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# Riemann's mapping theorem

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very preliminary version— more under way.

One dares say that the Riemann mapping theorem is one of more famous theorem in the whole science of mathematics. Together with its generalization to Riemann surfaces, the so called Uniformisation Theorem, it is with out doubt the corner-stone of function theory. The theorem classifies all simply connected Riemann-surfaces uo to biholomopisms; and list is astonishingly short. There are just three: The unit disk $\mathbb{D}$, the complex plane $\mathbb{C}$ and the Riemann sphere $\hat{\mathbb{C}}$!

Riemann announced the mapping theorem in his inaugural dissertation[1] which he defended in Göttingen in 1851. His version a was weaker than full version of today, in that he seems only to treat domains bounded by piecewise differentiable Jordan curves. His proof was not waterproof either, lacking arguments for why the Dirichlet problem has solutions. The fault was later repaired by several people, so his method is sound (of course!).

In the modern version there is no further restrictions on the domain than being simply connected. William Fogg Osgood was the first to give a complete proof of the theorem in that form (in 1900), but improvements of the proof continued to come during the first quarter of the 20th century. We present Carathéodory's version of the proof by Lipót Fejér and Frigyes Riesz, like Reinholdt Remmert does in his book [?], and we shall closely follow the presentation there.

This chapter starts with the legendary lemma of Schwarz' and a study of the biholomorphic automorphisms of the unit disk. In this course the lemma ended up in this

[1] The title of his thesis is "Grundlagen für eine allgemeine Theorie der Functionen einer veränderlichen complexen Grösse". It starts with the Cauchy-Riemann equations and ends with the Mapping Theorem.

chapter. It certainly deserves a much broader treatment, but time is short these days! Finally, it is may be not the worse place to include it.

The automorphisms of the disk play important role in the proof of the Mapping theorem, as in many other branches of function theory.

## Swartz' lemma and automorphisms of the disk

Both these themes could be the subject of a book, if not several. So this modest sections gives a short and bleak glimses of two utterly richt and manfold worlds.

We start by some examples of automorphism and then passe to prove Schwarz' lemma. With that lemma establish, we determine the group $\mathrm{Aut}(\mathbb{D})$. It consists of all Möbius transformations mapping the unit disk into itself, so the examples we gave generate the group.

### Some examples

We shall describe two classes of automorphisms of the unit disk, and in the end it will turn that these two classes generate all the automorphism of $\mathbb{D}$. To be precise, any automorphism is a product of one from each class.

(4.1) The first class of examples are the obvious automorphisms, namely the *rotations* about the origin. They are realized as multiplication by complex numbers of modulus one, *i.e.,* they are given as $z \mapsto \eta$, and if $\eta = e^{i\theta}$ the angle of rotation is $\theta$. Such a rotations is denoted by $\rho_\eta$ so that $\rho_\eta(z) = \eta z$. The rotations obviously form a subgroup of $\mathrm{Aut}(\mathbb{D})$ canonically isomorphic to the circle group $\mathbb{S}^1$. They of course all have 0 as a fixed point, and are, as we shall see, characterized by this.

(4.2) The other class of automorphism is less transparent, they will however all be Möbius transformations of a special kind. For any $a \in \mathbb{D}$ we define the function

$$\psi_a(z) = \frac{z-a}{\overline{a}z-1}.$$

It is a rational functions with a sole pol at $\overline{a}^{-1}$, and hence it is holomorphic in the unit disk. There are several ways to check that $\psi_a$ maps the unit disk into itself, one can for instance resort to the maximum principle. A more elementary way is to establish the inequality

$$1-|\psi_a(z)|^2 = \frac{(1-|z|^2)(1-|a|^2)}{|\overline{a}z-1|^2},$$

a matter of simple algebraic manipulations.

The map $\psi_a$ has two fixed points, one lying in the unit disk and the other one outside. The fixed points are determined by solving the equation $\psi_a(z) = z$, which is equivalent to the quadratic equation

$$\overline{a}z^2 - 2z + a = 0.$$

So the fixed points are the two points $\overline{a}^{-1}(1 \pm \sqrt{1 - |a|^2})$.

Clearly it holds true that $\psi_a(a) = 0$ and $\psi_a(0) = a$; so 0 and $a$ are swapped by $\psi_a$. Since a Möbius transformation that fixes three points equals the identity, it follows[2] that $\psi_a \circ \psi_a = \mathrm{id}$. One says that $\psi_a$ is an *involution*. The derivative $\psi_a'(z)$ is easily computed and is given as

$$\psi_a'(z) = \frac{1 - |a|^2}{(\overline{a}z - 1)^2}. \tag{4.1}$$

In particular we notice that the derivative at zero, $\psi_a'(0) = 1 - |a|^2$, is real and positive.

It is worth noticing that the four most important points for $\psi_a$, that is the zero, the pole and the two fixed points, all lie on same line through the origin. And in some sense they are pairwise "conjugated", the product of the pole and the zero, and the product of the two fixed points are both unimodular and equal to $a\overline{a}^{-1}$.

PROBLEM 4.1. Show that if $|\eta| = 1$ one has $\rho_\eta \circ \psi_{\overline{\eta}a} = \psi_a \circ \rho_\eta$. ✱

PROBLEM 4.2. Show that any Möbius transformation $\phi(z) = (az + b)(cz + d)^{-1}$ not reduced to the identity, has at least one but at most two fixed points. Show that it has one fixed point if and only if $(a + d)^2 \neq 4(ad - bc)$. Prove that two Möbius transformations coinciding in three points are equal. ✱

PROBLEM 4.3. Let $\psi = \rho_\eta \circ \psi_a$ with $\eta$ unimodular. Show that the product of the two fixed points equals $\eta a\overline{a}^{-1}$, and conclude that $\psi$ has at most one fixed point in $\mathbb{D}$ unless it reduces to the identity. ✱

PROBLEM 4.4. Show that if $\psi = \rho_\eta \circ \psi_a$ where $\eta$ is unimodular, one has

$$1 - |\psi(z)|^2 = (1 - |z|^2)\,|\psi'(z)|$$

for all $z \in \mathbb{D}$. Show that if $b$ is a fixed point for $\psi$, then the value $\psi_a'(b)$ of the derivative at $b$ is unimodular. ✱

PROBLEM 4.5. Show that the fixed points of $-\psi_a$ are the two square roots of $a\overline{a}^{-1}$. Hence these maps do not have fixed points in $\mathbb{D}$. ✱

PROBLEM 4.6. Let $f\colon \mathbb{D} \to \mathbb{D}$ be holomorphic and assume that $f(\mathbb{D})$ is realtively compact in $\mathbb{D}$ (the closure in $\mathbb{D}$ is compact). Show that $f$ has a fixed point. HINT: Use Rouché's theorem with the functiopns $f(z)$ and $f(z) - z$. ✱

## Schwarz' lemma

Karl Hermann Amandus Schwarz have given many significant contributions to complex function theory among them the result called "Schwarz' lemma". It appeared for the

[2] This could of course as well be viewed by a direct substitution.

first time in 1869 during a course Schwarz gave at ETH in Zürich about the Riemann mapping theorem, which at that time, although being in its infancy, was the cutting edge of mathematical science. It seems therefore appropriate to treat Schwarz' lemma in a chapter about Riemann's mapping theorem; that said, Schwarz' lemma has so many applications and extension that it certainly would have deserved its own proper chapter. Both the formulation and the proof of the lemma has developed, and it found its modern form in 1905, published by Carathéodory, though the proof of to day is due to Erhardt Schmidt.

**Theorem 4.1 (Schwarz's lemma)** *Let $f\colon \mathbb{D} \to \mathbb{D}$ be a holomorphic map having $0$ as a fixed point. Then it holds that $|f(z)| < |z|$ and $|f'(0)| <$ unless $f$ is a rotation, i.e., on the form $f(z) = \eta z$ with $|\eta| = 1$.*

Proof: The function $z^{-1}f(z)$ has a removable singularity at the origin since $f$ vanishes there. Hence if we let

$$g(z) = \begin{cases} z^{-1}f(z) & \text{when } z \neq 0 \\ f'(0) & \text{when } z = 0, \end{cases}$$

$g$ will be holomorphic in $\mathbb{D}$. The idea is to apply the maximum principle to $g$ in disks $D_r$ given by $|z| < r$ with $r < 1$. On the boundary $\partial D_r$ one has

$$|g(z)| = |f(z)|\, r^{-1} \leq r^{-1},$$

and consequently it holds that $|g(z)| \leq r^{-1}$ for $z \in D_r$. In the limit when $r$ tends to 1 one finds $|g(z)| \leq 1$.

To finish the proof, assume that $|g|$ takes the value 1 at a $a \in \mathbb{D}$. Then the modulus $|g|$ has a maximum at $a$; indeed, if $|g(b)| > 1$ for some other point in the unit disk, the above argument with $r > \max\{1/|g(b)|, |b|\}$ would give $|g(b)| < |g(b)|$. So by the maximum principle $g$ is a constant $\eta$, and clearly $|\eta| = 1$. ❏

Problem 4.7. Assume that $f$ is holomorphic that maps $\mathbb{D}$ to $\mathbb{D}$ and vanishes to the $n$-th order at the origin (that is $f$ and the derivatives $f^{(i)}$ vanish at $0$ for $i < n$). Show that $|f(z)| < |z|^n$ unless $f(z) = \eta z^n$ with $|\eta| = 1$. ✷

Problem 4.8. Let $f$ be a holomorphic map from $\mathbb{D}$ to $\mathbb{D}$, and let $a \in \mathbb{D}$ be any point. Study the function $\psi_{f(a)} \circ f \circ \psi_a$, which maps zero to zero, and prove that

$$\left| \frac{f(z) - f(a)}{1 - \overline{f(a)} f(z)} \right| \leq \left| \frac{z - a}{1 - \bar{a} z} \right| .$$

Let $z$ tend to $a$ to obtain

$$\frac{|f'(a)|}{1 - |f(a)|^2} \leq \frac{1}{1 - |a|^2}.$$

What happens in case of equality in a point? ✷

### The automorphisms of $\mathbb{D}$

Our first applications of Schwarz' lemma is to determine all automorphisms of the unit disk. They are all compositiuons of maps from the two calsses of examles we begun with. ne has

**Proposition 4.1** *A biholomorphic map $\psi\colon \mathbb{D} \to \mathbb{D}$ can be factored as $\psi(z) = \eta\psi_a(z)$ in a unique way. The numbers $a$ and $\eta$ are invariants of $\psi$ determined by $a$ being the only zero of $\psi$ and $\psi'(0) = \eta(1 - |a|^2)$.*

Proof: Since $\psi$ is bijective it has exactly one zero, call it $a$. Then the composition $\psi \circ \psi_a$ maps $\mathbb{D}$ to $\mathbb{D}$ and takes zero to zero. Therefore it suffices to show that an automorphism $\psi$ fixing zero is a rotation. To do that we apply Schwarz' lemma to both $\psi$ and to $\psi^{-1}$ and obtain the two inequalities

$$|\psi(z)| \leq |z| \text{ and } \left|\psi^{-1}(z)\right| \leq |z|\,.$$

Replacing $z$ by $\psi(z)$ in the first, we obtain

$$|z| = \left|\psi^{-1}(\psi(z))\right| \leq |\psi(z)| \leq |z|\,,$$

and so $|\psi(z)| = |z|$. From Schwarz' lemma we deduce then that $\psi$ is a rotation. The uniqueness follows since the function $\psi_a$ is the only one in its class that vanishes at $a$. The statemant about the derivative follows trivially from the formula (4.1) above. ❏

All elements in $\mathrm{Aut}(\mathbb{D})$ are therefore Möbius transformations, and $\mathrm{Aut}(\mathbb{D})$ can be described as the group of Möbius transformations that leave the unit disk invariant.

(4.3) The rotations are precisely those automorphisms that have zero as fixed point. A group theorist would say the the rotations constitute the *isotropy group* of the origin. It is of course isomorphic to the circle group $\mathbb{S}^1$.

Any other point $a$ in $\mathbb{D}$ has an isotropy group as well. It is denoted by $\mathrm{Aut}_a(\mathbb{D})$ and consists of the automorphisms leaving $a$ fixed. As generally true in groups acting transitively, different points have conjugate isotropy groups. Hence $\mathrm{Aut}_a(\mathbb{D})$ is conjugated to the group of rotations; indeed, $\psi_a \circ \psi \circ \psi_a$ fixes 0 if and only if $\psi$ fixes $a$.

Problem 4.9. Show that the map $\mathrm{Aut}_a(\mathbb{D}) \to \mathbb{C}$ defined by $\psi \mapsto \psi'(a)$ is a group homomorphism which induces an isomorphism between $\mathrm{Aut}_a(\mathbb{D})$ and the circle group $\mathbb{S}^1$. ✷

(4.4) The particular maps $\psi_a$ can be characterized among all the automorphisms in several ways. They are the only ones whose derivative at zero is real and positive. Indeed, if $\psi(z) = \eta\psi_a(z)$, one has $\psi'(a) = \eta\psi_a'(a) = \eta(1 - |a|^2)$ which is real and positive if and only if $\eta = 1$.

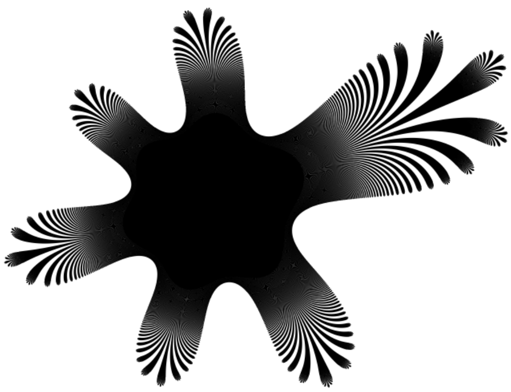
They are also the only involutions in $\mathrm{Aut}(\mathbb{D})$. To see this assume that $\psi$ is an involution, so that $\psi \circ \psi = \mathrm{id}$, and factor $\psi$ as $\psi = \rho_\eta \circ \psi_a$. Then $\psi(a) = 0$, and since $\psi$ is an involution, it follows that $\psi(0) = \psi(\psi(a)) = a$. This gives that $a = \psi(0) = \eta\psi_a(0) = \eta a$, and hence $\eta = 1$.

PROBLEM 4.10. Show that if $\psi$ and $\phi$ are two commuting automorphisms of $\mathbb{D}$ then they have a common fixed point. Show that if $\psi \in \mathrm{Aut_a}(\mathbb{D})$ is of finite order (as a group element) larger than 2, then $\psi$ is conjugate to the rotation $\rho_\eta$ with $\eta$ a $n$-th root of unity. ✷

## The Riemann mapping theorem

The theorem states that any simply connected domain in the complex plane, that is not the entire plane, is biholomorphic to the unit disk. Thinking about what enormously number of different simply connected domains there are and that they can be almost of infinite complexity, the theorem is at the least extremely deep and impressive. There is a generalization, of even greater depth, called the "Uniformization Theorem" stating that among the Riemann surfaces only $\mathbb{D}$, $\mathbb{C}$ and the Riemann sphere $\hat{\mathbb{C}}$ are simply connected. So the universal cover of any domain in $\mathbb{C}$ is either $\mathbb{C}$ or $\mathbb{D}$, and with some exceptions (that is, those with have $\mathbb{C}$ as the universal cover, but they are not so intricate) open sets in $\mathbb{C}$ are tightly connected to locally injective functions on $\mathbb{D}$! A clear indication that holomorphic function in $\mathbb{D}$ are worth a close study.

Camille Jordan was a great french mathematician, and proved the theorem that any closed, simple curve divides the plane in two connected components, and the bounded one is simply connected. Such curves are called *Jordan curves*, the theorem is called "Jordan's curve theorem". The domains bounded by Jordan curves form a very important class of simply connected domains, but they can also be extremely complicated. There are Jordan curves having positive area! Well, one should say positive two dimensional Lebesgue measure to be precise. In iteration theory beautiful and intricate simply connected domains appear. They are called Siegel domains and picture of them you can see everywhere (also here!)



### Examples

To give a modest indication of the depth of Riemann's mapping theorem, we offer a few concrete example of complicated simply connected domains — one could be tempted to call it a "horror show" of domains except that the author finds these examples as beautiful as the pictures! But, still, we can not resist showing John Lennon as a Jordan curve! The big question is what lies outside and what lies inside?

Figur 4.1: John Lennon as a Jordan curve

Example 4.1. Let $p/q$ be a positive rational number on reduced form so that $p$ and $q$ are positive relatively prime integers. We also assume that $p/q$ lies between zero and one.

Denote by $L_{p/q}$ the part of the ray making an angle $2\pi p/q$ with the real axis whose points have a distance from the origin larger than $q$. That is one has $L_{p/q} = \{\, re^{2\pi i p/q} \mid |z| \geq q \,\}$.

Clearly $L_{p/q}$ is closed, but the union $L = \bigcup_{p/q} L_{p/q}$ is also closed. The union being an infinite union, this is slightly subtle; the point is that given any complex number $z$ not in $L$, there is only finitely many rationals $p/q$ between zero and one such that $q \leq |z| + 1$. Hence if $D$ denotes the disk about $z$ of radius one, the intersection $L \cap D$ is a finite union of closed subset, and consequently is closed in $D$.

The complement $\mathbb{C} \setminus L$ is open and it is star-shaped with apex at the origin. Hence it is simply connected.

If one wants a finite version of this example here is one. We remove a "hedgehog-like" set from the open square $Q = < -1, 1 > \times < 0, 1 >$. Let $M_{p/q}$ be the ray $\{\, re^{2\pi i p/q} \mid 0 \leq r \leq 1/q \,\}$ and let $U$ be the sqaure $Q$ with all the rays $M_{p/q}$ that lie in $Q$ removed. ❄

EXAMPLE 4.2. A similar construction is as follows. Take a copy of the cantor set $\mathfrak{c}$ lying on the unit circle (for instance the image of $\mathfrak{c}$ by the standard parametisation) and let $L_c$ be the partial ray $L_c = \{\, rc \mid r \geq 1 \,\}$. The Cantor set being closed, it is not difficult to see that the union $L = \bigcup_{c\in\mathfrak{c}} L_c$ is a closed subset of $\mathbb{C}$, even if it is a uncountable union of closeds. And again, the complement $\mathbb{C} \setminus L$ is star-shaped with the origin as apex.

This domain also has the virtue that its complement in the Riemann sphere $\hat{\mathbb{C}}$ is connected, removing the point at infinity, the complement disintegrates into an uncountable union of components. ❄

EXAMPLE 4.3. The third example is the so called "comb-set". We again start with an open square, say $Q = <0,1> \times <0,1>$. and the set to be removed is the union of the sets $T_n = \{\, 1/n + yi \mid 0 \leq y \leq 1/2 \,\}$ for $n \in \mathbb{N}$. The result is an open domain that is simply connected being a deformatable to say the segment $<0,1> \times \{2/3\}$. ❄

### A motivation

Sometimes it is good strategy to explore a hypothetical solution to problem, to get a clue how to solve the problem. It turns out to be smart to somehow normalize the situation: Fix a point $a \in \Omega$ and confine the maps we are interested inn to those sending $a$ to zero.

So assume that $\Omega$ is a simply connected domain and assume $f$ is the solution we are striving for; a biholomorphic map $f\colon \Omega \to \mathbb{D}$ sending $a$ to 0. We want to compare it to any other holomorphic map $g\colon \Omega \to \mathbb{D}$ with $f(a) = 0$, and to that end, consider the composition $g \circ f^{-1}$. It sends the disk $\mathbb{D}$ into it self and fixes the origin. Hence it is prone to a treatment by Schwarz' lemma, that gives the inequality $g(f^{-1}(z)) \leq |z|$, or if one replaces $z$ by $f(z)$, it becomes $|g(z)| \leq |f(z)|$. The solution we seek is therefore a solution to a optimalisation problem: Find the function being maximal in modulus among the those mapping $\Omega$ to $\mathbb{D}$ and sending $a$ to 0.

### The formulaton and the proof

After these preliminary skirmishes it is high time formulation the theorem in a precise manner. The formulation includes a uniqueness statement that basically says that the Riemann mapping is unique up to automorphisms of the unite disk; so imposing normalization requirement on the function it will be unique.

**Theorem 4.2 (The Riemann mapping theorem)** *Let $\Omega$ be a simply connected plane domain is not the entire plane and let $a$ be a point in $\Omega$. Then there is a unique biholomorphic map $\phi\colon \Omega \to \mathbb{D}$ such that $f(a) = 0$ and $f'(a) > 0$*

PROOF: The crux of this proof is to search for functions $f\colon \Omega \to \mathbb{D}$ having maximal modulus in one point different from $a$. So choose a point $b \in \Omega$ other than $a$, and consider the set

$$\mathcal{P} = \{\, f\colon W \to \mathbb{D} \mid f(a) = 0, \text{ anf } f \text{ is injective } \,\}$$

where we audaciously also insist on the functions being injective.

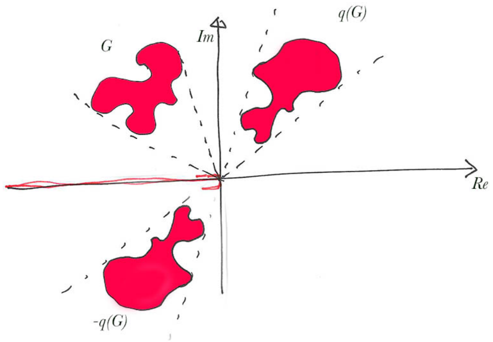
There are three steps on the road to the Riemann mapping theorem. i) Prove that the set $\mathcal{P}$ above is non-empty, ii) show that there is a function $f$ in $\mathcal{P}$ with $|f(b)|$ maximal, and iii) show that $f$ is biholomorphic.

(4.5) We start by showing that $\mathcal{P}$ is non-empty, *i.e.,* we are looking for an injective holomorphic map $f$ from $\Omega$ to $\mathbb{D}$; The requirement that $a$ maps to zero is easy to fulfil, we just follow up by an appropriate Möbius transformation sending $f(a)$ to 0.

Pick a point $c$ outside $\Omega$. Then $z-c$ never vanishes in $\Omega$ and since $\Omega$ is supposed to be a $Q$-domain, there is a well defined square root of $z-c$ in $\Omega$, that is a holomorphic function $q$ with $q(z)^2 = z - c$.

We claim that $q(\Omega)$ and $-q(\Omega)$ are disjoint. Assume the contrary that is $q(z) = -q(w)$ for two points $z$ and $w$ from $\Omega$. Squaring gives $z = w$, and hence $z = c$, which is impossible since $z$ lies in $\Omega$ but $c$ does not.

The set $-q(\Omega)$ is open by the open mapping theorem and therefore contains a disk$D$, say the disk $|z-d| < R$. Then the function $h(z) = R(z-d)^{-1}$ is an injective function, holomorphic for $z \neq d$ and mapping the complement of $D$, where $q(\Omega)$ lies, into $\mathbb{D}$. The composition $h \circ q$ is a function like we want.




(4.6) The next step is to prove there is a function in $\mathcal{P}$ with $|f(b)|$ maximal. By definition of the supremum there is a sequence of functions $f_\nu$ in $\mathcal{P}$ with $f_\nu(p)$ converging to $\alpha = \sup_{f\in\mathcal{P}} |f(p)|$. Montel's first theorem implies that the family $\mathcal{P}$, which is bounded, is a normal family. Hence there is subsequence of $f_\nu$ converging UPK in $\Omega$, and we may as well assume that the sequence $f_\nu$ itself converges. The limit function $f$ is holomorphic by Weierstrass' convergence theorem, it is injective by Hurwitz' injectivity theorem since each $f_\nu$ is, and it takes values in $\mathbb{D}$; *a priori* just in the closure $\overline{\mathbb{D}}$, but $f(\Omega)$ is open. Of course $f(a) = 0$, since $f_\nu(a) = 0$. So the limit function $f$ is our guy!

(4.7) Finally, we have come to the point where to show that $f$ is biholomorphic. By definition $f$ is injective so only the surjectivity is lacking. The salient point is that with the help of a (potential) point $d$ in $\mathbb{D}$, but not in $f(\Omega)$, one can construct an *expanding map* $h\colon f(\Omega) \to \mathbb{D}$. This is a holomorphic map sending $f\Omega$ to $\mathbb{D}$ whose main property is that $|h(z)| > |z|$, with the subsidiary properties of being injective and sending 0 to 0. Such a map would contradict the maximality of $|f(b)|$, since $h \circ f$ is a member of $\mathcal{P}$ and $|h(f(b))| > |f(b)|$.

An obvious expanding map in the unit disk is the square root. However there is the problem with the square root that it can not be a defined in the whole unit disk. To remedy this, we introduce the Möbius transformation $\psi_d$. It has a sole zero at $d$ and hence does not vanish in $f(\Omega)$.

As $\psi_d(f(\Omega))$ is a $Q$-domain, a branch $q$ of the square root is well defined there; that is, the composition $q \circ \psi_d$ is well defined in $f(\Omega)$. However, it does not send 0 to 0, but the function $h = \psi_{\sqrt{d}} \circ q \circ \psi_d$ does. This last function $h$ has as inverse the function $\psi_d \circ \kappa \circ \psi_{\sqrt{d}}$ (at least over $f(\Omega)$) where $\kappa$ is the quadratic function $\kappa(z) = z^2$. One easily checks this using that the $\psi_a$-s are involutions.

The inverse of a contracting map is expanding, and the function $\psi_d \circ \kappa \circ \psi_{\sqrt{d}}$ is indeed contracting! By Schwarz' lemma this is clear since it maps $\mathbb{D}$ into $\mathbb{D}$, sends 0 to 0 and is not a rotation! Hence $h$ is expanding, and it does the job. That finishes the proof of the existence part of Riemann's mapping theorem.

Finally the statement about the positivity of the derivative is easy to establish .One just follows $f$ by an appropriate rotation; one replaces $f$ by the function $\rho_\omega \circ f$ with $\theta = -\arg f$ which will have a positive derivative at $a$.

(4.8) To prove the uniqueness statement of the theorem assume that $f$ and $g$ are two biholomorphic maps from $\Omega$ to $\mathbb{D}$, both sending $a$ to 0 and both having maximal modulus at $b$. Then of course $|f(b)| = |g(b)|$.

The composition $f \circ g^{-1}$ maps $\mathbb{D}$ to $\mathbb{D}$ and have 0 as a fixed point, and moreover $|f(g^{-1}(g(b)))| = |f(b)| = |g(b)|$. Due to the last equality we deduce from Schwarz' lemma that the composition $f \circ g^{-1}$ is a rotation and one can write $f(z) = f(g^{-1}(g(z))) = \eta g(z)$ with $\eta \in \partial\mathbb{D}$. Taking derivatives we obtain $f'(b) = \eta g'(b)$ and as both $f'(b)$ and $g'(b)$ are real and positive, it follows that $\eta = 1$. ❑

PROBLEM 4.11. Show that $\psi(x) = \frac{z^2 - i}{z^2 + 1}$ maps the first quadrant biholomorphically onto the unit disk. Determine the inverse map. ✷

PROBLEM 4.12. Find a map that maps a half disk biholomorphically to a full disk. Do the same for a quarter of a disk. ✷