z_n)} dz$$

$= \#$ solutions of $f_n(z) = f_n(z_n)$ in $D(a,\rho) \geq 2$

This contradicts the fact that f is one-to-one.

7. Since all the factors are < 1 in absolute value it follows that $|B(z)| < 1$, i.e. B is bounded. We must check convergence.

For $\alpha \neq 0$, $|\alpha| < 1$ we have

$$1 - \frac{\alpha - z}{1 - \bar{\alpha} z} \cdot \frac{|\alpha|}{\alpha} = \frac{\alpha - |\alpha|^2 z - \alpha|\alpha| + |\alpha| z}{(1 - \bar{\alpha} z)\alpha}$$

$$= \frac{\alpha(1 - |\alpha|) + |\alpha| z (1 - |\alpha|)}{(1 - \bar{\alpha} z)\alpha} = \frac{(\alpha + |\alpha| z)(1 - |\alpha|)}{(1 - \bar{\alpha} z)\alpha} = \frac{1 + \frac{|\alpha|}{\alpha} z}{1 - \bar{\alpha} z}(1 - |\alpha|)$$

If $|z| \leq r < 1$, the absolute value of this is

$$< \frac{1 + r}{1 - r}(1 - |\alpha|)$$

Hence for $|z| \leq r$ we have

$$\sum_{n=1}^{\infty} \left| 1 - \frac{\alpha_n - z}{1 - \bar{\alpha}_n z} \frac{|\alpha_n|}{\alpha_n} \right| \leq \sum_{n=1}^{\infty} \frac{1 + r}{1 - r}(1 - |\alpha_n|) < \infty.$$

This proves that $\prod_{n=1}^{\infty} \frac{\alpha_n - z}{1 - \bar{\alpha}_n z} \frac{|\alpha_n|}{\alpha_n}$ converges uniformly on compacts and has a zero iff one of the factors is zero, i.e. $z = \alpha_n$ for some n.

8. 1.4. $\Gamma = \Gamma' \iff \exists A \in GL(2,\mathbb{Z})$, $\det A = \pm 1$ such that $\begin{pmatrix} w_1' \\ w_2' \end{pmatrix} = A \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$

Proof:

$\Leftarrow$ Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. The equation gives $w_1' = a w_1 + b w_2$ and $w_2' = c w_1 + d w_2$ which means that $\Gamma' \subset \Gamma$.

We can also solve for $\begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$: $\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = A^{-1} \begin{pmatrix} w_1' \\ w_2' \end{pmatrix}$

which gives $\Gamma \subset \Gamma'$. (Notice: $A^{-1} = (\det A) \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \in GL(2,\mathbb{Z})$).

$\Rightarrow$ The equation implies that the bases are integer linear combinations of the other one, i.e.

$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = A \begin{pmatrix} w_1' \\ w_2' \end{pmatrix}$, $\begin{pmatrix} w_1' \\ w_2' \end{pmatrix} = B \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$ with $A, B \in GL(2,\mathbb{Z})$

Hence $\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = AB \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$ so $AB = I$ hence $\det A \cdot \det B = 1$

$\det A = \pm 1$ since both are integers.

1.5 (a) Let $\pi : \mathbb{C} \to \mathbb{C}/\Gamma$ and $\pi' : \mathbb{C} \to \mathbb{C}/\Gamma'$ be the two projections. If $\alpha\Gamma \subset \Gamma'$ define $\varphi : \mathbb{C}/\Gamma \to \mathbb{C}/\Gamma'$ by $\varphi(\pi(z)) = \pi'(\alpha z)$. This is well defined since

$\pi(z) = \pi(z') \Rightarrow z - z' \in \Gamma \Rightarrow \alpha z - \alpha z' \in \Gamma' \Rightarrow \pi'(\alpha z) = \pi'(\alpha z')$.

Also, if $V' \subset \mathbb{C}$ is such that $V' \cap (V' + w) = \emptyset$ $\forall w \in \Gamma' \setminus \{0\}$, then $V = \frac{1}{\alpha} V'$ satisfies $V \cap (V + w) = \emptyset$ $\forall w \in \Gamma \setminus \{0\}$ and $\varphi : \pi(V) \to \pi'(V')$ is given in the local coordinates V, V' by $z \to \alpha z$, hence is holomorphic.

If $\alpha\Gamma \neq \Gamma'$ then there is $z' \in \Gamma'$, $z' \neq 0$ such that $\frac{1}{\alpha} z' \notin \Gamma$. Then $\pi(\frac{1}{\alpha} z') \neq \pi(0)$, but $\varphi(\pi(\frac{1}{\alpha} z')) = \pi'(z') = \pi'(0) = \varphi(\pi(0))$ so φ is not one-to-one.

If $\alpha\Gamma = \Gamma'$, then $\frac{1}{\alpha}\Gamma' = \Gamma$ and we may define $\psi : \mathbb{C}/\Gamma' \to \mathbb{C}/\Gamma$ by $\psi(\pi'(z)) = \pi(\frac{1}{\alpha}z)$. Clearly $\psi \circ \varphi = Id_{\mathbb{C}/\Gamma}$ and $\varphi \circ \psi = Id_{\mathbb{C}/\Gamma'}$, so φ is biholomorphic.

(b) We may assume $\operatorname{Im}\left(\frac{\omega_2}{\omega_1}\right) > 0$, otherwise just switch the indices. Let $\tau = \frac{\omega_2}{\omega_1}$ and $c = \frac{1}{\omega_1}$. Then

$$c\omega_1 = 1 \quad \text{and} \quad c\,\omega_2 = \tau$$

Hence $c\Gamma = \Gamma'$ where Γ' is spanned by 1 and τ. By (a) $\mathbb{C}/\Gamma$ is isomorphic to

$$\mathbb{C}/\Gamma' = \mathbb{C}/\mathbb{Z}+\mathbb{Z}\tau$$

(c) $X(\tau)$ is given by the lattice $\Gamma = \mathbb{Z}\cdot 1 + \mathbb{Z}\cdot\tau$. By (1.4) this is isomorphic (= biholomorphic) to $\mathbb{C}/\Gamma'$ where Γ' is spanned by

$$\begin{pmatrix} \omega_1' \\ \omega_2' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \tau \\ 1 \end{pmatrix} = \begin{pmatrix} a\tau + b \\ c\tau + d \end{pmatrix}$$

i.e. $\Gamma' = \mathbb{Z}\cdot(a\tau+b) + \mathbb{Z}\cdot(c\tau+d)$. Letting $\alpha = \frac{1}{c\tau+d}$ by (a) this is isomorphic to $\mathbb{C}/\Gamma''$ with

$$\Gamma'' = \alpha\Gamma' = \mathbb{Z}\cdot\frac{a\tau+b}{c\tau+d} + \mathbb{Z}\cdot 1 .$$

so $\mathbb{C}/\Gamma'' = X(\tau')$.