ALGEBRAIC TOPOLOGY III – MAT 9580 – SPRING 2015 INTRODUCTION TO THE ADAMS SPECTRAL SEQUENCE

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Contents

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1. The long exact sequence of a pair

Let (X, A) be a pair of spaces. The relationship between the homology groups $H_*(A)$, $H_*(X)$ and $H_*(X, A)$ is expressed by the long exact sequence

$$\dots \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X, A) \xrightarrow{\partial} H_{n-1}(X, A) \xrightarrow{i_*} \dots$$

Exactness at $H_n(X)$ amounts to the condition that

$$\operatorname{im}(i_* \colon H_n(A) \to H_n(X)) = \ker(j_* \colon H_n(X) \to H_n(X, A)) .$$

The homomorphism i_* induces a canonical isomorphism from

$$\begin{aligned} \operatorname{cok}(\partial \colon H_{n+1}(X, A) \to H_n(A)) &= \frac{H_n(A)}{\operatorname{im}(\partial \colon H_{n+1}(X, A) \to H_n(A))} \\ &= \frac{H_n(A)}{\ker(i_* \colon H_n(A) \to H_n(X))} \end{aligned}$$

to $\mathrm{im}(i_*\colon H_n(A) \to H_n(X))$. Exactness at $H_n(A)$ and at $H_n(X, A)$ amounts to similar conditions, and ∂ and j_* induce similar isomorphisms.

We can use the long exact sequence to get information about $H_*(X)$ from information about $H_*(A)$ and $H_*(X, A)$, if we can compute the kernel $\ker(\partial) = \mathrm{im}(j_*) \cong \mathrm{cok}(i_*)$ and the cokernel $\mathrm{cok}(\partial) \cong \mathrm{im}(i_*)$ of the boundary homomorphism ∂, and determine the extension

$$0 \to \mathrm{im}(i_*) \longrightarrow H_*(X) \longrightarrow \mathrm{cok}(i_*) \to 0$$

of graded abelian groups.

Let us carefully spell this out in a manner that generalizes from long exact sequences to spectral sequences.

We are interested in the graded abelian group $H_*(X)$. The map $i\colon A \to X$ induces the homomorphism $i_*\colon H_*(A) \to H_*(X)$, and we may consider the subgroup of $H_*(X)$ given by its image, $\mathrm{im}(i_*)$. We get a short increasing filtration

$$0 \subset \mathrm{im}(i_*) \subset H_*(X)\,.$$

More elaborately, we can let

$$F_s = \begin{cases} 0 & \text{for } s \le -1 \\ \mathrm{im}(i_*) & \text{for } s = 0 \\ H_*(X) & \text{for } s \ge 1 \end{cases}$$

for all integers s. We call s the *filtration degree.*

The possibly nontrivial filtration quotients are

$$\frac{\mathrm{im}(i_*)}{0} = \mathrm{im}(i_*) \qquad \text{and} \qquad \frac{H_*(X)}{\mathrm{im}(i_*)} = \mathrm{cok}(i_*)\,.$$

We find

$$fracF_sF_{s-1} = \begin{cases} 0 & \text{for } s \le -1 \\ \mathrm{im}(i_*) & \text{for } s = 0 \\ \mathrm{cok}(i_*) & \text{for } s = 1 \\ 0 & \text{for } s \ge 2. \end{cases}$$

The short exact sequence

$$0 \to \mathrm{im}(i_*) \longrightarrow H_*(X) \longrightarrow \mathrm{cok}(i_*) \to 0$$

expresses $H_*(X)$ as an extension of two graded abelian groups. This does not in general suffice to determine the group structure of $H_*(X)$, but it is often a tractable problem. More generally we have short exact sequences

$$0 \to F_{s-1} \longrightarrow F_s \longrightarrow F_s/F_{s-1} \to 0$$

for each integer s. If we can determine the previous filtration group F_{s-1}, say by induction on s, and if we know the filtration quotient F_s/F_{s-1}, then the short exact sequence above determines the next filtration group F_s, up to an extension problem.

In the present example $F_{-1} = 0$, $F_0 = \mathrm{im}(i_*)$ and $F_1 = H_*(X)$, so there is only one extension problem, from F_0 to F_1, given the quotient $F_1/F_0 = \mathrm{cok}(i_*)$.

We therefore need to understand $\mathrm{im}(i_*)$ and $\mathrm{cok}(i_*)$. By definition and exactness

$$\mathrm{im}(i_*) \cong \mathrm{cok}(\partial) \qquad \text{and} \qquad \mathrm{cok}(i_*) \cong \mathrm{im}(j_*) = \ker(\partial)\,,$$

so both of these graded abelian groups are determined by the connecting homomorphism

$$\partial\colon H_{*+1}(X, A) \longrightarrow H_*(A)\,.$$

If we assume that we know $H_*(A)$ and $H_{*+1}(X, A)$, we must therefore determine this homomorphism ∂, and compute its cokernel $\mathrm{cok}(\partial) = H_*(A)/\mathrm{im}(\partial)$ and its kernel $\ker(\partial) \subset H_{*+1}(X, A)$.

In view of the short exact sequences

$$0 \to \mathrm{im}(\partial) \longrightarrow H_*(A) \longrightarrow \mathrm{cok}(\partial) \to 0$$

and

$$0 \to \ker(\partial) \longrightarrow H_{*+1}(X, A) \longrightarrow \mathrm{im}(\partial) \to 0$$

we can say that the original groups $H_*(A)$ and $H_{*+1}(X, A)$ have been reduced to the subquotient groups $\mathrm{cok}(\partial)$ and $\ker(\partial)$, respectively, and that both groups have been reduced by the same factor, namely by $\mathrm{im}(\partial)$. This makes sense in terms of orders of groups if all of these groups are finite, but must be more carefully interpreted in general. The change between the old groups and the new groups is in each case created by the non-triviality of the homomorphism ∂.

1.1. **E^r-terms and d^r-differentials.** We can present the steps in this approach to calculating $H_*(X)$ using the following chart. First we place the known groups $H_*(A)$ and $H_*(X, A)$ in two columns of the (s, t)-plane:

$$\begin{array}{ccc}
 & \vdots & \vdots \\
t=2 & H_2(A) & H_3(X,A) \\
t=1 & H_1(A) & H_2(X,A) \\
t=0 & H_0(A) & H_1(X,A) \\
t=-1 & 0 & H_0(X,A) \\
 & s=0 & s=1
\end{array}$$

We call t the *internal degree*, even if this is not particularly meaningful in this example. The sum $s+t$ is called the *total degree*, and corresponds to the usual homological grading of $H_*(A)$, $H_*(X)$ and $H_*(X, A)$. This first page is called the E^1-term. It is a bigraded abelian group $E^1_{*,*}$, with

$$E^1_{0,t} = H_t(A) \qquad \text{and} \qquad E^1_{1,t} = H_{1+t}(X, A)$$

for all integers t. We extend the notation by setting $E^1_{s,t} = 0$ for $s \leq -1$ and for $s \geq 2$. This appears as follows

$$\begin{array}{cccccc}
 & & \vdots & & & \\
 & E^1_{-1,2} & E^1_{0,2} & E^1_{1,2} & E^1_{2,2} & \\
 & E^1_{-1,1} & E^1_{0,1} & E^1_{1,1} & E^1_{2,1} & \\
\cdots & E^1_{-1,0} & E^1_{0,0} & E^1_{1,0} & E^1_{2,0} & \cdots \\
 & E^1_{-1,-1} & E^1_{0,-1} & E^1_{1,-1} & E^1_{2,-1} & \\
 & & \vdots & & &
\end{array}$$

with nonzero groups only in the two central columns.

We next introduce the boundary homomorphism ∂. In the (s,t)-plane it has bidegree $(-1,0)$, i.e., maps one unit to the left. We can display it as follows:

$$
\begin{array}{ccc}
 & \vdots & \vdots \\
t=2 & H_2(A) \xleftarrow{\ \partial\ } & H_3(X,A) \\
t=1 & H_1(A) \xleftarrow{\ \partial\ } & H_2(X,A) \\
t=0 & H_0(A) \xleftarrow{\ \partial\ } & H_1(X,A) \\
t=-1 & 0 & H_0(X,A) \\
 & s=0 & s=1
\end{array}
$$

In spectral sequence parlance, this homomorphism is called the d^1-differential. It extends trivially to a homomorphism

$$d^1_{s,t} \colon E^1_{s,t} \longrightarrow E^1_{s-1,t}$$

for all integers s and t. In all other cases than those displayed above, this homomorphism is zero, since for $s \le 0$ the target is zero, for $s \ge 2$ the source is zero, and for $s = 1$ and $t \le -1$ the target is also zero.

We now replace each group $E^1_{0,t} = H_t(A)$ by its quotient group

$$\mathrm{cok}(\partial \colon H_{1+t}(X,A) \to H_t(A)) = \mathrm{cok}(d^1_{1,t})$$

and replace each group $E^1_{1,t} = H_{1+t}(X,A)$ by its subgroup

$$\ker(\partial \colon H_{1+t}(X,A) \to H_t(A)) = \ker(d^1_{1,t}) \,.$$

This leaves the following diagram

$$
\begin{array}{ccc}
 & \vdots & \vdots \\
t=2 & \mathrm{cok}(d^1_{1,2}) & \ker(d^1_{1,2}) \\
t=1 & \mathrm{cok}(d^1_{1,1}) & \ker(d^1_{1,1}) \\
t=0 & \mathrm{cok}(d^1_{1,0}) & \ker(d^1_{1,0}) \\
t=-1 & 0 & H_0(X,A) \\
 & s=0 & s=1
\end{array}
$$

We call this second page the E^2-term. It is a bigraded abelian group $E^2_{*,*}$, with

$$E^2_{0,t} = \mathrm{cok}(d^1_{1,t}) \qquad \text{and} \qquad E^2_{1,t} = \ker(d^1_{1,t})$$

for all integers t. As before, we extend the notation by setting $E^2_{s,t} = 0$ for $s \le -1$ and for $s \ge 2$.

What is the relation between the E^1-term and the E^2-term? This may be easier to see if we expand the diagram consisting of the E^1-term and the d^1-differential to also include the trivial groups surrounding the two interesting columns.

$$\begin{tikzcd}
 & \vdots & \vdots & \\
0 & H_2(A) \arrow[l] & H_3(X,A) \arrow[l, "\partial"'] & 0 \arrow[l] \\
0 & H_1(A) \arrow[l] & H_2(X,A) \arrow[l, "\partial"'] & 0 \arrow[l] \\
0 & H_0(A) \arrow[l] & H_1(X,A) \arrow[l, "\partial"'] & 0 \arrow[l] \\
0 & 0 \arrow[l] & H_0(X,A) \arrow[l, "\partial"'] & 0 \arrow[l]
\end{tikzcd}$$

In the other notation, this appears as follows:

$$\begin{tikzcd}
 & & \vdots & & & \\
 & 0 & E^1_{0,2} \arrow[l, "d^1_{0,2}"'] & E^1_{1,2} \arrow[l, "d^1_{1,2}"'] & 0 \arrow[l, "d^1_{2,2}"'] & \\
 & 0 & E^1_{0,1} \arrow[l, "d^1_{0,1}"'] & E^1_{1,1} \arrow[l, "d^1_{1,1}"'] & 0 \arrow[l, "d^1_{2,1}"'] & \\
\dots & 0 & E^1_{0,0} \arrow[l, "d^1_{0,0}"'] & E^1_{1,0} \arrow[l, "d^1_{1,0}"'] & 0 \arrow[l, "d^1_{2,0}"'] & \dots \\
 & 0 & E^1_{0,-1} \arrow[l, "d^1_{0,-1}"'] & E^1_{1,-1} \arrow[l, "d^1_{1,-1}"'] & 0 \arrow[l, "d^1_{2,-1}"'] & \\
 & & \vdots & & &
\end{tikzcd}$$

Now notice that each row $(E^1_{*,t}, d^1_{*,t})$ of the E^1-term with the d^1-differentials forms a chain complex, and the E^2-term is the homology of that chain complex:

$$E^2_{s,t} = \frac{\ker(d^1_{s,t})}{\operatorname{im}(d^1_{s+1,t})} = H_s(E^1_{*,t}, d^1_{*,t})$$

for all integers s and t. For $s = 0$ this is clear because

$$\ker(d^1_{0,t}) = E^1_{0,t} = H_t(A) \qquad \text{and} \qquad \operatorname{im}(d^1_{1,t}) = \operatorname{im}(\partial\colon H_{1+t}(X,A) \to H_t(A)) \,.$$

For $s = 1$ it is also clear, because

$$\ker(d^1_{1,t}) = \ker(\partial\colon H_{1+t}(X,A) \to H_t(A)) \qquad \text{and} \qquad \operatorname{im}(d^1_{2,t}) = 0 \,.$$

For the remaining values of s, all groups are trivial.

Having obtained the E^2-term as the homology of the E^1-term with respect to the d^1-differentials, we can now locate the short exact sequence

$$0 \to \operatorname{cok}(d^1_{1,n}) \longrightarrow H_n(X) \longrightarrow \ker(d^1_{1,n-1}) \to 0$$

within the diagram, for each n. This is nothing but the degree n part of the short exact sequences previously denoted

$$0 \to \operatorname{cok}(\partial) \longrightarrow H_*(X) \longrightarrow \ker(\partial) \to 0$$

and

$$0 \to \operatorname{im}(i_*) \longrightarrow H_*(X) \longrightarrow \operatorname{cok}(i_*) \to 0\,,$$

and is now written

$$0 \to E^2_{0,n} \longrightarrow H_n(X) \longrightarrow E^2_{1,n-1} \to 0\,.$$

These extensions appear along anti-diagonals in the E^2-term, or equivalently, along lines of slope -1:

$$\begin{tikzcd}
\vdots & & \vdots \\
E^2_{0,2} \arrow[dr, tail] & & E^2_{1,2} \\
& H_2(X) \arrow[dr, two heads] & \\
E^2_{0,1} \arrow[dr, tail] & & E^2_{1,1} \\
& H_1(X) \arrow[dr, two heads] & \\
E^2_{0,0} \arrow[dr, tail] & & E^2_{1,0} \\
& H_0(X) \arrow[dr, two heads] & \\
0 & & E^2_{1,-1}
\end{tikzcd}$$

In other words, the filtration quotients $(F_s/F_{s-1})_n$ associated to the increasing filtration

$$0 \subset \operatorname{im}(i_*) \subset H_n(X)$$

appear along the line in the (s,t)-plane where the total degree is $s+t=n$, starting with $(F_0)_n = \operatorname{im}(i_*)$ at $E^2_{0,n}$, and continuing with the filtration quotient $(F_1/F_0)_n = \operatorname{cok}(i_*)$ at $E^2_{1,n-1}$. The group we are interested in, $H_n(X)$, is realized as an extension of the two parts of the E^2-term in bidegrees $(0,n)$ and $(1,n-1)$.

This indexing system is standard for the Serre spectral sequence.

1.2. **Adams indexing.** In some cases it is more convenient to collect the terms contributing to a single degree in the answer, in our case the terms $E^2_{0,n}$ and $E^2_{1,n-1}$ contributing to $H_n(X)$, in a single column. This means that the terms $E^1_{0,n}$ and $E^1_{1,n-1}$ are also placed in a single column, and the d^1-differential will map diagonally to the left and upwards. The E^1-term is then displayed as follows, in the (n,s)-plane:

$$\begin{array}{cccccc}
s=1 & H_0(A) & H_1(A) & H_2(A) & H_3(A) & \dots \\
\\
s=0 & H_0(X,A) & H_1(X,A) & H_2(X,A) & H_3(X,A) & \dots \\
\\
 & n=0 & n=1 & n=2 & n=3 &
\end{array}$$

The orientation of the s-axis has also been switched, so that $H_0(X, A)$ rather than $H_0(A)$ sits at the origin, and the total degree n is related to the filtration degree s and the internal degree t by $n = t - s$ instead of $n = s + t$. We will discuss this more precisely later. The d^1-differential is still the connecting homomorphism ∂:

$$\begin{tikzcd}
H_0(A) & H_1(A) & H_2(A) & H_3(A) & \dots \\
H_0(X,A) & H_1(X,A) \arrow[ul, "\partial"'] & H_2(X,A) \arrow[ul, "\partial"'] & H_3(X,A) \arrow[ul, "\partial"'] & \dots \arrow[ul, "\partial"']
\end{tikzcd}$$

The E^2-term is the homology of the E^1-term with respect to the d^1-differential:

$$\begin{matrix}
\mathrm{cok}(\partial)_0 & \mathrm{cok}(\partial)_1 & \mathrm{cok}(\partial)_2 & \mathrm{cok}(\partial)_3 & \dots \\
\\
H_0(X,A) & \ker(\partial)_1 & \ker(\partial)_2 & \ker(\partial)_3 & \dots
\end{matrix}$$

The end product, known as the abutment, of the spectral sequence, is now determined up to an extension problem, by the following vertical short exact sequences:

$$\begin{tikzcd}
\mathrm{cok}(\partial)_0 \arrow[d, tail] & \mathrm{cok}(\partial)_1 \arrow[d, tail] & \mathrm{cok}(\partial)_2 \arrow[d, tail] & \mathrm{cok}(\partial)_3 \arrow[d, tail] & \dots \\
H_0(X) \arrow[d, two heads] & H_1(X) \arrow[d, two heads] & H_2(X) \arrow[d, two heads] & H_3(X) \arrow[d, two heads] & \\
H_0(X,A) & \ker(\partial)_1 & \ker(\partial)_2 & \ker(\partial)_3 & \dots
\end{tikzcd}$$

This indexing system is standard for the Adams spectral sequence, and we refer to it as Adams indexing.

2. Spectral sequences

Definition 2.1. A *homological spectral sequence* is a sequence $(E^r, d^r)_r$ of bigraded abelian groups and differentials, together with isomorphisms

$$E^{r+1} \cong H(E^r, d^r) = \frac{\ker(d^r)}{\mathrm{im}(d^r)},$$

for all natural numbers r. Each $E^r = E^r_{*,*} = (E^r_{s,t})_{s,t}$ is a bigraded abelian group, called the E^r-term of the spectral sequence. The r-th differential is a homomorphism $d^r \colon E^r_{*,*} \to E^r_{*,*}$ of bidegree $(-r, r-1)$, satisfying $d^r \circ d^r = 0$. We write

$$d^r_{s,t} \colon E^r_{s,t} \to E^r_{s-r,t+r-1}$$

for the component of d^r starting in bidegree (s,t). The isomorphism

$$E^{r+1}_{s,t} \cong H_{s,t}(E^r, d^r) = \frac{\ker(d^r_{s,t} \colon E^r_{s,t} \to E^r_{s-r,t+r-1})}{\mathrm{im}(d^r_{s+r,t-r+1} \colon E^r_{s+r,t-r+1} \to E^r_{s,t})}$$

is part of the data.

Here is the typical E^1-term and d^1-differential, depicted in the (s,t)-plane:

$$\begin{tikzcd}
\cdots & E^1_{-1,2} \arrow[l] & E^1_{0,2} \arrow[l] & E^1_{1,2} \arrow[l] & E^1_{2,2} \arrow[l] & E^1_{3,2} \arrow[l] & \cdots \arrow[l] \\
\cdots & E^1_{-1,1} \arrow[l] & E^1_{0,1} \arrow[l] & E^1_{1,1} \arrow[l] & E^1_{2,1} \arrow[l] & E^1_{3,1} \arrow[l] & \cdots \arrow[l] \\
\cdots & E^1_{-1,0} \arrow[l] & E^1_{0,0} \arrow[l] & E^1_{1,0} \arrow[l] & E^1_{2,0} \arrow[l] & E^1_{3,0} \arrow[l] & \cdots \arrow[l] \\
\cdots & E^1_{-1,-1} \arrow[l] & E^1_{0,-1} \arrow[l] & E^1_{1,-1} \arrow[l] & E^1_{2,-1} \arrow[l] & E^1_{3,-1} \arrow[l] & \cdots \arrow[l]
\end{tikzcd}$$

Each row is a chain complex, and the homology of this chain complex is isomorphic to the E^2-term. That E^2-term, together with the d^2-differentials, appears as follows:

$$\begin{tikzcd}
\cdots & E^2_{-1,2} & E^2_{0,2} & E^2_{1,2} & E^2_{2,2} & E^2_{3,2} & \cdots \\
\cdots & E^2_{-1,1} & E^2_{0,1} \arrow[ull] & E^2_{1,1} \arrow[ull] & E^2_{2,1} \arrow[ull] & E^2_{3,1} \arrow[ull] & \cdots \arrow[ull] \\
\cdots & E^2_{-1,0} & E^2_{0,0} \arrow[ull] & E^2_{1,0} \arrow[ull] & E^2_{2,0} \arrow[ull] & E^2_{3,0} \arrow[ull] & \cdots \arrow[ull] \\
\cdots & E^2_{-1,-1} & E^2_{0,-1} \arrow[ull] & E^2_{1,-1} \arrow[ull] & E^2_{2,-1} \arrow[ull] & E^2_{3,-1} \arrow[ull] & \cdots \arrow[ull]
\end{tikzcd}$$

(The differentials entering or leaving the displayed part are not shown.) Each line of slope $-1/2$ is a chain complex, with homology isomorphic to the E^3-term. That E^3-term, together with the d^3-differentials, appears as follows:

$$\begin{tikzcd}
\cdots & E^3_{-1,2} & E^3_{0,2} & E^3_{1,2} & E^3_{2,2} & E^3_{3,2} & \cdots \\
\cdots & E^3_{-1,1} & E^3_{0,1} & E^3_{1,1} & E^3_{2,1} & E^3_{3,1} & \cdots \\
\cdots & E^3_{-1,0} & E^3_{0,0} & E^3_{1,0} \arrow[uulll] & E^3_{2,0} \arrow[uulll] & E^3_{3,0} \arrow[uulll] & \cdots \arrow[uulll] \\
\cdots & E^3_{-1,-1} & E^3_{0,-1} & E^3_{1,-1} \arrow[uulll] & E^3_{2,-1} \arrow[uulll] & E^3_{3,-1} \arrow[uulll] & \cdots \arrow[uulll]
\end{tikzcd}$$

Each line of slope $-2/3$ is a chain complex, with homology isomorphic to the E^4-term:

$$\begin{tikzcd}
\cdots & E^4_{-1,2} & E^4_{0,2} & E^4_{1,2} & E^4_{2,2} & E^4_{3,2} & \cdots \\
\cdots & E^4_{-1,1} & E^4_{0,1} & E^4_{1,1} & E^4_{2,1} & E^4_{3,1} & \cdots \\
\cdots & E^4_{-1,0} & E^4_{0,0} & E^4_{1,0} & E^4_{2,0} & E^4_{3,0} & \cdots \\
\cdots & E^4_{-1,-1} & E^4_{0,-1} & E^4_{1,-1} & E^4_{2,-1} \arrow[uuullll] & E^4_{3,-1} \arrow[uuullll] & \cdots \arrow[uuullll]
\end{tikzcd}$$

Each line of slope $-3/4$ is a chain complex, with homology isomorphic to the E^5-term:

$$\begin{array}{ccccccc}
\dots & E^5_{-1,2} & E^5_{0,2} & E^5_{1,2} & E^5_{2,2} & E^5_{3,2} & \dots \\
\\
\dots & E^5_{-1,1} & E^5_{0,1} & E^5_{1,1} & E^5_{2,1} & E^5_{3,1} & \dots \\
\\
\dots & E^5_{-1,0} & E^5_{0,0} & E^5_{1,0} & E^5_{2,0} & E^5_{3,0} & \dots \\
\\
\dots & E^5_{-1,-1} & E^5_{0,-1} & E^5_{1,-1} & E^5_{2,-1} & E^5_{3,-1} & \dots
\end{array}$$

At this point there is not room for any further differentials within the finite part of the spectral sequence that is displayed. There may of course always be longer differentials that enter or leave the displayed region.

2.1. **E^∞-terms.** We now want to give sense to the limiting term, the E^∞-term $E^\infty = E^\infty_{*,*}$, of a spectral sequence. This is a bigraded abelian group, and we would like to make sense of $E^\infty_{s,t}$ as an algebraic limit of the abelian groups $E^r_{s,t}$ as $r \to \infty$.

In many cases the spectral sequence is *locally eventually constant*, in the sense that for each fixed bidegree (s,t) there is a natural number $m(s,t)$ such that the homomorphisms

$$d^r : E^r_{s,t} \to E^r_{s-r,t+r-1} \qquad \text{and} \qquad d^r : E^r_{s+r,t-r+1} \to E^r_{s,t}$$

are zero for all $r \geq m(s,t)$. Then $E^{r+1}_{s,t} \cong E^r_{s,t}$ for all $r \geq m(s,t)$, and we define

$$E^\infty_{s,t} = E^{m(s,t)}_{s,t}$$

to be this common value. If there is a fixed bound m that works for each bidegree (s,t), so that $d^r = 0$ for all $r \geq m$ and $E^{r+1} \cong E^r$ for all $r \geq m$, we say that the spectral sequence *collapses* at the E^m-term. In this case $E^\infty = E^m$.

In general, a spectral sequence determines a descending sequence of *r-cycles*

$$\dots \subset Z^{r+1} \subset Z^r \subset \dots \subset Z^2 = \ker(d^1) \subset Z^1 = E^1$$

and an increasing sequence of *r-boundaries*

$$0 = B^1 \subset \operatorname{im}(d^1) = B^2 \subset \dots \subset B^r \subset B^{r+1} \subset \dots \subset E^1 \,,$$

with $B^r \subset Z^r$ and $E^r \cong Z^r/B^r$ for all $r \geq 1$. (This is Boardman's indexing convention. Other authors like Mac Lane (1963) have $E^{r+1} \cong Z^r/B^r$.) We then define the bigraded abelian groups of infinite cycles and infinite boundaries to be

$$Z^\infty = \bigcap_r Z^r = \lim_r Z^r \qquad \text{and} \qquad B^\infty = \bigcup_r B^r = \operatorname*{colim}_r B^r \,,$$

respectively, and set $E^\infty = Z^\infty/B^\infty$. This definition is reasonable if the limit system of r-cycles is well-behaved, i.e., if the left derived limit $\operatorname{Rlim}_r Z^r$ vanishes. In the case of a locally eventually constant spectral sequence, the general definition agrees with the previous definition, since $Z^\infty_{s,t} = Z^r_{s,t}$ and $B^r_{s,t} = B^\infty_{s,t}$ for all $r \geq m(s,t) - 1$. [[More about this later.]]

2.2. **Filtrations.**

Definition 2.2. An *increasing filtration* of an abelian group G is a sequence $\{F_s\}_s$ of subgroups

$$\dots \subset F_{s-1} \subset F_s \subset \dots \subset G \,.$$

The filtration is *exhaustive* if the canonical map

$$\operatorname*{colim}_s F_s \longrightarrow G$$

is an isomorphism. The filtration is *Hausdorff* if

$$\lim_s F_s = 0 \,,$$

and it is *complete* if

$$\operatorname*{Rlim}_s F_s = 0\,.$$

Here $\mathrm{colim}_s F_s \cong \bigcup_s F_s$, so the filtration is exhaustive precisely if each element in G lies in some F_s. We can think of the F_s as specifying neighborhoods of 0 in a (linear) topology on G. Since $\bigcap_s F_s = \lim_s F_s$, this topology is Hausdorff if and only if the filtration is Hausdorff. An Cauchy sequence is an element in $\lim_s G/F_s$, so the topology is complete exactly when the canonical map $G \to \lim_s G/F_s$ is surjective, i.e., when $\mathrm{Rlim}_s F_s = 0$. [[More about this later.]]

In the case of a *finite filtration*, these conditions are easily verified. If there are integers $a \le b$ such that $F_s = 0$ for $s < a$ and $F_s = G$ for $s \ge b$, then the filtration has the form

$$0 \subset F_a \subset \cdots \subset F_b = G\,.$$

Clearly $\mathrm{colim}_s F_s = G$, $\lim_s F_s = 0$ and $\mathrm{Rlim}_s F_s = 0$. In this case, the only nontrivial filtration quotients are the F_s/F_{s-1} for integers s in the finite interval $[a, b]$.

In the case of a finite filtration, the group G appears as the filtration subquotient F_b/F_{a-1}. Under the three conditions above, G is also algebraically determined by the finite filtration subquotients F_s/F_{s-r}.

Lemma 2.3. *If $\{F_s\}_s$ is an exhaustive complete Hausdorff filtration of G, then*

$$G \cong \operatorname*{colim}_s \lim_r F_s/F_{s-r}$$

so that G can be recovered from the subquotients F_s/F_{s-r} of a filtration.

Proof. For each s, there is a tower of short exact sequences

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & F_{s-r} \arrow[r] \arrow[d] & F_s \arrow[r] \arrow[d] & F_s/F_{s-r} \arrow[r] \arrow[d] & 0 \\
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & F_{s-1} \arrow[r] & F_s \arrow[r] & F_s/F_{s-1} \arrow[r] & 0
\end{tikzcd}$$

giving rise to the six-term exact sequence

$$0 \to \lim_r F_{s-r} \to F_s \to \lim_s F_s/F_{s-r} \longrightarrow \operatorname*{Rlim}_r F_{s-r} \to 0 \to \operatorname*{Rlim}_r F_s/F_{s-r} \to 0\,.$$

By the complete Hausdorff assumption, $\lim_r F_{s-r} = 0$ and $\mathrm{Rlim}_r F_{s-r} = 0$, so

$$F_s \xrightarrow{\cong} \lim_s F_s/F_{s-r}$$

is an isomorphism. Passing to the colimit over s, and using the exhaustive assumption, we get the asserted formula. □

A filtration of a graded abelian group is a filtration in each degree.

Definition 2.4. A homological spectral sequence $(E^r, d^r)_r$ *converges* to a (graded) abelian group G if there is an increasing exhaustive Hausdorff filtration $\{F_s\}_s$ of G, and isomorphisms of (graded) abelian groups

$$E^\infty_{s,*} \cong F_s/F_{s-1}$$

for all integers s. The spectral sequence *converges strongly* if the filtration is also complete. In these cases we write

$$E^r \Longrightarrow G\,.$$

We call G the *target*, or the *abutment*, of the spectral sequence.

If it is necessary to emphasize the filtration degree s, we write $E^r \Longrightarrow_s G$. We may also make the bigrading explicit, as in $E^r_{*,*} \Longrightarrow G_*$ or $E^r_{s,t} \Longrightarrow G_{s+t}$.

A strongly convergent spectral sequence determines its abutment, up to questions about differentials and extensions. If we know the E^m-term for some $m \geq 1$, and can determine the d^r-differentials for all $r \geq m$, then we know the E^r-terms for all $r \geq m$, and can pass to the limit to determine the E^∞-term. [[Elaborate on how the Z^r and B^r are found, and how they specify Z^∞ and B^∞.]]

By convergence, this determines the filtration quotients $E^\infty_{s,*} \cong F_s/F_{s-1}$ for each s. There are short exact sequences

$$0 \to F_{s-r}/F_{s-r-1} \longrightarrow F_s/F_{s-r-1} \longrightarrow F_s/F_{s-r} \to 0$$

for all $r \geq 1$ and integers s, so if we inductively have determined F_s/F_{s-r}, and know $F_{s-r}/F_{s-r-1} = E^\infty_{s-r,*}$, then only an extension problem of abelian groups remains in our quest to determine F_s/F_{s-r-1}. This gives the input for the next inductive step, over r.

In the case of a finite filtration, this process gives us G after a finite number of steps. In the general case, assuming strong convergence, passing to limits over r and colimits over s recovers the abutment G.

3. The spectral sequence of a triple

To illustrate the general definitions in the first case that does not reduce to a long exact sequence, let us consider a triple (X, B, A) of spaces, and aim to understand the relationship between the homology groups $H_*(A)$, $H_*(B, A)$, $H_*(X, B)$ and $H_*(X)$. The essential pairs and maps appear in the diagram

$$\begin{tikzcd}
A \arrow[r, "i"] & B \arrow[r, "i"] \arrow[d, "j"] & X \arrow[d, "j"] \\
 & (B, A) & (X, B)\,,
\end{tikzcd}$$

but can be more systematically embedded in the larger diagram

$$\begin{tikzcd}
\dots \arrow[r, "i", "="'] & \emptyset \arrow[r, "i"] \arrow[d, "j"] & A \arrow[r, "i"] \arrow[d, "j"] & B \arrow[r, "i"] \arrow[d, "j"] & X \arrow[r, "i", "="'] \arrow[d, "j"] & X \arrow[r, "i", "="'] \arrow[d, "j"] & \dots \\
 & (\emptyset, \emptyset) & (A, \emptyset) & (B, A) & (X, B) & (X, X)\,.
\end{tikzcd}$$

We have two long exact sequences, associated to the pairs (B, A) and (X, A), respectively:

$$\dots \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(B) \xrightarrow{j_*} H_n(B, A) \xrightarrow{\partial} H_{n-1}(A) \xrightarrow{i_*} \dots$$

and

$$\dots \xrightarrow{\partial} H_n(B) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X, B) \xrightarrow{\partial} H_{n-1}(B) \xrightarrow{i_*} \dots\,.$$

We can also display these two long exact sequences together, as follows, where ∂ has degree -1 and each triangle is exact.

$$\begin{tikzcd}
H_*(A) \arrow[r, "i_*"] & H_*(B) \arrow[r, "i_*"] \arrow[d, "j_*"] & H_*(X) \arrow[d, "j_*"] \\
 & H_*(B, A) \arrow[ul, "\partial"] & H_*(X, B) \arrow[ul, "\partial"]
\end{tikzcd}$$

Again, this is the essential part of the bigger diagram

$$\begin{tikzcd}
\dots \arrow[r, "i_*"] & 0 \arrow[r, "i_*"] \arrow[d, "j_*"] & H_*(A) \arrow[r, "i_*"] \arrow[d, "j_*", "="'] & H_*(B) \arrow[r, "i_*"] \arrow[d, "j_*"] & H_*(X) \arrow[r, "i_*", "="'] \arrow[d, "j_*"] & H_*(X) \arrow[r, "i_*", "="'] \arrow[d, "j_*"] & \dots \\
 & 0 \arrow[ul, "\partial"] & H_*(A) \arrow[ul, "\partial"] & H_*(B, A) \arrow[ul, "\partial"] & H_*(X, B) \arrow[ul, "\partial"] & 0 \arrow[ul, "\partial"] & \dots \arrow[ul, "\partial"]
\end{tikzcd}$$

Our aim is to construct a spectral sequence starting with an E^1-term given by the homology groups in the lower row of this diagram, namely, $H_*(A)$, $H_*(B, A)$ and $H_*(X, B)$, and converging to the homology group $G = H_*(X)$, equipped with the finite filtration

$$0 \subset F_0 \subset F_1 \subset F_2 = G$$

where

$$
\begin{aligned}
F_0 &= \operatorname{im}(i^2_*\colon H_*(A) \to H_*(X)) \\
F_1 &= \operatorname{im}(i_*\colon H_*(B) \to H_*(X)) \\
F_2 &= H_*(X)\,.
\end{aligned}
$$

To emphasize the grading we may write $(F_s)_n$ for the part of F_s in degree n. Convergence means that there should be isomorphisms

$$E^\infty_{0,t} \cong (F_0)_t \quad , \quad E^\infty_{1,t} \cong (F_1/F_0)_{1+t} \quad \text{and} \quad E^\infty_{2,t} \cong (F_2/F_1)_{2+t}\,.$$

In fact, this spectral sequence will collapse at the E^3-term, so that there are nonzero d^1- and d^2-differentials, but $d^r = 0$ for $r \ge 3$ and $E^3 = E^\infty$.

First, the E^1-term is given by

$$E^1_{0,t} = H_t(A) \quad , \quad E^1_{1,t} = H_{1+t}(B,A) \quad \text{and} \quad E^1_{2,t} = H_{2+t}(X,B)\,,$$

and the d^1-differential is defined to be the composite $d^1 = j_* \circ \partial$. More explicitly,

$$d^1_{1,t} = \partial\colon H_{1+t}(B,A) \longrightarrow H_t(A) \qquad \text{and} \qquad d^1_{2,t} = j_*\partial\colon H_{2+t}(X,B) \longrightarrow H_{1+t}(B,A)$$

are visible in the diagrams above. Note that $d^1_{s-1,t} \circ d^1_{s,t} = 0$ for all s and t, since $\partial j_* = 0$.

The (E^1, d^1)-chart appears as follows:

$$
\begin{array}{ccccc}
\vdots & & & & \\
H_2(A) \xleftarrow{\ \partial\ } & H_3(B,A) \xleftarrow{\ j_*\partial\ } & H_4(X,B) & & \\
H_1(A) \xleftarrow{\ \partial\ } & H_2(B,A) \xleftarrow{\ j_*\partial\ } & H_3(X,B) & & \\
H_0(A) \xleftarrow{\ \partial\ } & H_1(B,A) \xleftarrow{\ j_*\partial\ } & H_2(X,B) & & \dots \\
0 & H_0(B,A) \xleftarrow{\ j_*\partial\ } & H_1(X,B) & & \\
0 & 0 & H_0(X,B) & &
\end{array}
$$

Passing to homology, we get to the E^2-term.

In column $s = 0$, we compute

$$
\begin{aligned}
E^2_{0,t} = \frac{\ker(d^1_{0,t})}{\operatorname{im}(d^1_{1,t})} &= \frac{H_t(A)}{\operatorname{im}(\partial\colon H_{1+t}(B,A) \to H_t(A))} \\
&= \frac{H_t(A)}{\ker(i_*\colon H_t(A) \to H_t(B))} \cong \operatorname{im}(i_*\colon H_t(A) \to H_t(B)) = \operatorname{im}(i_*)_t
\end{aligned}
$$

using exactness at $H_t(A)$. The right hand isomorphism is induced by i_*.

In column $s = 1$, we find

$$
\begin{aligned}
E^2_{1,t} = \frac{\ker(d^1_{1,t})}{\operatorname{im}(d^1_{2,t})} &= \frac{\ker(\partial\colon H_{1+t}(B,A) \to H_t(A))}{\operatorname{im}(j_*\partial\colon H_{2+t}(X,B) \to H_{1+t}(B,A))} \\
&= \frac{\operatorname{im}(j_*\colon H_{1+t}(B) \to H_{1+t}(B,A))}{\operatorname{im}(j_*\partial\colon H_{2+t}(X,B) \to H_{1+t}(B,A))} \\
&\cong \frac{H_{1+t}(B)}{\ker(j_*\colon H_{1+t}(B) \to H_{1+t}(B,A)) + \operatorname{im}(\partial\colon H_{2+t}(X,B) \to H_{1+t}(B))}
\end{aligned}
$$

using exactness at $H_{1+t}(B,A)$. The last isomorphism is induced by j_*. To verify it, it is clear that j_* induces a surjection from $H_{1+t}(B)$ to the quotient $\operatorname{im}(j_*)/\operatorname{im}(j_*\partial)$ (on the preceding line). Its kernel consists of the elements that map under j_* to elements in the image of $j_*\partial$. These differ by elements in $\ker(j_*)$ from elements in $\operatorname{im}(\partial)$, hence are in the sum $\ker(j_*) + \operatorname{im}(\partial)$. This is an internal sum of subgroups of $H_{1+t}(B)$, not necessarily a direct sum. Using exactness at $H_{1+t}(B)$ in two different exact sequences, we can rewrite this as follows:

$$
\begin{aligned}
\dots &= \frac{H_{1+t}(B)}{\operatorname{im}(i_*\colon H_{1+t}(A) \to H_{1+t}(B)) + \ker(i_*\colon H_{1+t}(B) \to H_{1+t}(X))} \\
&\cong \frac{\operatorname{im}(i_*\colon H_{1+t}(B) \to H_{1+t}(X))}{\operatorname{im}(i_*^2\colon H_{1+t}(A) \to H_{1+t}(X))} = (F_1/F_0)_{1+t}
\end{aligned}
$$

The second isomorphism is induced by i_*, and is formally of the same type as the one we just discussed: The homomorphism i_* induces a surjection from $H_{1+t}(B)$ to $\operatorname{im}(i_*)/\operatorname{im}(i_*^2)$, with kernel given by the internal sum of $\ker(i_*\colon H_{1+t}(B) \to H_{1+t}(X))$ and $\operatorname{im}(i_*\colon H_{1+t}(A) \to H_{1+t}(B))$.

In column $s = 2$, we calculate

$$
\begin{aligned}
E^2_{2,t} = \frac{\ker(d^1_{2,t})}{\operatorname{im}(d^1_{3,t})} &= \frac{\ker(j_*\partial\colon H_{2+t}(X, B) \to H_{1+t}(B, A))}{0} \\
&= \partial^{-1} \ker(j_*\colon H_{1+t}(B) \to H_{1+t}(B, A)) \\
&= \partial^{-1} \operatorname{im}(i_*\colon H_{1+t}(A) \to H_{1+t}(B)) = \partial^{-1}(\operatorname{im}(i_*)_{1+t})
\end{aligned}
$$

using exactness at $H_{1+t}(B)$. This is the subgroup of $H_{2+t}(X, B)$ consisting of elements x with $\partial(x) \in H_{1+t}(B)$ lying in the image of $i_*\colon H_{1+t}(A) \to H_{1+t}(B)$.

The d^2-differential acting on the E^2-term is now defined to be the homomorphism

$$
d^2_{2,t}\colon E^2_{2,t} = \partial^{-1}(\operatorname{im}(i_*)_{1+t}) \longrightarrow \operatorname{im}(i_*)_{1+t} = E^2_{0,t+1}
$$

induced by ∂, mapping a class $x \in E^2_{2,t}$ with $\partial(x) \in \operatorname{im}(i_*)_{1+t}$ to the class $d^2(x) = \partial(x) \in E^2_{0,1+t}$.

The (E^2, d^2)-chart appears as follows:

$$
\begin{tikzcd}
\vdots & & & \\
\operatorname{im}(i_*)_2 & (F_1/F_0)_3 & \partial^{-1}(\operatorname{im}(i_*)_3) \arrow[ull, "d^2_{2,2}"'] & \\
\operatorname{im}(i_*)_1 & (F_1/F_0)_2 & \partial^{-1}(\operatorname{im}(i_*)_2) \arrow[ull, "d^2_{2,1}"'] & \\
\operatorname{im}(i_*)_0 & (F_1/F_0)_1 & \partial^{-1}(\operatorname{im}(i_*)_1) \arrow[ull, "d^2_{2,0}"'] & \dots \\
0 & (F_1/F_0)_0 & \partial^{-1}(\operatorname{im}(i_*)_0) \arrow[ull, "d^2_{2,-1}"'] & \\
0 & 0 & H_0(X, B) &
\end{tikzcd}
$$

Passing to homology once more, we get to the E^3-term.

In column $s = 0$, the E^3-term is

$$
\begin{aligned}
E^3_{0,t} = \frac{\ker(d^2_{0,t})}{\operatorname{im}(d^2_{2,t-1})} &\cong \frac{\operatorname{im}(i_*\colon H_t(A) \to H_t(B))}{\operatorname{im}(\partial\colon \partial^{-1}(\operatorname{im} i_*)_t \to \operatorname{im}(i_*)_t)} \\
&= \frac{\operatorname{im}(i_*\colon H_t(A) \to H_t(B))}{\operatorname{im}(\partial\colon H_{1+t}(X, B) \to H_t(B)) \cap \operatorname{im}(i_*\colon H_t(A) \to H_t(B))} \\
&= \frac{\operatorname{im}(i_*\colon H_t(A) \to H_t(B))}{\ker(i_*\colon H_t(B) \to H_t(X)) \cap \operatorname{im}(i_*\colon H_t(A) \to H_t(B))} \\
&\cong \operatorname{im}(i_*^2\colon H_t(A) \to H_t(X)) = (F_0)_t
\end{aligned}
$$

using the definition of $d^2_{2,t-1}$ and exactness at $H_t(B)$. The last isomorphism is induced by $i_* \colon H_t(B) \to H_t(X)$.

Column $s = 1$ is not affected by the d^2-differentials, so

$$E^3_{1,t} = \frac{\ker(d^2_{1,t})}{\operatorname{im}(d^2_{3,t-1})} = \frac{E^2_{1,t}}{0} \cong (F_1/F_0)_{1+t} \,.$$

In column $s = 2$, the E^3-term is

$$\begin{aligned} E^3_{2,t} = \frac{\ker(d^2_{2,t})}{\operatorname{im}(d^2_{4,t-1})} &= \frac{\ker(d^2_{2,t} \colon \partial^{-1}(\operatorname{im}(i_*)_{1+t}) \to \operatorname{im}(i_*)_{1+t})}{0} \\ &= \ker(\partial \colon H_{2+t}(X,B) \to H_{1+t}(B)) \\ &= \operatorname{im}(j_* \colon H_{2+t}(X) \to H_{2+t}(X,B)) \\ &\cong \frac{H_{2+t}(X)}{\operatorname{im}(i_* \colon H_{2+t}(B) \to H_{2+t}(X))} = (F_2/F_1)_{2+t} \end{aligned}$$

by the definition of the d^2-differential, and exactness at $H_{2+t}(X,B)$ and at $H_{2+t}(X)$.

The E^3-term appears as follows:

$$\begin{array}{cccc} \vdots & & & \\ & & & \\ (F_0)_2 & (F_1/F_0)_3 & (F_2/F_1)_4 & \\ (F_0)_1 & (F_1/F_0)_2 & (F_2/F_1)_3 & \\ (F_0)_0 & (F_1/F_0)_1 & (F_2/F_1)_2 & \dots \\ 0 & (F_1/F_0)_0 & (F_2/F_1)_1 & \\ 0 & 0 & (F_2/F_1)_0 & \end{array}$$

There is no room for further nonzero differentials, since d^r for $r \geq 3$ must involve columns three or more units apart. Hence this spectral sequence collapses at the E^3-term, and $E^\infty = E^3$ is as displayed above.

In view of our calculations, we have isomorphisms

$$E^\infty_{s,t} \cong (F_s/F_{s-1})_{s+t}$$

in all bidegrees (s,t), which proves that the spectral sequence we have constructed, with the given E^1-term, d^1-differential and d^2-differential, indeed converges strongly to the abutment $H_*(X)$, with the finite filtration given by

$$(F_s)_n = \begin{cases} 0 & \text{for } s \leq -1 \\ \operatorname{im}(i^2_* \colon H_n(A) \to H_n(X)) & \text{for } s = 0 \\ \operatorname{im}(i_* \colon H_n(B) \to H_n(X)) & \text{for } s = 1 \\ H_n(X) & \text{for } s \geq 2. \end{cases}$$

Hence we can conclude that there is a strongly convergent spectral sequence

$$E^1_{s,t} \Longrightarrow_s H_{s+t}(X)$$

with three nonzero columns

$$E^1_{s,t} = \begin{cases} 0 & \text{for } s \le -1 \\ H_t(A) & \text{for } s = 0 \\ H_{1+t}(B, A) & \text{for } s = 1 \\ H_{2+t}(X, B) & \text{for } s = 2 \\ 0 & \text{for } s \ge 3. \end{cases}$$

[[Illustrate with an example?]]

[[The K-theory based Adams spectral sequence is an interesting three-line spectral sequence (Adams–Baird, Bousfield, Dwyer–Mitchell).]]

4. Cohomological spectral sequences

So far we have focused on so-called *homological* spectral sequences, where the differentials reduce total degrees and filtration indices. If one applies cohomology to the same diagrams of spaces, one instead obtains a *cohomological* spectral sequence.

Definition 4.1. A *cohomological spectral sequence* is a sequence $(E_r, d_r)_r$ of bigraded abelian groups and differentials, together with isomorphisms

$$E_{r+1} \cong H(E_r, d_r)$$

for all $r \ge 1$. Each E_r-term is a bigraded abelian group $E_r = E_r^{*,*} = (E_r^{s,t})_{s,t}$, and each d^r-differential is a homomorphism $d_r \colon E_r^{*,*} \to E_r^{*,*}$ of bidegree $(r, 1-r)$, satisfying $d_r \circ d_r = 0$.

Definition 4.2. A *decreasing filtration* of an abelian group G is a sequence $\{F^s\}_s$ of subgroups

$$G \supset \cdots \supset F^s \supset F^{s+1} \supset \dots .$$

It is *exhaustive* if $\operatorname{colim}_s F^s \cong G$, *Hausdorff* if $\lim_s F^s = 0$ and *complete* if $\operatorname{Rlim}_s F^s = 0$.

Definition 4.3. A cohomological spectral sequence $(E^r, d^r)_r$ *converges* to a graded abelian group G if there is a decreasing exhaustive Hausdorff filtration $\{F^s\}_s$ of G, and isomorphisms of (graded) abelian groups

$$E_\infty^{s,*} \cong F^s/F^{s+1}$$

for all integers s. The spectral sequence *converges strongly* if the filtration is also complete.

The algebraic structure in a cohomological spectral sequence is really the same as in a homological spectral sequence; the difference only lies in the sign conventions for the grading. To each homological spectral sequence $(E^r, d^r)_r$ there is an associated cohomological spectral sequence $(E_r, d_r)_r$ with

$$E_r^{s,t} = E^r_{-s,-t}$$

for all integers s and t, and with

$$d_r^{s,t} = d^r_{-s,-t} \,.$$

To each increasing filtration $\{F_s\}_s$ of an abelian group G there is an associated decreasing filtration $\{F^s\}_s$ of the same group, with

$$F^s = F_{-s} \,.$$

The spectral sequence $(E^r, d^r)_r$ converges (strongly) to the abutment G, filtered by $\{F_s\}_s$, if and only if the associated cohomological spectra sequence $(E_r, d_r)_r$ converges (strongly) to the abutment G, filtered by $\{F^s\}_s$.

The sign change in the bidegree of the spectral sequence differentials implies that the direction of the arrows in an (E_r, d_r)-chart is reversed in comparison with the direction in an (E^r, d^r)-chart. For instance, an (E_2, d_2)-term typically appears as follows. (Compare with the (E^2, d^2)-term displayed earlier.)

$$\begin{tikzcd}
\dots \arrow[drr] & E_2^{-1,2} \arrow[drr] & E_2^{0,2} \arrow[drr] & E_2^{1,2} \arrow[drr] & E_2^{2,2} \arrow[drr] & E_2^{3,2} & \dots \\
\dots \arrow[drr] & E_2^{-1,1} \arrow[drr] & E_2^{0,1} \arrow[drr] & E_2^{1,1} \arrow[drr] & E_2^{2,1} \arrow[drr] & E_2^{3,1} & \dots \\
\dots \arrow[drr] & E_2^{-1,0} \arrow[drr] & E_2^{0,0} \arrow[drr] & E_2^{1,0} \arrow[drr] & E_2^{2,0} \arrow[drr] & E_2^{3,0} & \dots \\
\dots & E_2^{-1,-1} & E_2^{0,-1} & E_2^{1,-1} & E_2^{2,-1} & E_2^{3,-1} & \dots
\end{tikzcd}$$

One reason for switching from a homological to a cohomological indexing occurs when the spectral sequence occupies a quadrant, or a half-plane. If the homological spectral sequence $E^r_{s,t}$ is nonzero only for $s \le 0$ and $t \le 0$ (or for $s \le 0$), then the associated cohomological spectral sequence $E_r^{s,t}$ is nonzero only for $s \ge 0$ and $t \ge 0$ (or for $s \ge 0$). It tends to be notationally easier to work with the latter conventions. We refer to such a spectral sequence as a *first quadrant* (or *right half-plane*) spectral sequence. [[Formalize this definition?]]

[[Another reason for working with cohomology has to do with product structures. The cup product in cohomology can be well respected by the spectral sequence.]]

[[Also mention Adams indexing. Since this will be our main focus, once we get the basic formalism for spectral sequences in place, we will return to this in more detail later.]]

5. Example: The Serre spectral sequence

5.1. **Serre fibrations.** A *Serre fibration* is a map $p\colon E \to B$ with the homotopy lifting property for CW complexes (or, equivalently, for polyhedra), cf. Serre (1951). This means that for any CW complex X, map $f\colon X \to E$ and homotopy $H\colon X \times I \to B$ such that $H(x,0) = pf(x)$, there exists a homotopy $\tilde{H}\colon X \times I \to E$ with $\tilde{H}(x,0) = f(x)$ and $p\tilde{H} = H$.

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "i_0"'] & E \arrow[d, "p"] \\
X \times I \arrow[r, "H"'] \arrow[ur, dashed, "\tilde{H}"] & B
\end{tikzcd}$$

Any fiber bundle over a paracompact base space is a Serre fibration, cf. Spanier (1981, Theorem 2.7.13). Suppose that B is a connected CW complex, and choose a base point $b_0 \in B$. Let $F = p^{-1}(b_0)$ be the fiber above that base point. The fundamental group $\pi_1(B, b_0)$ acts on the homology $H_*(F)$ of that fiber.

$$\begin{tikzcd}
F \arrow[r] \arrow[d] & E \arrow[d, "p"] \\
\{b_0\} \arrow[r] & B
\end{tikzcd}$$

5.2. **The homological Serre spectral sequence.** The homological Serre spectral sequence for $F \to E \to B$ is a spectral sequence converging to the homology $H_*(E)$ of the total space. It has E^1-term

$$E^1_{s,t} = C_s(B; \mathscr{H}_t(F))$$

given in bidegree (s,t) by the cellular s-chains of B with coefficients in the local coefficient system $\mathscr{H}_t(F)$ associated to the action of the fundamental group on the homology of the fiber, and

$$E^2_{s,t} = H_s(B; \mathscr{H}_t(F))$$

is given by the cellular homology of B with these local coefficients. If B is simply-connected, or more generally, if the action is trivial, then this is the ordinary cellular homology of B with coefficients in the abelian group $H_t(F)$. Notice that the E^2-term, unlike the E^1-term, does not depend on the chosen CW structure on B. Hence the remaining terms of the spectral sequence are topological invariants of the Serre fibration $p\colon E \to B$. Notice also that $E^2_{s,t}$ can only be nonzero for $s \ge 0$ and $t \ge 0$, hence the same holds for every later term $E^r_{s,t}$. It follows that the Serre spectral sequence, like any other first quadrant spectral sequence, is locally eventually constant, because $d^r_{s,t} = 0$ when $s - r < 0$ and $d^r_{s+r,t-r+1} = 0$ when $t - r + 1 < 0$, so both of these differentials vanish whenever $r \ge m(s,t) = \max\{s+1, t+2\}$. The Serre spectral sequence converges strongly to the homology of the total space:

$$E^2_{s,t} = H_s(B; \mathscr{H}_t(F)) \Longrightarrow_s H_{s+t}(E)\,.$$

5.3. **The cohomological Serre spectral sequence.** There is also a cohomological Serre spectral sequence, with E_1-term

$$E_1^{s,t} = C^s(B; \mathscr{H}^t(F))$$

given by the cellular s-cochains on B with coefficients in the local coefficient system $\mathscr{H}^t(F)$. The E_2-term

$$E_2^{s,t} = H^s(B; \mathscr{H}^t(F))$$

is given by the cellular cohomology with the same coefficients, and the spectral sequence converges strongly to the cohomology of the total space:

$$E_2^{s,t} = H^s(B; \mathscr{H}^t(F)) \Longrightarrow_s H^{s+t}(E)\,.$$

5.4. **Killing homotopy groups.** We illustrate by an example, based on the method of "killing homotopy groups", which was used by Serre [[and others?]] to determine several of the first nontrivial homotopy groups of spheres, i.e., the homotopy groups $\pi_i(S^j)$ for varying i and j. It is quite easy to show that $\pi_i(S^j) = 0$ for $i < j$. In the case $i = j$ the Hurewicz theorem shows that $\pi_i(S^i) \cong H_i(S^i) \cong \mathbb{Z}$ for $i \ge 1$. The cases $i > j$ remain. When $j = 1$ we know that the contractible space $\mathbb{R}$ is the universal covering space of S^1, so $\pi_i(\mathbb{R}) \cong \pi_i(S^1)$ for all $i \ge 2$, hence $\pi_i(S^1) = 0$ for $i \ge 2$. The cases $j \ge 2$ are significantly harder. There is a fiber sequence

$$S^1 \longrightarrow S^3 \xrightarrow{\eta} S^2\,,$$

where η is the complex Hopf fibration, and the associated long exact sequence of homotopy groups tells us that $\eta_* \colon \pi_i(S^3) \to \pi_i(S^2)$ is an isomorphism for $i \ge 3$. Hence the cases $j = 2$ and $j = 3$ are practically equivalent.

It turns out to be most convenient to start the analysis with the space S^3. As already mentioned, the first homotopy groups of S^3 are $\pi_i(S^3) = 0$ for $i < 3$ and $\pi_3(S^3) \cong H_3(S^3) \cong \mathbb{Z}$, by the Hurewicz theorem. To calculate $\pi_4(S^3)$, we shall construct the 3-connected cover E of S^3, i.e., a map $g \colon E \to S^3$ such that $\pi_i(E) = 0$ for $i \le 3$ and $g_* \colon \pi_i(E) \to \pi_i(S^3)$ is an isomorphism for $i > 3$, in such a way that we can calculate the homology $H_*(E)$ using a Serre spectral sequence. First we construct a map $h \colon S^3 \to K$, where $h_* \colon \pi_i(S^3) \to \pi_i(K)$ is an isomorphism for $i \le 3$ and $\pi_i(K) = 0$ for $i > 3$. The space K will then be an Eilenberg–Mac Lane space of type $K(\mathbb{Z}, 3)$.

To construct K, start with $K^{(4)} = S^3$ and attach 5-cells to kill the non-zero classes in $\pi_4(S^3)$. Then attach 6-cells to kill the non-zero classes in π_5 of the resulting CW complex $K^{(5)}$. Inductively, suppose we have constructed a CW pair $(K^{(n)}, S^3)$, such that $\pi_i(S^3) \cong \pi_i(K^{(n)})$ for $i \le 3$ and $\pi_i(K^{(n)}) = 0$ for $3 < i < n$. Attach $(n+1)$-cells to $K^{(n)}$ to kill the non-zero classes in $\pi_n(K^{(n)})$, and call the result $K^{(n+1)}$. Continuing, we can let $K = \bigcup_n K^{(n)} = \operatorname{colim}_n K^{(n)}$, and the inclusion $h \colon S^3 \to K$ has the properties described above. To prove that this works, use the homotopy excision theorem. [[Reference in Hatcher?]]

[[Also comment on Pontryagin and Whitehead's early work using framed bordism, and its pitfalls.]]

5.5. **The 3-connected cover of S^3.** Let $g \colon E \to S^3$ be the homotopy fiber of $h \colon S^3 \to K$. By the long exact sequence in homotopy, E is the 3-connected cover of S^3. Furthermore, g is a Serre fibration. Let $f \colon F \to E$ be the homotopy fiber of $g \colon E \to S^3$.

$$F \xrightarrow{f} E \xrightarrow{g} S^3 \xrightarrow{h} K\,.$$

By a general result for such Puppe fiber sequences, we know that $F \simeq \Omega K$, so F is an Eilenberg–Mac Lane space of type $K(\mathbb{Z}, 2)$. In other words, $F \simeq \mathbb{C}P^\infty$. Hence we have a homotopy fiber sequence

$$\mathbb{C}P^\infty \longrightarrow E \xrightarrow{g} S^3\,.$$

The associated homological Serre spectral sequence has E^2-term

$$E^2_{s,t} = H_s(S^3; H_t(\mathbb{C}P^\infty)) \cong \begin{cases} \mathbb{Z} & \text{for } s \in \{0,3\} \text{ and } t \ge 0 \text{ even,} \\ 0 & \text{otherwise,} \end{cases}$$

and converges strongly to $H_{s+t}(E)$. We can display the E^2-term as in Figure 1. Notice that there is

	$\vdots$					
$t=5$	0	0	0	0	0	
$t=4$	$\mathbb{Z}$	0	0	$\mathbb{Z}$	0	
$t=3$	0	0	0	0	0	
$t=2$	$\mathbb{Z}$	0	0	$\mathbb{Z}$	0	
$t=1$	0	0	0	0	0	
$t=0$	$\mathbb{Z}$	0	0	$\mathbb{Z}$	0	$\dots$
	$s=0$	$s=1$	$s=2$	$s=3$	$s=4$	

FIGURE 1. Serre E^2-term for $H_*(E)$

no room for nonzero d^2-differentials, since $d^2_{s,t}$ can only originate from a nonzero group $E^2_{s,t}$ if $s = 0$ or $s = 3$, and in either case the target group $E^2_{s-2,t+1}$ is zero.

Hence $d^2 = 0$ in this spectral sequence, which implies that $E^3 = E^2$. There is, however, room for d^3-differentials, which we display in the E^3-term as in Figure 2. The difficulty now is to determine these homomorphisms $d^3_{3,t}$ for $t \ge 0$ even. At this point we can already deduce that each group $H_n(E)$ is a finitely generated abelian group (of rank 0 or 1), since whatever the d^r-differentials are, only a trivial, finite cyclic or infinite cyclic group will be left at the E^∞-term in each bidegree (s,t) with $s \in \{0,3\}$ and $t \ge 0$ even. Since there is at most one nontrivial group in each total degree of $E^\infty_{*,*}$, we can conclude that the abutment $H_*(E)$ is also either trivial, finite cyclic or infinite cyclic in each degree.

5.6. **The first differential.** By looking a bit ahead and working backwards, we can prove that the first differential, $d^3_{3,0}$, is an isomorphism. This is because at the E^4-term the only possibly nonzero groups in total degree $s + t \le 3$ will be

$$E^4_{0,2} = \mathrm{cok}(d^2_{3,0}) \qquad \text{and} \qquad E^4_{3,0} = \ker(d^2_{3,0})\,.$$

Since the spectral sequence is concentrated in the two columns $s = 0$ and $s = 3$, i.e., is zero for all other s, there is no room for any longer differentials than the d^3-differentials. Hence $d^r = 0$ for $r \ge 4$, and $E^4 = E^\infty$. So if the cokernel or kernel of $d^2_{3,0}$ is nonzero, then it will survive to the E^∞-term of the spectral sequence, in total degree 2 or 3, respectively. The spectral sequence converges to $H_*(E)$, where E by construction is 3-connected. Hence, by the Hurewicz theorem, $H_n(E) = 0$ for $0 < n \le 3$. It follows that $(F_s)_n = 0$ and $(F_s/F_{s-1})_n = 0$ for all s and all $0 < n \le 3$. Convergence of the spectral sequence thus implies that $E^\infty_{s,t} = 0$ for $0 < s+t \le 3$. In particular, $E^\infty_{0,2} = \mathrm{cok}(d^2_{3,0}) = 0$ and $E^4_{3,0} = \ker(d^2_{3,0}) = 0$. This is equivalent to the assertion that $d^2_{3,0}$ is an isomorphism.

5.7. **The cohomological version.** How do we proceed from here to determine the second differential, $d^3_{3,2}$? It is not clear how to do this using only the additive structure in homology. Instead, we will pass to cohomology, and use the multiplicative structure in the cohomological Serre spectral sequence, related to the cup product in cohomology, to calculate all the later cohomological d_3-differentials from the first d_3-differential, dual to the homological d^3-differential that we just identified.

Let us use the notations $H^*(\mathbb{C}P^\infty) = \mathbb{Z}[y]$ and $H^*(S^3) = \mathbb{Z}[z]/(z^2)$, with algebra generators y in degree $|y| = 2$, and z in degree $|z| = 3$. The cohomological Serre spectral sequence for $\mathbb{C}P^\infty \to E \to S^3$

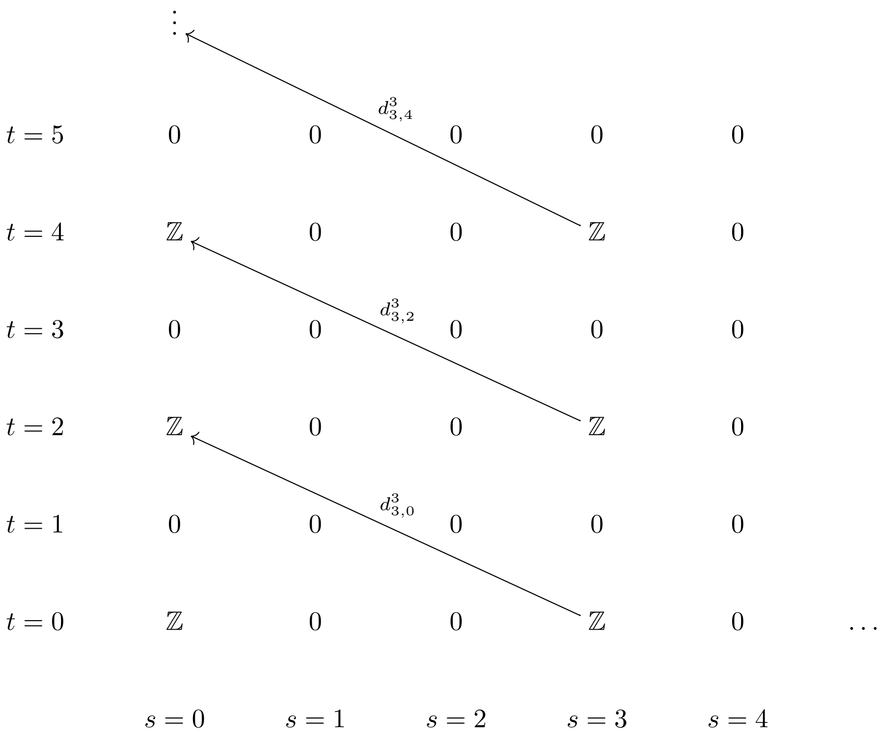


FIGURE 2. Serre E^3-term for $H_*(E)$

has E_2-term

$$E_2^{s,t} = H^s(S^3; H^t(\mathbb{C}P^\infty)) = \begin{cases} \mathbb{Z} & \text{for } s \in \{0,3\} \text{ and } t \geq 0 \text{ even,} \\ 0 & \text{otherwise,} \end{cases}$$

and converges strongly to $H^{s+t}(E)$. So far, this looks just like in homology. However, the cohomological Serre spectral sequence has the additional property of being an *algebra spectral sequence*, meaning that each E_r-term is a graded algebra, and each d_r-differential is a derivation with respect to this algebra structure. This means that d_r satisfies a Leibniz rule, of the form

$$d_r(ab) = d_r(a)b + (-1)^{|a|} a d_r(b)\,,$$

for classes $a, b \in E_r$ and their (cup) product $ab = a \cup b$. We shall return to the precise definition and interpretation of multiplicative structures in spectral sequences, later.

For now, we just observe that the algebra structure of the cohomological Serre spectral sequence E_2-term can be written as

$$E_2^{*,*} = H^*(S^3; H^*(\mathbb{C}P^\infty)) \cong \mathbb{Z}[z]/(z^2) \otimes \mathbb{Z}[y]\,.$$

Since the spectral sequence is concentrated in the columns $s = 0$ and $s = 3$, there is only room for d_3-differentials, so $E_2 = E_3$ and $E_4 = E_\infty$. We now display the cohomological E_3-term and the d_3-differentials, in Figure 3. Note that the direction of the differentials is reversed, compared to the homological case. Note also that we can now give names to the additive generators in the various bidegrees, as products of powers of y and z.

We can now argue as before, that $\ker(d_3^{0,2}) = E_4^{0,2} = E_\infty^{0,2}$ and $\mathrm{cok}(d_3^{0,2}) = E_4^{3,0} = E_\infty^{3,0}$ must contribute to $H^2(E)$ and $H^3(E)$, respectively, and since the latter two groups are trivial, hence so is the kernel and cokernel of $d_3^{0,2}$. Alternatively, one can appeal to a Kronecker pairing of spectral sequences, evaluating the cohomological spectral sequence on the homological one, to deduce that

$$d_3^{0,2} \colon H^0(S^3; H^2(\mathbb{C}P^\infty)) \longrightarrow H^3(S^3; H^0(\mathbb{C}P^\infty))$$

is dual to

$$d^3_{3,0} \colon H_3(S^3; H_0(\mathbb{C}P^\infty)) \longrightarrow H_0(S^3; H_2(\mathbb{C}P^\infty))\,,$$

$$
\begin{tikzcd}
 & \vdots \arrow[rrrdd, "d_3^{0,6}"] & & & & & \\
t=5 & 0 & 0 & 0 & 0 & 0 & \\
t=4 & \mathbb{Z}\{y^2\} \arrow[rrrdd, "d_3^{0,4}"] & 0 & 0 & \mathbb{Z}\{zy^2\} & 0 & \\
t=3 & 0 & 0 & 0 & 0 & 0 & \\
t=2 & \mathbb{Z}\{y\} \arrow[rrrdd, "d_3^{0,2}"] & 0 & 0 & \mathbb{Z}\{zy\} & 0 & \\
t=1 & 0 & 0 & 0 & 0 & 0 & \\
t=0 & \mathbb{Z}\{1\} & 0 & 0 & \mathbb{Z}\{z\} & 0 & \dots \\
 & s=0 & s=1 & s=2 & s=3 & s=4 &
\end{tikzcd}
$$

FIGURE 3. Serre E_3-term for $H^*(E)$

in the sense of the universal coefficient theorem. This leads to the same conclusion. Hence we find that

$$d_3(y) = z\,,$$

up to a possible sign. If necessary we replace y or z by its negative, to make sure that the formula above holds.

5.8. **The remaining differentials.** At this point, the algebra structure comes to our aid. The Leibniz rule for d_3 applied with $a = y$ and $b = y$ asserts that

$$d_3(y^2) = d_3(y)y + yd_3(y) = zy + yz = 2zy\,.$$

By induction, it follows that

$$d_3(y^k) = kzy^{k-1}$$

for all $k \geq 1$. Hence the homomorphism

$$d_3^{0,2k} \colon \mathbb{Z}\{y^k\} \longrightarrow \mathbb{Z}\{zy^{k-1}\}$$

is given by multiplication by k, with respect to this basis. This lets us calculate the $E_4 = E_\infty$-term

$$E_\infty^{s,t} \cong \begin{cases} \mathbb{Z} & \text{for } s = t = 0, \\ \mathbb{Z}/k & \text{for } s = 3 \text{ and } t = 2k-2, \\ 0 & \text{otherwise.} \end{cases}$$

It appears as in Figure 4. The group $\mathbb{Z}/1\{z\}$ in bidegree $(3, 0)$ is of course trivial. This leads to the conclusion

$$H^n(E) \cong \begin{cases} \mathbb{Z} & \text{for } n = 0, \\ \mathbb{Z}/k & \text{for } n = 2k+1, \\ 0 & \text{otherwise.} \end{cases}$$

For instance, in total degree $n = 5$, we have a finite descending filtration

$$H^5(E) = (F^0)^5 \supset (F^1)^5 \supset (F^2)^5 \supset (F^3)^5 \supset (F^4)^5 \supset (F^5)^5 \supset (F^6)^5 = 0\,,$$

$$
\begin{array}{ccccccc}
 & \vdots & & & & & \\
t=5 & 0 & 0 & 0 & 0 & 0 & \\
t=4 & 0 & 0 & 0 & \mathbb{Z}/3\{zy^2\} & 0 & \\
t=3 & 0 & 0 & 0 & 0 & 0 & \\
t=2 & 0 & 0 & 0 & \mathbb{Z}/2\{zy\} & 0 & \\
t=1 & 0 & 0 & 0 & 0 & 0 & \\
t=0 & \mathbb{Z}\{1\} & 0 & 0 & 0 & 0 & \dots \\
 & s=0 & s=1 & s=2 & s=3 & s=4 &
\end{array}
$$

FIGURE 4. Serre E_∞-term for $H^*(E)$

with filtration quotients

$$(F^s/F^{s+1})^5 = (F^s)^5/(F^{s+1})^5 \cong E_\infty^{s,5-s}$$

for all $0 \le s \le 5$. Hence

$$H^5(E) = (F^0)^5 = (F^1)^5 = (F^2)^5 = (F^3)^5 \qquad \text{and} \qquad (F^4)^5 = (F^5)^5 = (F^6)^5 = 0$$

while $(F^3)^5/(F^4)^5 \cong E_\infty^{3,2} = \mathbb{Z}/2\{zy\}$. Thus $H^5(E) \cong \mathbb{Z}/2$. In this case there were no (non-obvious) extension questions, since there was at most one nontrivial group in each total degree of the E_∞-term.

5.9. **Conclusions about homotopy groups.** We observed from the homological Serre spectral sequence that $H_*(E)$ is of *finite type*, i.e., a finitely generated abelian group in each degree, so the universal coefficient theorem allows us to determine these homology groups from the corresponding cohomology groups. We obtain

$$H_n(E) \cong \begin{cases} \mathbb{Z} & \text{for } n = 0, \\ \mathbb{Z}/k & \text{for } n = 2k, \\ 0 & \text{otherwise.} \end{cases}$$

Hence the first nontrivial homology group of the 3-connected cover E of S^3 is $H_4(E) \cong \mathbb{Z}/2$. By the Hurewicz theorem, $\pi_4(E) \cong H_4(E)$, and by the defining property of a 3-connected cover, $\pi_4(S^3) \cong \pi_4(E)$.

Theorem 5.1. $\pi_4(S^3) \cong \mathbb{Z}/2$.

By a refinement of these methods, it is possible to concentrate on a prime p, such as 2, 3 or 5, and to calculate the p-localized homology and homotopy groups of all the spaces involved. For instance, we write $H_n(E)_{(p)}$ for the localization of $H_n(E)$ at p, which means the result of making multiplication by each prime other than p invertible in $H_n(E)$. More explicitly,

$$H_n(E)_{(p)} \cong H_n(E) \otimes \mathbb{Z}_{(p)}$$

where $\mathbb{Z}_{(p)}$ is the ring of p-local integers, i.e., the ring of rational numbers a/b where p does not divide b. We find that $(\mathbb{Z}/k)_{(p)} = \mathbb{Z}/(k,p)$ for $k \ge 1$, where (k,p) denotes the greatest common divisor of k and

p. This equals the p-Sylow subgroup of $\mathbb{Z}/k$, which is only nontrivial if p divides k. Hence

$$H_n(E)_{(p)} \cong \begin{cases} \mathbb{Z}_{(p)} & \text{for } n = 0, \\ \mathbb{Z}/(k,p) & \text{for } n = 2k, \\ 0 & \text{otherwise,} \end{cases}$$

and the first nontrivial p-local homology group of E is $H_{2p}(E)_{(p)} \cong \mathbb{Z}/p$. By the p-local Hurewicz theorem, $\pi_{2p}(E)_{(p)} \cong H_{2p}(E)_{(p)}$, and by the defining property of E, $\pi_{2p}(S^3)_{(p)} \cong \pi_{2p}(E)_{(p)}$.

Theorem 5.2. *$\pi_i(S^3)_{(p)} = 0$ for $3 < i < 2p$, and $\pi_{2p}(S^3)_{(p)} \cong \mathbb{Z}/p$.*

The formalism for working with p-local homology and homotopy groups is a special case of a more general theory of localizations. Its first incarnation, which suffices for the computation above, is known as the theory of "Serre classes", cf. Spanier (1981, Section 9.6). [[References for later work: Sullivan, Bousfield–Kan (1972).]]

5.10. **Stable homotopy groups.** There are suspension homomorphisms

$$\pi_i(S^j) \xrightarrow{\Sigma} \pi_{i+1}(S^{j+1})$$

taking the homotopy class of a map $\alpha\colon S^i \to S^j$ to the homotopy class of its suspension, $\Sigma\alpha\colon S^{i+1} = \Sigma S^i \to \Sigma S^j = S^{j+1}$. The homomorphism Σ is often denoted E, for the German "Einhängung". By Freudenthal's suspension theorem, Σ is an isomorphism if $i \le 2j-2$, and it is surjective if $i = 2j-1$. Iterating, and passing to the colimit, we come to the stable homotopy groups of spheres, also known as the *stable stems*:

$$\pi_n^S = \operatorname*{colim}_j \pi_{j+n}(S^j) \,.$$

By Freudenthal's theorem, the colimit system consists of isomorphisms for $j \ge n+2$, so we have isomorphisms $\pi_{j+n}(S^j) \cong \pi_n^S$ for all $j \ge n+2$, and a surjection $\Sigma\colon \pi_{2n+1}(S^{n+1}) \to \pi_n^S$.

In particular, $\pi_3(S^2) \cong \mathbb{Z}\{\eta\}$ surjects onto $\pi_4(S^3) \cong \pi_1^S$, so the first stable stem $\pi_1^S \cong \mathbb{Z}/2$ is generated by (the suspensions of) the Hopf map η. The Freudenthal theorem does not suffice to prove that $\pi_{2p}(S^3) \to \pi_{2p-3}^S$ becomes an isomorphism after p-localization, but this is true:

Theorem 5.3. *$(\pi_n^S)_{(p)} = 0$ for $0 < n < 2p-3$ and $(\pi_{2p-3}^S)_{(p)} \cong \mathbb{Z}/p$.*

For each odd prime p, the generator of $(\pi_{2p-3}^S)_{(p)}$ given by the suspensions of the generator of $\pi_{2p}(S^3)_{(p)}$ is usually denoted α_1. It is the first class in the first of the so-called *Greek letter families* in the stable homotopy groups of spheres. [[Reference to Ravenel.]]

6. Example: The Adams spectral sequence

6.1. **Eilenberg–Mac Lane spectra and the Steenrod algebra.** Let X and Y be spectra, i.e., objects in one of the categories modeling stable homotopy theory. The Adams spectral sequence is a tool for analyzing the homotopy classes $[X,Y]_n$ of spectrum maps $\Sigma^n X \to Y$, for all integers n, starting with the mod p cohomology groups $H^*(X;\mathbb{F}_p)$ and $H^*(Y;\mathbb{F}_p)$ as modules over the mod p Steenrod algebra $\mathscr{A}$. For instance, if $X = Y = S$ are both equal to the sphere spectrum, with k-th space S^k, then the group

$$\pi_n(S) = [S,S]_n$$

equals the n-th stable stem π_n^S.

Let $H = H\mathbb{F}_p$ denote the mod p Eilenberg–Mac Lane spectrum, representing mod p cohomology. There is a natural isomorphism

$$[X,H]_{-*} \cong H^*(X;\mathbb{F}_p)$$

for all spectra X. By the Yoneda lemma, the natural graded transformations

$$H^*(X;\mathbb{F}_p) \longrightarrow H^*(X;\mathbb{F}_p) \,,$$

i.e., the mod p cohomology operations for spectra, are in one-to-one correspondence with the elements of

$$\mathscr{A} = [H,H]_{-*} \cong H^*(H;\mathbb{F}_p) \,.$$

This graded endomorphism algebra of the spectrum H is the mod p Steenrod algebra. It is concentrated in non-negative cohomological degrees, i.e., is only nonzero for $* \ge 0$ in the notation above.

6.2. **The d-invariant.** Each spectrum map $f\colon X \to Y$ induces a homomorphism $f^*\colon H^*(Y;\mathbb{F}_p) \to H^*(X;\mathbb{F}_p)$, which by the discussion above is a homomorphism of $\mathscr{A}$-modules. Hence the rule that takes the homotopy class $[f]$ to the $\mathscr{A}$-module homomorphism f^* is a homomorphism

$$\begin{aligned} d\colon [X,Y]_* &\longrightarrow \operatorname{Hom}^*_{\mathscr{A}}(H^*(Y;\mathbb{F}_p), H^*(X;\mathbb{F}_p)) \\ [f] &\longmapsto f^*\,. \end{aligned}$$

This is sometimes called the *d-invariant*, by analogy with the name "degree" for the integer $\deg(f)$ such that $f_*[M] = \deg(f)[N]$, where $f\colon M \to N$ is a map of oriented closed n-manifolds with fundamental classes $[M] \in H_n(M)$ and $[N] \in H_n(N)$.

The (cohomological) grading of $\operatorname{Hom}_{\mathscr{A}}$-groups works as follows: An element $[f] \in [X,Y]_t$ is the homotopy class of a spectrum map $f\colon \Sigma^t X \to Y$, which induces a homomorphism

$$f^*\colon H^*(Y;\mathbb{F}_p) \to H^*(\Sigma^t X;\mathbb{F}_p) \cong H^{*-t}(X;\mathbb{F}_p)\,.$$

By definition this is an $\mathscr{A}$-module homomorphism of degree t, i.e., an element of

$$\operatorname{Hom}^t_{\mathscr{A}}(H^*(Y;\mathbb{F}_p), H^*(X;\mathbb{F}_p))\,,$$

taking elements in $H^i(Y;\mathbb{F}_p)$ to elements of $H^{i-t}(X;\mathbb{F}_p)$, for all integers i.

6.3. **Wedge sums of suspended Eilenberg–Mac Lane spectra.** In the special case $Y = H$ we have $H^*(Y;\mathbb{F}_p) = \mathscr{A}$, and the d-invariant

$$d\colon [X,H]_t \longrightarrow \operatorname{Hom}^t_{\mathscr{A}}(\mathscr{A}, H^*(X;\mathbb{F}_p))$$

is an isomorphism for each t, since both sides are naturally isomorphic to $H^{-t}(X;\mathbb{F}_p)$. More generally, suppose that

$$Y \simeq \bigvee_u \Sigma^{n_u} H$$

is a wedge sum of suspensions of mod p Eilenberg–Mac Lane spectra. If $\pi_*(Y) = \bigoplus_u \Sigma^{n_u}\mathbb{F}_p$ is bounded below and of finite type, or equivalently, if $n_u \to \infty$ as $u \to \infty$, then the canonical map

$$\bigvee_u \Sigma^{n_u} H \longrightarrow \prod_u \Sigma^{n_u} H$$

is an equivalence. In this case the d-invariant is also an isomorphism, since

$$[X,Y]_t \cong [X, \prod_u \Sigma^{n_u} H]_t \cong \prod_u H^{n_u - t}(X;\mathbb{F}_p)$$

is naturally isomorphic to

$$\operatorname{Hom}^t_{\mathscr{A}}(H^*(Y;\mathbb{F}_p), H^*(X;\mathbb{F}_p)) \cong \operatorname{Hom}^t_{\mathscr{A}}(\bigoplus_u \Sigma^{n_u}\mathscr{A}, H^*(X;\mathbb{F}_p)) \cong \prod_u \operatorname{Hom}^t_{\mathscr{A}}(\Sigma^{n_u}\mathscr{A}, H^*(X;\mathbb{F}_p))$$

for each integer t.

In general d is not an isomorphism. For instance, the Hopf map $\eta\colon S^3 \to S^2$ induces the zero homomorphism in (reduced) cohomology, but stabilizes to a nontrivial homotopy class of maps $S^1 = \Sigma S \to S$. Furthermore, the target of d is always a graded $\mathbb{F}_p$-vector space, while the source may be any graded abelian group.

6.4. **Two-stage extensions.** If Y is an extension of two wedge sums of suspended Eilenberg–Mac Lane spectra, so that there is a cofiber sequence of spectra

$$K_1 \xrightarrow{i} Y \xrightarrow{j} K_0$$

with

$$K_0 \simeq \bigvee_u \Sigma^{n_u} H \qquad \text{and} \qquad K_1 \simeq \bigvee_v \Sigma^{n_v} H\,,$$

then there are long exact sequences

$$\dots \longrightarrow [X,K_1]_* \xrightarrow{i_*} [X,Y]_* \xrightarrow{j_*} [X,K_0]_* \xrightarrow{\partial} [X,K_1]_{*-1} \to \dots$$

and

$$\dots \to H^*(K_0;\mathbb{F}_p) \xrightarrow{j^*} H^*(Y;\mathbb{F}_p) \xrightarrow{i^*} H^*(K_1;\mathbb{F}_p) \xrightarrow{\delta} H^{*+1}(K_0;\mathbb{F}_p) \to \dots\,,$$

but the complex

$$\dots \longrightarrow \mathrm{Hom}^*_{\mathscr{A}}(H^*(K_1;\mathbb{F}_p), H^*(X;\mathbb{F}_p)) \xrightarrow{(i^*)^\#} \mathrm{Hom}^*_{\mathscr{A}}(H^*(Y;\mathbb{F}_p), H^*(X;\mathbb{F}_p))$$
$$\xrightarrow{(j^*)^\#} \mathrm{Hom}^*_{\mathscr{A}}(H^*(K_0;\mathbb{F}_p), H^*(X;\mathbb{F}_p)) \xrightarrow{\delta^\#} \mathrm{Hom}^*_{\mathscr{A}}(H^{*-1}(K_1;\mathbb{F}_p), H^*(X;\mathbb{F}_p)) \longrightarrow \dots$$

is typically not exact. Here $\delta^\#$ denotes the value of the contravariant functor $\mathrm{Hom}^*_{\mathscr{A}}(-, H^*(X;\mathbb{F}_p))$ applied to the homomorphism δ, and likewise for $(i^*)^\#$ and $(j^*)^\#$.

Now suppose that j^* is surjective, which is equivalent to asking that i^* is zero. Then there is instead a short exact sequence of $\mathscr{A}$-modules

$$0 \to H^{*-1}(K_1;\mathbb{F}_p) \xrightarrow{\delta} H^*(K_0;\mathbb{F}_p) \xrightarrow{j^*} H^*(Y;\mathbb{F}_p) \to 0\,.$$

If $\pi_*(K_0)$ and $\pi_*(K_1)$ are bounded below and of finite type, as before, then the left hand and middle $\mathscr{A}$-modules are free, so there is an associated exact sequence

$$0 \to \mathrm{Hom}^t_{\mathscr{A}}(H^*(Y;\mathbb{F}_p), H^*(X;\mathbb{F}_p)) \xrightarrow{(j^*)^\#} \mathrm{Hom}^t_{\mathscr{A}}(H^*(K_0;\mathbb{F}_p), H^*(X;\mathbb{F}_p))$$
$$\xrightarrow{\delta^\#} \mathrm{Hom}^t_{\mathscr{A}}(H^{*-1}(K_1;\mathbb{F}_p), H^*(X;\mathbb{F}_p)) \longrightarrow \mathrm{Ext}^{1,t}_{\mathscr{A}}(H^*(Y;\mathbb{F}_p), H^*(X;\mathbb{F}_p)) \to 0\,.$$

Here $\mathrm{Ext}^1_{\mathscr{A}}$ denotes the first right derived functor of $\mathrm{Hom}_{\mathscr{A}}$. More generally we write $\mathrm{Ext}^{s,t}_{\mathscr{A}}$ for the internal degree t part of the s-th derived functor of $\mathrm{Hom}^*_{\mathscr{A}}$, for each $s \ge 0$. Recall that $\mathrm{Ext}^0_{\mathscr{A}} = \mathrm{Hom}_{\mathscr{A}}$. The groups

$$\mathrm{Ext}^{s,t}_{\mathscr{A}}(H^*(Y;\mathbb{F}_p), H^*(X;\mathbb{F}_p))$$

for $s \ge 2$ are zero for the Y that we are presently considering, due to the existence of the short exact sequence of $\mathscr{A}$-modules above.

Under the d-invariant isomorphisms associated to K_0 and K_1, the homomorphism $\partial\colon [X, K_0]_* \to [X, K_1]_{*-1}$ corresponds to the homomorphism $\delta^\#$ above:

$$\begin{tikzcd}
{[X, K_0]_t} \arrow[r, "\partial"] \arrow[d, "d"', "\cong"] & {[X, K_1]_{t-1}} \arrow[d, "d"', "\cong"] \\
\mathrm{Hom}^t_{\mathscr{A}}(H^*(K_0;\mathbb{F}_p), H^*(X;\mathbb{F}_p)) \arrow[r, "\delta^\#"] & \mathrm{Hom}^{t-1}_{\mathscr{A}}(H^*(K_1;\mathbb{F}_p), H^*(X;\mathbb{F}_p))\,,
\end{tikzcd}$$

so there are isomorphisms

$$\ker(\partial)_t \cong \mathrm{Hom}^t_{\mathscr{A}}(H^*(Y;\mathbb{F}_p), H^*(X;\mathbb{F}_p))$$
$$\mathrm{cok}(\partial)_{t-1} \cong \mathrm{Ext}^{1,t}_{\mathscr{A}}(H^*(Y;\mathbb{F}_p), H^*(X;\mathbb{F}_p))$$

for all integers t. Hence the short exact sequence

$$0 \to \mathrm{cok}(\partial)_t \longrightarrow [X, Y]_t \longrightarrow \ker(\partial)_t \to 0$$

can be rewritten as

$$0 \to \mathrm{Ext}^{1,t+1}_{\mathscr{A}}(H^*(Y;\mathbb{F}_p), H^*(X;\mathbb{F}_p)) \longrightarrow [X, Y]_t \xrightarrow{d} \mathrm{Hom}^t_{\mathscr{A}}(H^*(Y;\mathbb{F}_p), H^*(X;\mathbb{F}_p)) \to 0\,,$$

for each integer t. In particular d is surjective in these cases. The homomorphism

$$e\colon \ker(d)_t \longrightarrow \mathrm{Ext}^{1,t+1}_{\mathscr{A}}(H^*(Y;\mathbb{F}_p), H^*(X;\mathbb{F}_p))\,,$$

which in this case is an isomorphism, is often called the *e-invariant*. Here e refers to "extension", and goes well with d.

This extension can be presented as a spectral sequence with E_2-term

$$E_2^{s,t} = \begin{cases} \mathrm{Hom}^t_{\mathscr{A}}(H^*(Y;\mathbb{F}_p), H^*(X;\mathbb{F}_p)) & \text{for } s = 0, \\ \mathrm{Ext}^{1,t}_{\mathscr{A}}(H^*(Y;\mathbb{F}_p), H^*(X;\mathbb{F}_p)) & \text{for } s = 1, \\ 0 & \text{otherwise,} \end{cases}$$

that collapses at the $E_2 = E_\infty$-term, and which converges to a finite decreasing filtration

$$[X, Y]_t = F^0 \supset F^1 \supset F^2 = 0$$

in the sense that

$$(F^0/F^1)_t = E_\infty^{0,t} \qquad \text{and} \qquad (F^1)_t = E_\infty^{1,t+1}\,.$$

Thus for such Y the d-invariant is surjective, and $F^1 = \ker(d)$ is its kernel.

This discussion suggests that in order to get a better approximation to the graded abelian group $[X, Y]_*$, it is necessary to take the derived functors of $\mathrm{Hom}_{\mathscr{A}}$ into account.

6.5. **The mod p Adams spectral sequence.** The mod p Adams spectral sequence for X and Y has E_2-term

$$E_2^{s,t} = \mathrm{Ext}_{\mathscr{A}}^{s,t}(H^*(Y; \mathbb{F}_p), H^*(X; \mathbb{F}_p)) .$$

If X is a finite CW spectrum, and Y is bounded below and of finite type, then it converges strongly to the p-completion

$$([X, Y]_{t-s})_p^\wedge$$

of the abelian group $[X, Y]_{t-s}$, equipped with a decreasing filtration

$$([X, Y]_{t-s})_p^\wedge = F^0 \supset F^1 \supset \cdots \supset F^s \supset \dots$$

called the Adams filtration. For a finitely generated abelian group G the p-completion can be defined as the limit

$$G_p^\wedge = \lim_n G/p^n G .$$

If $G = \mathbb{Z}$, this equals the p-adic integers $\mathbb{Z}_p$. If G is finite, this is a quotient group of G that is isomorphic to the p-Sylow subgroup of G.

In the special case $X = S$ we get the E_2-term of the mod p Adams spectral sequence for Y, namely

$$E_2^{s,t} = \mathrm{Ext}_{\mathscr{A}}^{s,t}(H^*(Y; \mathbb{F}_p), \mathbb{F}_p) .$$

When Y is bounded below and of finite type it converges strongly to the p-completed homotopy groups

$$\pi_{t-s}(Y)_p^\wedge = ([S, Y]_{t-s})_p^\wedge$$

of Y. In particular, the mod p Adams spectral sequence for the sphere spectrum itself has E_2-term

$$E_2^{s,t} = \mathrm{Ext}_{\mathscr{A}}^{s,t}(\mathbb{F}_p, \mathbb{F}_p) ,$$

and converges strongly to

$$(\pi_{t-s}^S)_p^\wedge = \pi_{t-s}(S)_p^\wedge = ([S, S]_{t-s})_p^\wedge ,$$

i.e., the p-completed stable stems.

By the Hurewicz theorem, $\pi_n^S = 0$ for $n < 0$ and $\pi_0^S \cong \mathbb{Z}$, via the isomorphisms $\pi_j(S^j) \cong H_j(S^j)$ for all $j \geq 1$. Using the theory of Serre classes, one can prove that each stable stem π_n^S for $n \geq 1$ is a finite abelian group. Hence it is the product of its p-Sylow subgroups, or equivalently, of the groups $(\pi_n^S)_p^\wedge$, which we can hope to calculate using the corresponding mod p Adams spectral sequence. This will be the aim of much of the remainder of these lectures.

6.6. **Endomorphism ring spectra and their modules.** Working at the spectrum level, without passing to homotopy classes of maps, we can instead consider the function spectra $F(X, Y)$, $X^H = F(X, H)$, $Y^H = F(Y, H)$ and $R = F(H, H)$, with $\pi_* F(X, Y) = [X, Y]_*$, $\pi_{-*} F(X, H) = H^*(X; \mathbb{F}_p)$, $\pi_{-*} F(Y, H) = H^*(Y; \mathbb{F}_p)$ and $\pi_{-*} F(H, H) = \mathscr{A}$. The endomorphism spectrum $R = F(H, H)$ is a ring spectrum, with product corresponding to the composition of cohomology operations. The map $F(X, Y) \to F(Y^H, X^H)$ factors through the spectrum of R-module maps, so that there is a spectrum level degree map

$$d \colon F(X, Y) \longrightarrow F_R(Y^H, X^H) .$$

This turns out to be an equivalence in a wider range of cases than that for which the group level degree map is an isomorphism, and to amount to a p-completion map of the source in an even wider range of cases. Passing to homotopy groups, there is a spectral sequence

$$E_2^{s,t} = \mathrm{Ext}_{\mathscr{A}}^{s,t}(H^*(Y; \mathbb{F}_p), H^*(X; \mathbb{F}_p))$$

converging [[conditionally? strongly?]] to π_{t-s} of the target of the spectrum level degree map. [[This is the Adams spectral sequence converging to $\pi_{t-s} F(X, Y)_p^\wedge$.]]

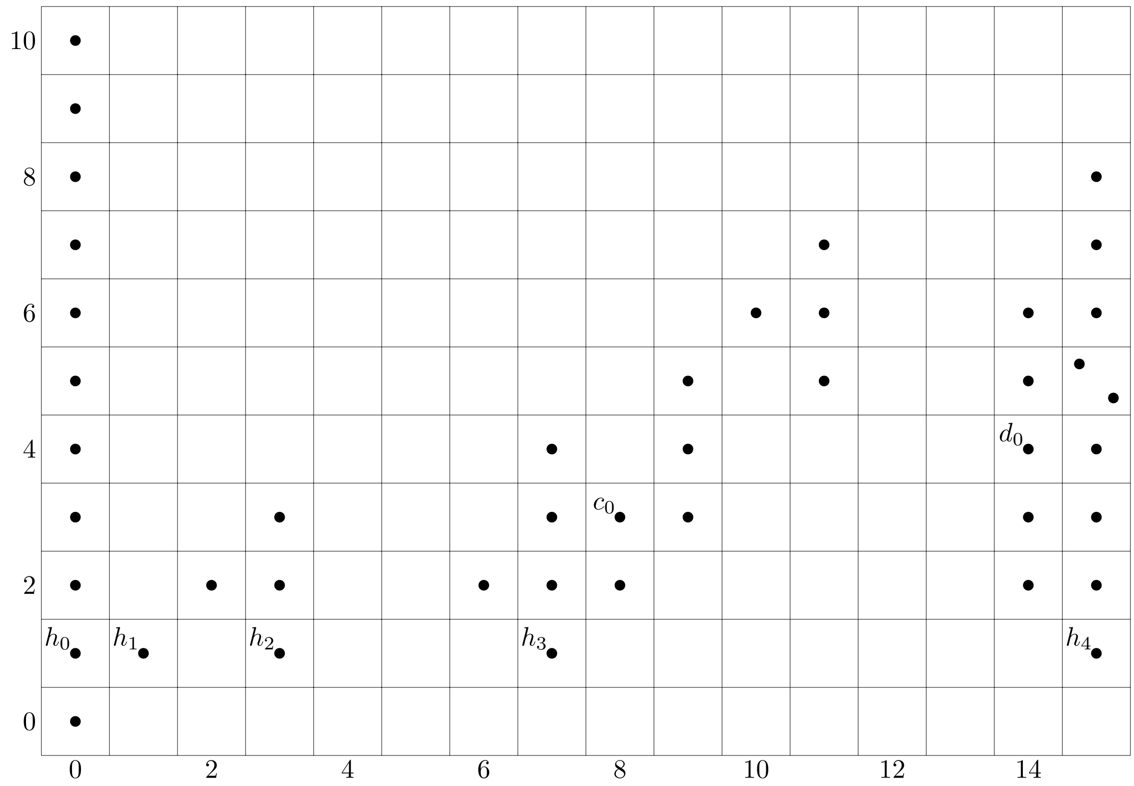


FIGURE 5. Adams E_2-term for $t-s \le 15$

6.7. **The mod 2 Adams spectral sequence for the sphere.** Let us look more closely at the mod 2 Adams spectral sequence for the sphere:

$$E_2^{s,t} = \mathrm{Ext}_{\mathscr{A}}^{s,t}(\mathbb{F}_2, \mathbb{F}_2) \Longrightarrow_s \pi_{t-s}(S)_2^\wedge \,.$$

The E_2-term is an $\mathbb{F}_2$-vector space in each bidegree, concentrated in the region $0 \le s \le t$, or equivalently, in the region $t-s \ge 0$ and $s \ge 0$. We display the part where $0 \le t-s \le 15$ and $0 \le s \le 10$ of this E_2-term in Figure 5, using the Adams indexing with the topological degree $t-s$ on the horizontal axis and the filtration degree s on the vertical axis. This picture is usually called an *Adams chart*, and we refer to $(t-s, s)$ as the *Adams bidegree*.

The dots in a square corresponding to a given $(t-s, s)$-bidegree represent the elements of a basis for the $\mathbb{F}_2$-vector space in that bidegree. Empty bidegrees correspond to 0-dimensional vector spaces, bidegrees with a single dot correspond to 1-dimensional vector spaces, and so on. In this range of bidegrees the only 2-dimensional vector space is $E_2^{5,20}$, located at $(t-s, s) = (15, 5)$. Some of the generators are labeled with their standard names. We will explain later how these $\mathrm{Ext}_{\mathscr{A}}$-groups can be calculated, at least in a limited range.

The chart continues upward and to the right. In the upward direction, only the groups in the zeroth column are nonzero, while the groups in columns $1 \le t-s \le 15$ for $s \ge 9$ are all zero. There is much more structure present in this E_2-term, and in the subsequent terms of the spectral sequence, than that of a bigraded $\mathbb{F}_2$-vector space, but let us introduce these structures one by one.

The d_2-differentials in the Adams spectral sequence are homomorphisms

$$d_2^{s,t} \colon E_2^{s,t} \longrightarrow E_2^{s+2,t+1} \,,$$

mapping bidegree $(t-s, s)$ to bidegree $(t-s-1, s+2)$, i.e., one unit to the left and two units upwards in the $(t-s, s)$-plane. Looking at the chart, the d_2-differentials that could possibly be nonzero are those originating in bidegrees $(t-s, s) = (1,1)$, $(8,2)$, $(15,1)$, $(15,2)$, $(15,3)$ and $(15,4)$. See Figure 6.

More generally, the d_r-differentials in the Adams spectral sequence are homomorphisms

$$d_r^{s,t} \colon E_r^{s,t} \longrightarrow E_r^{s+r,t+r-1} \,,$$

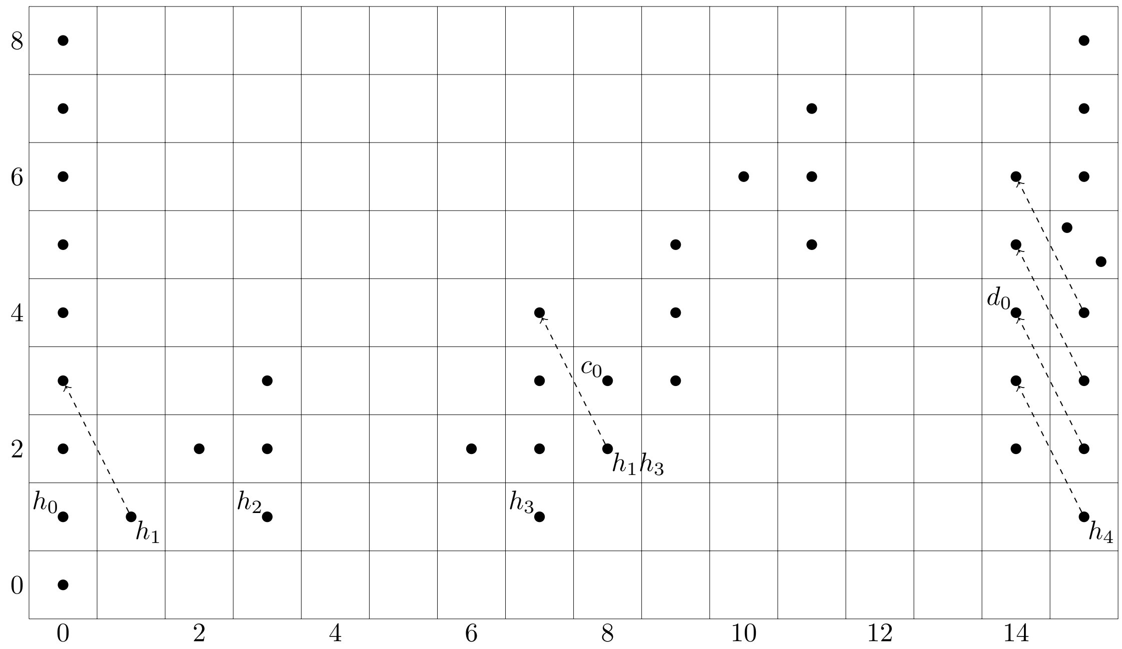


FIGURE 6. Possible d_2-differentials

mapping bidegree $(t-s, s)$ to bidegree $(t-s-1, s+r)$, i.e., one unit to the left and r units upwards in the $(t-s, s)$-plane. For $r \geq 3$, the only possible d_r-differentials are those originating in bidegrees $(t-s, s) = (1, 1)$, $(15, 1)$, $(15, 2)$ and $(15, 3)$.

The Adams E_∞-term is a subquotient of the E_2-term, hence is zero in all the bidegrees where $E_2^{s,t} = 0$. By strong convergence, there is a descending complete Hausdorff filtration

$$\pi_n(S)_2^\wedge = (F^0)_n \supset (F^1)_n \supset \cdots \supset (F^s)_n \supset \dots ,$$

the *Adams filtration*, and isomorphisms

$$(F^s/F^{s+1})_{t-s} \cong E_\infty^{s,t}$$

for all s and t. For each integer n, the groups in the E_∞-term that contribute as filtration quotients to the filtration of $\pi_n(S)_2^\wedge$ are the groups $E_\infty^{s,t}$ with $t-s=n$, i.e., the groups in the n-th column. Thus the Adams indexing has the feature that all of the terms that contribute to the same topological degree are aligned in the same column in the $(t-s, s)$-plane.

In fact the d_r-differentials originating on the class $h_1 \in E_2^{1,2}$ in Adams bidegree $(1, 1)$ are all zero. In other words, h_1 is an infinite cycle, and *survives* to the E_∞-term. To see this, we might start from our knowledge that $\pi_1(S) = \pi_1^S \cong \mathbb{Z}/2$. If some d_r-differential on h_1 were nonzero, then h_1 would not survive to $E_{r+1}^{1,2}$, so $E_{r+1}^{1,2} = 0$ and $E_\infty^{1,2} = 0$. Hence every group $E_\infty^{s,s+1}$ for $s \geq 0$ would be zero, and the filtration of $\pi_1(S)_2^\wedge$ would have to be constant:

$$\pi_1(S)_2^\wedge = (F^0)_1 = (F^1)_1 = \cdots = (F^s)_1 = \dots .$$

Since the filtration is Hausdorff, $\lim_s (F^s)_1 = 0$, so this implies that each $(F^s)_1 = 0$. In particular $\pi_1(S)_2^\wedge$ would be zero, contradicting our earlier calculation.

Theorem 6.1. *$d_r(h_1) = 0$ for all $r \geq 2$.*

Hence the Adams E_2-term equals the Adams E_∞-term in topological degrees $t-s \leq 6$. In degree 0, the Adams filtration

$$\pi_0(S)_2^\wedge = (F^0)_0 \supset (F^1)_0 \supset \cdots \supset (F^s)_0 \supset \dots$$

has $(F^s/F^{s+1})_0 \cong E_\infty^{s,s} \cong \mathbb{Z}/2$, so $(F^{s+1})_0$ has index 2 in $(F^s)_0$, for each $s \geq 0$. Hence this complete Hausdorff filtration is equal to the 2-adic filtration

$$\mathbb{Z}_2^\wedge = \mathbb{Z}_2 \supset 2\mathbb{Z}_2 \supset \cdots \supset 2^s\mathbb{Z}_2 \supset \dots$$

of the 2-adic integers. Note that $\mathbb{Z}_2/2^s\mathbb{Z}_2 \cong \mathbb{Z}/2^s$, and $\mathbb{Z}_2 \cong \lim_s \mathbb{Z}/2^s$.

In degree 1, the Adams filtration has

$$\pi_1(S)^\wedge_2 = (F^0)_1 = (F^1)_1 \qquad \text{and} \qquad (F^2)_1 = \dots = (F^s)_1 = 0$$

for $s \ge 2$, so

$$\pi_1(S)^\wedge_2 \cong (F^1/F^2)_1 \cong E_\infty^{1,2} \cong \mathbb{Z}/2\{h_1\}\,.$$

The generator of $\pi_1(S) \cong \mathbb{Z}/2$, represented by the Hopf map η, has d-invariant $d(\eta) = 0$, hence lifts to Adams filtration 1 and is represented in $(F^1/F^2)_1 \cong E_\infty^{1,2}$ by the infinite cycle h_1 in $E_2^{1,2}$.

We can now go beyond what we already knew. In degree 2, the only nonzero class in the E_∞-term, i.e., in the groups $E_\infty^{s,s+2}$ for $s \ge 0$, is the generator of $E_2^{2,4} = E_\infty^{2,4}$ in Adams bidegree $(2,2)$. Foreshadowing the existence of a multiplicative structure on the Adams E_2-term, this generator is usually called h_1^2. The Adams filtration has

$$\pi_2(S)^\wedge_2 = (F^0)_2 = (F^1)_2 = (F^2)_2 \qquad \text{and} \qquad (F^3)_2 = \dots = (F^s)_2 = 0$$

for $s \ge 3$, so

$$\pi_2(S)^\wedge_2 \cong (F^2/F^3)_1 \cong E_\infty^{2,4} \cong \mathbb{Z}/2\{h_1^2\}\,.$$

The generator of $\pi_2(S) \cong \mathbb{Z}/2$, represented by the square η^2 of the Hopf map, lifts to Adams filtration 2 and is represented in $(F^2/F^3)_2 \cong E_\infty^{2,4}$ by the infinite cycle h_1^2.

In degree 3, there are three generators of the $E_2 = E_\infty$-term, namely h_2 generating $E_2^{1,4}$, a class we call h_0h_2 generating $E_2^{2,5}$, and a class we call $h_0^2h_2$ generating $E_2^{3,6}$. The Adams filtration has

$$\begin{aligned} \pi_3(S)^\wedge_2 &= (F^0)_3 = (F^1)_3 \\ (F^1/F^2)_3 &\cong \mathbb{Z}/2\{h_2\} \\ (F^2/F^3)_3 &\cong \mathbb{Z}/2\{h_0h_2\} \\ (F^3/F^4)_3 &\cong \mathbb{Z}/2\{h_0^2h_2\} \\ (F^4)_3 &= \dots = (F^s)_3 = 0 \end{aligned}$$

for $s \ge 4$. This proves that $(F^3)_3 \cong \mathbb{Z}/2\{h_0^2h_2\}$, but without further information we have two ambiguous extension problems

$$0 \to (F^3)_3 \longrightarrow (F^2)_3 \longrightarrow \mathbb{Z}/2\{h_0h_2\} \to 0 \tag{1}$$

and

$$0 \to (F^2)_3 \longrightarrow (F^1)_3 \longrightarrow \mathbb{Z}/2\{h_2\} \to 0\,. \tag{2}$$

It is clear that $(F^2)_3$ is an abelian group of order four, and that $\pi_3(S)^\wedge_2 = (F^1)_3$ is an abelian group of order eight. In fact both of these extensions are nontrivial, and $\pi_3(S)^\wedge_2$ is cyclic of order eight, but we will need to refer to the multiplicative structure in the spectral sequence to deduce this. [[Relate h_2 to the quaternionic Hopf fibration ν?]]

In degrees 4 and 5 the $E_2 = E_\infty$-term only contains trivial groups, so $\pi_4(S)^\wedge_2 = 0$ and $\pi_5(S)^\wedge_2$ are both trivial.

In degree 6, the only nonzero class in the E_∞-term is the generator of $E_2^{2,8}$ in Adams bidegree $(2,6)$, which is usually called h_2^2. The Adams filtration has

$$\pi_6(S)^\wedge_2 = (F^0)_6 = (F^1)_6 = (F^2)_6 \qquad \text{and} \qquad (F^3)_6 = \dots = (F^s)_6 = 0$$

for $s \ge 3$, so

$$\pi_6(S)^\wedge_2 \cong (F^2/F^3)_6 \cong E_\infty^{2,8} \cong \mathbb{Z}/2\{h_2^2\}\,.$$

The generator of $\pi_6(S) \cong \mathbb{Z}/2$, represented by the square lifts to Adams filtration 2 and is represented in $(F^2/F^3)_6 \cong E_\infty^{2,8}$ by the infinite cycle h_2^2.

In degree 7, we can see that $\pi_7(S)^\wedge_2$ has order $2^3 = 8$ or $2^4 = 16$, but in order to decide between these two cases, we need to determine the possibly nonzero differential $d_2^{2,10} \colon E_2^{2,10} \to E_2^{4,11}$.

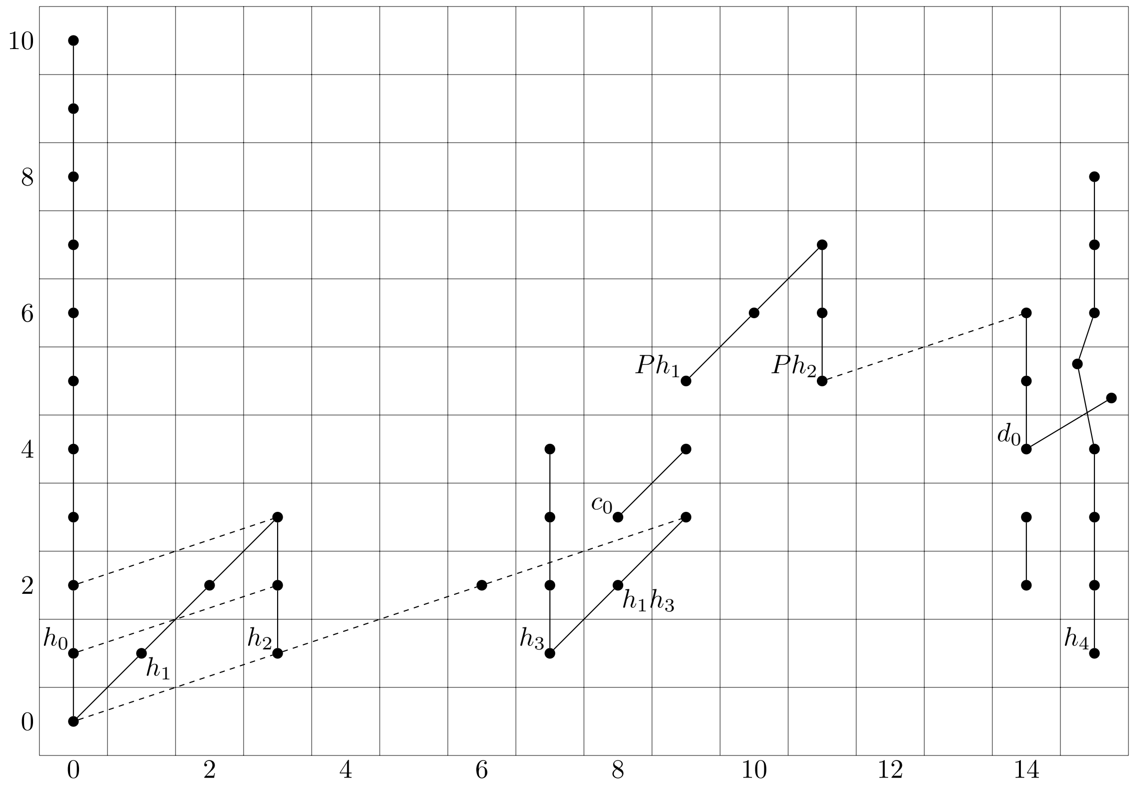


FIGURE 7. Adams E_2-term, with h_i-multiplications

6.8. **Multiplicative structure.** The sphere spectrum S is a homotopy commutative ring spectrum, or in fact a "homotopy everything ring spectrum", more technically known as an E_∞ ring spectrum, or as a commutative structured ring spectrum. This implies that $\pi_*(S)$ is a graded commutative ring, with the pairing

$$\wedge\colon \pi_m(S) \otimes \pi_n(S) \longrightarrow \pi_{m+n}(S)$$

mapping $[f] \otimes [g]$ to $[f \wedge g]$, where $f\colon S^m \to S$ and $g\colon S^n \to S$ are spectrum maps, with smash product $f \wedge g\colon S^{m+n} \cong S^m \wedge S^n \to S \wedge S = S$.

This graded commutative ring structure is reflected in the Adams spectral sequence. There is a Yoneda pairing

$$\circ\colon \operatorname{Ext}_{\mathscr{A}}^{s_1,t_1}(\mathbb{F}_p, \mathbb{F}_p) \otimes \operatorname{Ext}_{\mathscr{A}}^{s_2,t_2}(\mathbb{F}_p, \mathbb{F}_p) \longrightarrow \operatorname{Ext}_{\mathscr{A}}^{s_1+s_2,t_1+t_2}(\mathbb{F}_p, \mathbb{F}_p)$$

making the Adams E_2-term a graded commutative $\mathbb{F}_p$-algebra, and in fact the Adams spectral sequence is an algebra spectral sequence, in the sense that each E_r-term is a graded (commutative) algebra, each d_r-differential is a derivation, and these multiplicative structures are compatible under the isomorphism $E_{r+1} \cong H(E_r, d_r)$. Furthermore, the convergence of the Adams spectral sequence is compatible with the multiplicative structure, in the sense that the Adams filtration $\{F^s\}_s$ of $\pi_*(S)^\wedge_p$ is such that the smash pairing takes $F^{s_1} \otimes F^{s_2}$ into $F^{s_1+s_2}$, and the induced pairing

$$\wedge\colon F^{s_1}/F^{s_1+1} \otimes F^{s_2}/F^{s_2+1} \longrightarrow F^{s_1+s_2}/F^{s_1+s_2+1}$$

agrees with the pairing

$$\circ\colon E_\infty^{s_1,*} \otimes E_\infty^{s_2,*} \longrightarrow E_\infty^{s_1+s_2,*}$$

under the isomorphisms $F^s/F^{s+1} \cong E_\infty^s$.

The class h_0 generating $E_2^{1,1} = E_\infty^{1,1}$, in Adams bidegree $(t-s, s) = (0, 1)$, represents 2 times the generator ι in $\pi_0(S)^\wedge_2 \cong \mathbb{Z}_2$. Thus 2ι has Adams filtration 1. If an infinite cycle $x \in E_\infty^{s,t}$ represents a class $[f] \in \pi_{t-s}(S)^\wedge_2$, in Adams filtration s, then the product $h_0 \circ x = h_0 x \in E_\infty^{s+1,t+1}$ represents the product $2\iota \wedge [f] = 2[f] \in \pi_{t-s}(S)^\wedge_2$, in Adams filtration $s+1$, modulo classes in Adams filtration $s+2$ (or greater).

We can use this to determine much of the additive structure of the groups $\pi_n(S)^\wedge_2$ in this range. In Figure 7, a vertical line of length 1, from a class x in Adams bidegree $(t-s, s)$ to a class y in Adams bidegree $(t-s, s+1)$, indicates that $h_0 \circ x = y$, i.e., the line connects x to $h_0 x$. We say that these vertical lines show the h_0-multiplications. If $h_0 x = 0$, no line is drawn.

In the same way, the class h_1 generating $E_2^{1,2} = E_\infty^{1,2}$, in Adams bidegree $(t-s, s) = (1,1)$, represents the generator η of $\pi_1(S)_2^\wedge \cong \mathbb{Z}/2$. If $x \in E_\infty^{s,t}$ represents a class $[f] \in \pi_{t-s}(S)_2^\wedge$ in filtration s, then $h_1 x$ represents $\eta[f] \in \pi_{t-s+1}(S)_2^\wedge$ in filtration $s+1$, modulo Adams filtration $s+2$. This is indicated in Figure 7 by a line of slope 1, from x in Adams bidegree $(t-s, s)$ to $h_1 x$ in Adams bidegree $(t-s+1, s+1)$. If $h_1 x = 0$, no line is drawn.

The dashed lines of slope 1/3 correspond to multiplications by h_2, the class generating $E_2^{1,4} = E_\infty^{1,4}$. [[Relate to ν?]] We could add lines of slope 1/7 corresponding to multiplications by h_3, the class generating $E_2^{1,8} = E_\infty^{1,8}$, but these tend to clutter the diagram too much. [[Relate to σ?]]

Using the multiplicative structure, we can now deduce that $d_r^{s,t} = 0$ for all $r \ge 2$ and $t - s \le 13$, so that $E_2^{s,t} = E_\infty^{s,t}$ for all $t - s \le 13$.

It is clear that $d_r(h_0) = 0$ for all $r \ge 2$, since these differentials land in trivial groups. The product $h_0 h_1 = 0$ vanishes for the same reason. Hence if h_1 survives to the E_r-term, we have

$$0 = d_r(0) = d_r(h_0 h_1) = d_r(h_0) h_1 + h_0 d_r(h_1) = h_0 d_r(h_1)$$

by the Leibniz rule. But $d_r(h_1)$ lies in the bidegree $(0, r+1)$ generated by h_0^{r+1}, and multiplication by h_0 acts injectively on this bidegree. Hence $d_r(h_1) = 0$, also for all $r \ge 2$.

We can also use the multiplicative structure to deduce that $d_2^{2,10} = 0$. The group $E_2^{2,10}$ is generated by the product $h_1 h_3$. We know that $d_2(h_3) = 0$, for bidegree reasons, so by the Leibniz rule $d_2(h_1 h_3) = d_2(h_1) h_3 + h_1 d_2(h_3) = 0$, as claimed.

6.9. **The first 13 stems.** The h_i-multiplications seen at the E_2-term, allow us to determine the group structures of $\pi_n(S)_2^\wedge$ for $0 \le n \le 13$.

Theorem 6.2. (1) *$\pi_1(S)_2^\wedge \cong \mathbb{Z}/2$ generated by η represented by h_1.*
(2) *$\pi_2(S)_2^\wedge \cong \mathbb{Z}/2$ generated by η^2 represented by h_1^2.*
(3) *$\pi_3(S)_2^\wedge \cong \mathbb{Z}/8$ generated by ν represented by h_2. Here 2ν is represented by $h_0 h_2$, and $4\nu = \eta^3$ is represented by $h_0^2 h_2 = h_1^3$.*
(4) *$\pi_4(S)_2^\wedge = 0$.*
(5) *$\pi_5(S)_2^\wedge = 0$.*
(6) *$\pi_6(S)_2^\wedge \cong \mathbb{Z}/2$ generated by ν^2 represented by h_2^2.*
(7) *$\pi_7(S)_2^\wedge \cong \mathbb{Z}/16$ generated by σ represented by h_3. Here 2σ is represented by $h_0 h_3$, 4σ is represented by $h_0^2 h_3$, and 8σ is represented by $h_0^3 h_3$.*
(8) *$\pi_8(S)_2^\wedge \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$ generated by $\eta\sigma$ and ϵ, represented by $h_1 h_3$ and c_0, respectively.*
(9) *$\pi_9(S)_2^\wedge \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2$ generated by $\eta^2\sigma$, $\eta\epsilon$ and μ, represented by $h_1^2 h_3$, $h_1 c_0$ and Ph_1, respectively. [[Explain Ph_1. No hidden additive extensions.]]*
(10) *$\pi_{10}(S)_2^\wedge \cong \mathbb{Z}/2$ generated by $\eta\mu$ represented by $h_1 Ph_1$.*
(11) *$\pi_{11}(S)_2^\wedge \cong \mathbb{Z}/8$ generated by ζ represented by Ph_2. Here 2ζ is represented by $h_0 Ph_2$, and $4\zeta = \eta^2\mu$ is represented by $h_0^2 Ph_2 = h_1^2 Ph_1$.*
(12) *$\pi_{12}(S)_2^\wedge = 0$.*
(13) *$\pi_{13}(S)_2^\wedge = 0$.*

Remark 6.3. To remember the nomenclature in $\pi_*(S)_2^\wedge$, one may note that h_1, h_2 and h_3 represent classes η, ν and σ, which are the Greek letters expressing the beginning sounds in 'ichi', 'ni' and 'san', the Japanese words for 'one', 'two' and 'three'. The identity map of S corresponds to the unit class ι.

Proof. Let $\nu \in \pi_3(S)_2^\wedge$ be a class represented by h_2 in $E_2^{1,4} = E_\infty^{1,4}$. [[We may prove later that any class in $\pi_3(S)$ of Hopf invariant 1 mod 2 has this property, for instance, the stable class $S^3 \to S$ of the quaternionic Hopf fibration $S^7 \to S^4$. The product $2\nu = 2\iota \wedge \nu$ is then represented by $h_0 h_3$, and 4ν is represented by $h_0^2 h_3$. Hence both extensions in (1) and (2) are nontrivial, with $(F^2)_3 \cong \mathbb{Z}/4$ generated by 2ν and $(F^1)_3 \cong \mathbb{Z}/8$ generated by ν.

Let $\sigma \in \pi_7(S)_2^\wedge$ be a class represented by h_3 in $E_2^{1,8} = E_\infty^{1,8}$. [[We may prove later that any class in $\pi_7(S)$ of Hopf invariant 1 mod 2 has this property, for instance, the stable class $S^7 \to S$ of the octonionic Hopf fibration $S^{15} \to S^8$. The product $2\sigma = 2\iota \wedge \sigma$ is then represented by $h_0 h_4$, 4σ is represented by $h_0^2 h_4$, and 8σ is represented by $h_0^3 h_4$. Hence $(F^4)_7 = \mathbb{Z}/2$ is generated by 8σ, $(F^3)_7 = \mathbb{Z}/4$ is generated by 4σ, $(F^2)_7 = \mathbb{Z}/8$ is generated by 2σ, and $\pi_7(S)_2^\wedge = (F^1)_7 = \mathbb{Z}/16$ is generated by σ.

In the 8-stem, we have an extension

$$0 \to \mathbb{Z}/2\{c_0\} \longrightarrow \pi_8(S)_2^\wedge \longrightarrow \mathbb{Z}/2\{h_1 h_3\} \to 0\,.$$

The element ϵ in $\pi_8(S)^\wedge_2$ that is represented by c_0 in Adams filtration 3 is uniquely defined by this property. The product $\eta\sigma = \eta \wedge \sigma$, represented by h_1h_3 in Adams filtration 2, modulo Adams filtration 3, is also well defined, since the ambiguity in the definition of σ is given by the even multiples of σ, and $\eta \wedge 2\sigma = 0$ since $2\eta = 0$. The latter relation also implies that the extension above is split, so $\pi_8(S)^\wedge_2 \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$ is a Klein four-group, not a cyclic group of order four.

In the 9-stem, we have a well-defined element $\mu \in \pi_9(S)^\wedge_2$ that is represented by the generator $Ph_1 \in E^{5,14}_\infty$. The notation refers to an operator P called the *Adams periodicity operator*, which is defined in part of the E_2-term, and which takes h_1 to Ph_1 and h_2 to Ph_2. The product classes $\eta\epsilon$ and $\eta^2\sigma$ are well defined, and are represented by h_1c_0 and $h_1^2h_3$, modulo the Adams filtration. Hence $(F^5)_7 = \mathbb{Z}/2$ is generated by μ, the extension

$$0 \to (F^5)_7 \longrightarrow (F^4)_7 \longrightarrow \mathbb{Z}/2\{\eta\epsilon\} \to 0$$

splits, and so does the extension

$$0 \to (F^4)_7 \longrightarrow \pi_9(S)^\wedge_2 \longrightarrow \mathbb{Z}/2\{\eta^2\sigma\} \to 0\,.$$

The additive extensions in the 11-stem are all nontrivial, just like in the 3-stem. The generator ζ is only defined up to an odd multiple, much like the case of ν. $\square$

We can also deduce most of the product structure on $\pi_*(S)^\wedge_2$ in this range.

Theorem 6.4. *Multiplication by η satisfies the relations $\eta\nu = 0$, $\eta^3\sigma = 0$, $\eta^2\epsilon = 0$ (!), $\eta^3\mu = 0$, $\eta\zeta = 0$.*

Multiplication by ν satisfies the relations $\nu\sigma = 0$ (!), $\nu^3 = \eta^2\sigma + \eta\epsilon$ (!), $\nu\epsilon = 0$ (!) and $\nu\mu = 0$.

Proof. [[Why is $\nu^3 = \eta^2\sigma + \eta\epsilon$? Use $e\colon S \to j$ to deduce that $\eta^2\epsilon = 0$ and $\nu\epsilon = 0$. How about $\nu\sigma$?]] $\square$

6.10. **The first Adams differential.** Recall that $\sigma \in \pi_7(S)^\wedge_2$ denotes a class represented by h_3 in $E_2^{1,8} = E_\infty^{1,8}$, e.g. the stable octonionic Hopf fibration. By graded commutativity of $\pi_*(S)^\wedge_2$ we know that $\sigma \wedge \sigma = -\sigma \wedge \sigma$, since σ is in an odd degree, so $2\sigma^2 = 0$ in $\pi_{14}(S)^\wedge_2$. Here σ^2 is represented by h_3^2 in $E_2^{2,16} = E_\infty^{2,16}$, so $2\sigma^2$ is represented by $h_0h_3^2$ in $E_\infty^{3,17}$, modulo Adams filtration 4. Since $2\sigma^2 = 0$, it follows that $h_0h_3^2$ must be equal to 0 at the E_∞-term. Since this product is not 0 at the E_2-term (and $d_r(h_0h_3^2) = 0$ for all $r \ge 2$ by the Leibniz rule), the only way to explain this is that $h_0h_3^2$ is a boundary, i.e., is hit by a differential. For bidegree reasons, the only possibility candidate is the d_2-differential originating at h_4 in $E_2^{1,16}$. Hence the "first" nonzero differential in the mod 2 Adams spectral sequence is

$$d_2(h_4) = h_0h_3^2\,.$$

There are in fact also nonzero d_3-differentials on h_0h_4 and $h_0^2h_4$, from Adams bidegrees $(15, 2)$ and $(15, 3)$, but these are harder to establish.

7. Exact couples

Following Massey (1952, 1953) and Boardman (1981 preprint, 1999), we introduce the notion of an *exact couple*, and show how to use it to construct a spectral sequence. [[First additive, then convergence, then perhaps products.]]

7.1. The spectral sequence associated to an unrolled exact couple.

Definition 7.1. An *unrolled exact couple* of homological type is a diagram

$$\begin{tikzcd}
\dots \arrow[r] & A_{s-2} \arrow[r, "i"] & A_{s-1} \arrow[r, "i"] \arrow[d, "j"] & A_s \arrow[r, "i"] \arrow[d, "j"] & A_{s+1} \arrow[r, "i"] \arrow[d, "j"] & \dots \\
\dots & & E_{s-1} \arrow[ul, "k"] & E_s \arrow[ul, "k"] & E_{s+1} \arrow[ul, "k"] & \dots
\end{tikzcd}$$

of graded abelian groups and homomorphisms, in which each triangle

$$\dots \to A_{s-1} \xrightarrow{i} A_s \xrightarrow{j} E_s \xrightarrow{k} A_{s-1} \to \dots$$

is a long exact sequence.

Usually i is of internal degree 0, while j and k are of internal degree 0 and -1, in one order or the other.

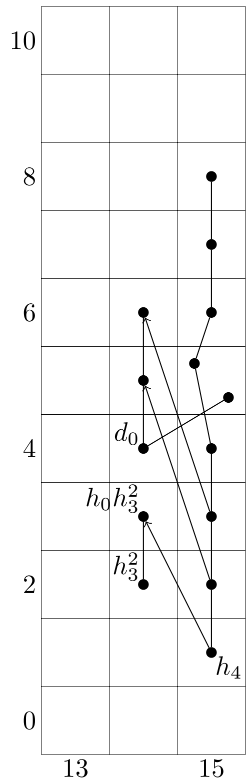


FIGURE 8. Adams E_2-term, differentials near $t - s = 14$

Definition 7.2. For $r \geq 1$, let

$$Z_s^r = k^{-1}(\mathrm{im}(i^{r-1} \colon A_{s-r} \to A_{s-1}))$$

be the *r-th cycle subgroup* of E_s, and let

$$B_s^r = j(\ker(i^{r-1} \colon A_s \to A_{s+r-1}))$$

be the *r-th boundary subgroup*. We have inclusions

$$0 = B_s^1 \subset \cdots \subset B_s^r \subset B_s^{r+1} \subset \cdots \subset \mathrm{im}(j) = \ker(k) \subset \cdots \subset Z_s^{r+1} \subset Z_s^r \subset \cdots \subset Z_s^1 = E_s$$

of graded abelian groups, for each filtration index s. Let

$$E_s^r = Z_s^r / B_s^r$$

be the *E^r-term* of the spectral sequence, and let the *d^r-differential*

$$d_s^r \colon E_s^r \longrightarrow E_{s-r}^r$$

be defined by $d_s^r([x]) = [j(y)]$, where $x \in Z_s^r$, $y \in A_{s-r}$ and $k(x) = i^{r-1}(y)$.

To see that the definition of the d^r-differential makes sense, note that for each $x \in Z_s^r$, $k(x)$ lies in the image of i^{r-1}, so there exists a $y \in A_{s-r}$ with $k(x) = i^{r-1}(y)$. If y' is another class with $k(x) = i^{r-1}(y')$, then $y' - y \in \ker(i^{r-1})$, so $j(y') - j(y)$ lies in B_s^r, so the class of $j(y)$ in E_{s-r}^r is well-defined. If $x \in B_s^r$, then $x \in \mathrm{im}(j) = \ker(k)$, so $k(x) = 0$ and we may take $y = 0$ in this case, with $[j(y)] = 0$. In general it follows that $[j(y)]$ only depends on the class $[x]$ of x in E_s^r. To see that d^r is a differential, i.e., that $d_{s-r}^r \circ d_s^r = 0$, just note that with notation as above, $kj(y) = 0$.

For $r = 1$ we identify $E_s^1 = Z_s^1/B_s^1 = E_s/0$ with E_s, and note that $d_s^1 \colon E_s^1 \to E_{s-1}^1$ equals the composite $jk \colon E_s \to E_{s-1}$. Hence the E^2-term is the homology of the chain complex

$$\cdots \leftarrow E_{s-1} \xleftarrow{jk} E_s \xleftarrow{jk} E_{s+1} \leftarrow \ldots .$$

Proposition 7.3. $\ker(d_s^r) = Z_s^{r+1}/B_s^r$ *and* $\mathrm{im}(d_{s+r}^r) = B_s^{r+1}/B_s^r$*, so there is a canonical isomorphism*

$$H_s(E^r, d^r) = \frac{\ker(d_s^r)}{\mathrm{im}(d_{s+r}^r)} = \frac{Z_s^{r+1}/B_s^r}{B_s^{r+1}/B_s^r} \cong \frac{Z_s^{r+1}}{B_s^{r+1}} = E_s^{r+1} \,,$$

for each $r \geq 1$ and each s.

We call $(E^r, d^r)_r$ the spectral sequence associated to the unrolled exact couple in Definition 7.1.

Proof. If $x \in Z_s^r$ satisfies $d_s^r([x]) = 0$, then $k(x) = i^{r-1}(y)$ for and $y \in A_{s-r}$ with $j(y) \in B_{s-r}^r$. Hence $j(y) = j(y')$ for some $y' \in A_{s-r}$ with $i^{r-1}(y') = 0$. Thus $j(y - y') = 0$, so $y - y' = i(z)$ for some $z \in A_{s-r-1}$, and $k(x) = i^{r-1}(y) = i^{r-1}(y - y') = i^r(z)$ is in $\mathrm{im}(i^r)$. Hence $x \in Z_s^{r+1}$.

Conversely, if $x \in Z_s^{r+1}$, then $k(x) = i^r(z)$ for some $z \in A_{s-r-1}$, so $k(x) = i^{r-1}(y)$ with $y = i(z)$, and $j(y) = ji(z) = 0$. Thus $d_s^r([x]) = [j(y)] = 0$.

$$\begin{tikzcd}
A_{s-r-1} \arrow[r, "i"] & A_{s-r} \arrow[r, "i^{r-1}"] \arrow[d, "j"] & A_{s-1} \arrow[r, "i"] & A_s \arrow[r, "i^{r-1}"] \arrow[d, "j"] & A_{s+r-1} \arrow[r, "i"] & A_{s+r} \arrow[d, "j"] \\
& E_{s-r} \arrow[ul, "k"] & & E_s \arrow[ul, "k"] & & E_{s+r} \arrow[ul, "k"]
\end{tikzcd}$$

If $u \in Z_s^r$ satisfies $[u] = d_s^r([v])$ for some $v \in Z_{s+r}^r$, then $[u] = [j(w)]$ for some $w \in A_{s+1}$ with $k(v) = i^{r-1}(w)$. Then $i^r(w) = ik(v) = 0$, so $j(w) \in B_s^{r+1}$. Hence $u \in B_s^{r+1}$.

Conversely, if $u \in B_s^{r+1}$, then $u = j(w)$ for some $w \in A_s$ with $i^r(w) = 0$. Then $i^{r-1}(w) \in A_{s+r-1}$ lies in $\ker(i) = \mathrm{im}(k)$, so $i^{r-1}(w) = k(v)$ for some $v \in E_{s+r}$. This relation shows that $v \in Z_{s+r}^r$, and by definition, $d_{s+r}^r([v]) = [j(w)] = [u]$, so $[u] \in \mathrm{im}(d_{s+r}^r)$. ☐

7.2. **E^∞-terms and target groups.**

Definition 7.4. Let
$$Z_s^\infty = \lim_r Z_s^r = \bigcap_r Z_s^r$$
be the subgroup of *infinite cycles* in E_s, and let
$$B_s^\infty = \operatorname*{colim}_r B_s^r = \bigcup_r B_s^r$$
be the subgroup of *infinite boundaries*. Let
$$E_s^\infty = Z_s^\infty / B_s^\infty$$
be the E^∞*-term* of the spectral sequence. For later use, let
$$RE_s^\infty = \operatorname*{Rlim}_r Z_s^r$$
denote the *derived* E^∞*-term*.

To justify the notation RE_s^∞ in place of RZ_s^∞, note that if the boundary group $B_{s,t}^r$ in a fixed bidegree (s,t) is independent of r for $r \ge m = m(s,t)$, then $\mathrm{Rlim}_r Z_{s,t}^r \cong \mathrm{Rlim}_r Z_{s,t}^r / B_{s,t}^m = \mathrm{Rlim}_r E_{s,t}^r$. If the spectral sequence collapses at a finite stage, or is locally eventually constant, then $RE_s^\infty = 0$ for all s.

In particular, we have inclusions
$$B_s^\infty \subset \mathrm{im}(j) = \ker(k) \subset Z_s^\infty$$
of (graded) subgroups of E_s, and an associated short exact sequence
$$0 \to \frac{\mathrm{im}(j)}{B_s^\infty} \longrightarrow \frac{Z_s^\infty}{B_s^\infty} \longrightarrow \frac{Z_s^\infty}{\ker(k)} \to 0 \tag{3}$$
expressing the E^∞-term as an extension.

If $A_{s-r} = 0$ for r sufficiently large, then $Z_s^r = \ker(k)$ for all these r, so that $Z_s^\infty / \ker(k) = 0$ and $\mathrm{im}(j)/B_s^\infty \cong E_s^\infty$. We shall give other sufficient conditions for the vanishing of this group in the next subsection. On the other hand, if $A_{s+r-1} = 0$ for r sufficiently large, then $B_s^r = \mathrm{im}(j)$ for all these r, so that $\mathrm{im}(j)/B_s^\infty = 0$ and $E_s^\infty \cong Z_s^\infty / \ker(k)$.

Definition 7.5. Let
$$\begin{aligned} A_{-\infty} &= \lim_s A_s \\ RA_{-\infty} &= \operatorname*{Rlim}_s A_s \\ A_\infty &= \operatorname*{colim}_s A_s \end{aligned}$$
be the limit, derived limit and colimit of the bi-infinite sequence $(A_s)_s$.

We consider two possible target groups for the spectral sequence; the colimit A_∞ and the limit $A_{-\infty}$. Each comes with a natural increasing filtration.

Definition 7.6. Let $F_sA_\infty = \mathrm{im}(A_s \to A_\infty)$ and $F_sA_{-\infty} = \ker(A_{-\infty} \to A_s)$, for each integer s.

Lemma 7.7. *The filtration $\{F_sA_\infty\}_s$ of A_∞ is exhaustive, and the filtration $\{F_sA_{-\infty}\}_s$ of $A_{-\infty}$ is complete Hausdorff.*

Proof. The first claim is clear. For the second claim, use the lim-Rlim exact sequences for

$$0 \to F_sA_{-\infty} \longrightarrow A_{-\infty} \longrightarrow \mathrm{im}(A_{-\infty} \to A_s) \to 0$$

and

$$0 \to \mathrm{im}(A_{-\infty} \to A_s) \longrightarrow A_s \longrightarrow \mathrm{cok}(A_{-\infty} \to A_s) \to 0\,.$$

□

Proposition 7.8. *There are natural isomorphisms*

$$\frac{F_sA_\infty}{F_{s-1}A_\infty} \cong \frac{\mathrm{im}(j)}{B_s^\infty} \qquad \textit{and} \qquad \frac{F_sA_{-\infty}}{F_{s-1}A_{-\infty}} \cong [[ETC]]\,.$$

Proof. Consider the diagram

$$\begin{tikzcd}
A_{-\infty} \arrow[r] & A_{s-1} \arrow[r, "i"] & A_s \arrow[r] \arrow[d, "j"] & A_\infty \\
& & E_s\,. \arrow[ul, "k"] &
\end{tikzcd}$$

The homomorphisms $A_s \to A_\infty$ and $j\colon A_s \to E_s$ induce isomorphisms

$$\frac{F_sA_\infty}{F_{s-1}A_\infty} \cong \frac{A_s}{\ker(A_s \to A_\infty) + \mathrm{im}(i\colon A_{s-1} \to A_s)}$$

and

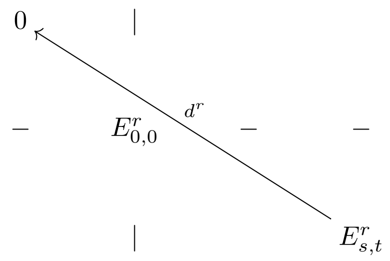
$$\frac{\mathrm{im}(j\colon A_s \to E_s)}{j(\ker(A_s \to A_\infty))} \cong \frac{A_s}{\ker(A_s \to A_\infty) + \ker(j\colon A_s \to E_s)}\,,$$

respectively, and the right hand sides are equal. Finally, $j(\ker(A_s \to A_\infty)) = B_s^\infty$ by passage to colimits over r from the definition $j(\ker(A_s \to A_{s+r-1})) = B_s^r$.

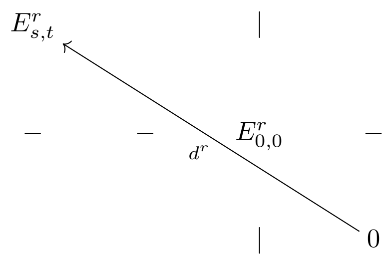
[[ETC, limit case]] □

7.3. Conditional convergence.

Definition 7.9. A *homological right half-plane spectral sequence* is a spectral sequence such that $E^r_{s,t} = 0$ for all $s < 0$. More generally, a *spectral sequence with exiting differentials* is a spectral sequence such that in each bidegree (s,t) only finitely many of the differentials starting in that bidegree map to nonzero groups.



A *homological left half-plane spectral sequence* is a spectral sequence such that $E^r_{s,t} = 0$ for all $s > 0$. More generally, a *spectral sequence with entering differentials* is a spectral sequence such that in each bidegree (s,t) only finitely many of the differentials ending in that bidegree map from nonzero groups.



Let $(E^r, d^r)_r$ be the homological spectral sequence associated to an unrolled exact couple, as in Definition 7.1 and Proposition 7.3. In the right half-plane case, the following classical theorem suffices.

Theorem 7.10 (Cartan–Eilenberg(?))**.** *Suppose that $E_s = 0$ for all $s < 0$, so that $A_{-\infty} \cong A_s$ for all $s < 0$ and $(E^r, d^r)_r$ is a spectral sequence with exiting differentials.*

(1) *If $A_{-\infty} = 0$ then the spectral sequence converges strongly to the colimit A_∞.*
(2) *If $A_\infty = 0$ then the spectral sequence converges strongly to the limit $A_{-\infty} \cong A_{-1}$.*

In the homological left half-plane case, as well as in the case of whole-plane spectral sequences, the utility of the following definition was explained by Boardman (1981 preprint, 1999). [[Check what Adams writes in the Chicago lecture notes, or his 1971 survey of 1960s algebraic topology.]]

Definition 7.11. The spectral sequence $(E^r, d^r)_r$ associated to an unrolled exact couple *converges conditionally to the colimit A_∞* if $A_{-\infty} = 0$ and $RA_{-\infty} = 0$. It *converges conditionally to the limit $A_{-\infty}$* if $A_\infty = 0$.

Theorem 7.12 (Adams(?), Boardman)**.** *Suppose that $E_s = 0$ for all $s > 0$, so that $A_s \cong A_\infty$ for all $s \geq 0$ and $(E^r, d^r)_r$ is a spectral sequence with entering differentials.*

(1) *If the spectral sequence converges conditionally to the colimit A_∞, and if $RE^\infty = 0$, then the spectral sequence converges strongly to that colimit.*
(2) *If the spectral sequence converges conditionally to the limit $A_{-\infty}$, and if $RE^\infty = 0$, then the spectral sequence converges strongly to that limit.*

Conditional convergence is a property of the unrolled exact couple, which can often be verified in terms of its construction. The additional assumption that $RE^\infty = 0$ can often be verified in concrete cases, e.g., in the presence of finiteness assumptions. The theorem asserts that it combination, these two properties suffice to ensure strong convergence. We shall only discuss a minimal path towards this result. The reader should consult Boardman (1999) for a much more complete story, including comparison theorems and results about when the sufficient conditions are also necessary.

Proof. In view of equation (3) and Proposition 7.8, in order to prove that $F_s A_\infty / F_{s-1} A_\infty \cong E^\infty_s$ it suffices to prove that $Z^\infty_s / \ker(k) = 0$. To establish strong convergence, we also need to prove that the filtration $\{F_s A_\infty\}_s$ is complete and Hausdorff.

Definition 7.13. Let $Q_s = \lim_r \operatorname{im}(i^r \colon A_{s-r} \to A_s)$ and $RQ_s = \operatorname{Rlim}_r \operatorname{im}(i^r \colon A_{s-r} \to A_s)$ be the limit and the derived limit, respectively, of the image filtration

$$\cdots \subset \operatorname{im}(i^r \colon A_{s-r} \to A_s) \subset \cdots \subset \operatorname{im}(i \colon A_{s-1} \to A_s) \subset A_s \,.$$

Lemma 7.14. *There is a six term exact sequence*

$$0 \to \frac{Z^\infty_s}{\ker(k)} \xrightarrow{k} Q_{s-1} \xrightarrow{i} Q_s \longrightarrow RE^\infty_s \xrightarrow{k} RQ_{s-1} \xrightarrow{i} RQ_s \to 0 \,.$$

Proof. For each r and s, there is a short exact sequence

$$0 \to \frac{Z^r_s}{\ker(k)} \xrightarrow{k} \operatorname{im}(i^{r-1} \colon A_{s-r} \to A_{s-1}) \xrightarrow{i} \operatorname{im}(i^r \colon A_{s-r} \to A_s) \to 0 \,.$$

Passing to limits over r, we get the asserted six term exact sequence. □

Corollary 7.15. *If $RE^\infty = 0$, then each $i \colon Q_{s-1} \to Q_s$ is surjective and each $i \colon RQ_{s-1} \to RQ_s$ is an isomorphism. Hence $\lim_s Q_s \to Q_m$ is surjective and $\lim_s RQ_s \to RQ_m$ is an isomorphism, for each m.*

By assumption $A_0 = A_\infty$, so we have

$$Q_0 = \lim_r \operatorname{im}(A_{-r} \to A_0) = \lim_s F_s A_\infty$$

and

$$RQ_0 = \operatorname{Rlim}_r \operatorname{im}(A_{-r} \to A_0) = \operatorname{Rlim}_s F_s A_\infty \,.$$

Hence proving that $Q_0 = 0$ and $RQ_0 = 0$ is equivalent to proving that $\{F_s A_\infty\}_s$ is complete Hausdorff. By the corollary, when $RE^\infty = 0$ it will suffice to prove that $\lim_s Q_s = 0$ and $\lim_s RQ_s = 0$. This will then also imply that each $Q_{s-1} = 0$, so $Z^\infty_s / \ker(k) = 0$, as desired. By the following lemma, these properties follow from the assumptions $A_{-\infty} = 0$ and $RA_{-\infty} = 0$. □

Lemma 7.16. *If $A_{-\infty} = 0$ and $RA_{-\infty} = 0$ then $\lim_s Q_s = 0$ and $\lim_s RQ_s = 0$.*

Proof. Consider the double limit system

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \\
\dots \arrow[r, "i"] & \operatorname{im}(i^r\colon A_{s-1-r} \to A_{s-1}) \arrow[r, "i"] \arrow[d] & \operatorname{im}(i^r\colon A_{s-r} \to A_s) \arrow[r, "i"] \arrow[d] & \dots \\
 & \vdots \arrow[d] & \vdots \arrow[d] & \\
\dots \arrow[r, "i"] & A_{s-1} \arrow[r, "i"] & A_s \arrow[r, "i"] & \dots
\end{tikzcd}$$

where the vertical maps are inclusions. The limit of the s-th column is by definition Q_s. The limit of the r-th row maps identically to the limit of the bottom row, i.e., to $A_{-\infty}$. Hence

$$\lim_s Q_s = \lim_s \lim_r \operatorname{im}(i^r\colon A_{s-r} \to A_s) \cong \lim_r \lim_s \operatorname{im}(i^r\colon A_{s-r} \to A_s) \cong \lim_r A_{-\infty} \cong A_{-\infty}\,.$$

For each s let

$$\widehat{A}_s = \lim_r \frac{A_s}{\operatorname{im}(i^r\colon A_{s-r} \to A_s)}$$

be the completion of A_s with respect to the image filtration. The lim-Rlim sequence of the r-indexed system of short exact sequences

$$0 \to \operatorname{im}(i^r\colon A_{s-r} \to A_s) \longrightarrow A_s \longrightarrow \frac{A_s}{\operatorname{im}(i^r\colon A_{s-r} \to A_s)} \to 0$$

contains the exact sequence

$$0 \to Q_s \longrightarrow A_s \longrightarrow \widehat{A}_s \longrightarrow RQ_s \to 0\,,$$

which breaks into the two s-indexed systems of short exact sequences

$$0 \to Q_s \longrightarrow A_s \longrightarrow A_s/Q_s \to 0$$

and

$$0 \to A_s/Q_s \longrightarrow \widehat{A}_s \longrightarrow RQ_s \to 0\,.$$

These in turn give rise to the exact lim-Rlim sequences

$$0 \to \lim_s Q_s \longrightarrow A_{-\infty} \longrightarrow \lim_s A_s/Q_s \longrightarrow \operatorname*{Rlim}_s Q_s \longrightarrow RA_{-\infty} \longrightarrow \operatorname*{Rlim}_s A_s/Q_s \to 0$$

and

$$0 \to \lim_s A_s/Q_s \longrightarrow \lim_s \widehat{A}_s \longrightarrow \lim_s RQ_s \longrightarrow \operatorname*{Rlim}_s A_s/Q_s \longrightarrow \operatorname*{Rlim}_s \widehat{A}_s \longrightarrow \operatorname*{Rlim}_s RQ_s \to 0\,.$$

Here

$$\lim_s \widehat{A}_s = \lim_s \lim_r \frac{A_s}{\operatorname{im}(i^r\colon A_{s-r} \to A_s)} \cong \lim_r \lim_s \frac{A_s}{\operatorname{im}(i^r\colon A_{s-r} \to A_s)} = \lim_r 0 = 0\,,$$

since for each fixed r, the r-fold composite

$$i^r\colon \frac{A_{s-r}}{\operatorname{im}(i^r\colon A_{s-2r} \to A_{s-r})} \longrightarrow \frac{A_s}{\operatorname{im}(i^r\colon A_{s-r} \to A_s)}$$

is zero.

The assumptions $A_{-\infty} = 0$ and $RA_{-\infty} = 0$ now yield $\lim_s Q_s = 0$, $\lim_s A_s/Q_s \cong \operatorname{Rlim}_s Q_s$ and $\operatorname{Rlim}_s A_s/Q_s = 0$. Combined with the vanishing of $\lim_s \widehat{A}_s$, this implies $\lim_s A_s/Q_s = 0$, $\lim_s RQ_s = 0$ and $\operatorname{Rlim}_s \widehat{A}_s \cong \operatorname{Rlim}_s RQ_s$. In fact $\operatorname{Rlim}_s RQ_s = 0$, since $i\colon RQ_{s-1} \to RQ_s$ is surjective for each s. $\square$

Boardman in fact proves the following more precise result, the middle part of which he refers to as the Mittag–Leffler exact sequence. This is a special case of the Grothendieck spectral sequence for the composite of two limit functors, which was first (?) analyzed by Roos.

Proposition 7.17. $\lim_s Q_s \cong A_{-\infty}$, *there is a short exact sequence*

$$0 \to \operatorname*{Rlim}_s Q_s \longrightarrow RA_{-\infty} \longrightarrow \lim_s RQ_s \to 0\,,$$

and $\operatorname{Rlim}_s RQ_s = 0$.

8. Examples of exact couples

8.1. **Homology of sequences of cofibrations.** Generalizing the examples from Section 1 and Section 3, consider a sequence of spaces

$$\emptyset = X_{-1} \subset X_0 \subset \cdots \subset X_{s-1} \subset X_s \subset \cdots \subset X$$

where each inclusion $i\colon X_{s-1} \to X_s$ is a cofibration and $X = \operatorname{colim}_s X_s \simeq \operatorname{hocolim}_s X_s$ has the weak (colimit) topology. For instance, X might be a CW complex and X_s its s-skeleton. Applying homology, we obtain an unrolled exact couple

$$\begin{tikzcd}
\cdots \arrow[r] & H_*(X_{s-2}) \arrow[r, "i_*"] & H_*(X_{s-1}) \arrow[r, "i_*"] \arrow[d, "j_*"] & H_*(X_s) \arrow[r, "i_*"] \arrow[d, "j_*"] & H_*(X_{s+1}) \arrow[r, "i_*"] \arrow[d, "j_*"] & \cdots \\
& \cdots & H_*(X_{s-1}, X_{s-2}) \arrow[ul, "\partial"] & H_*(X_s, X_{s-1}) \arrow[ul, "\partial"] & H_*(X_{s+1}, X_s) \arrow[ul, "\partial"] & \cdots
\end{tikzcd}$$

with $A_s = H_*(X_s)$ and $E_s = H_*(X_s, X_{s-1})$. Each triangle is the long exact sequence of a pair, hence is exact. The homomorphisms $i = i_*$ and $j = j_*$ preserve the internal grading, while $k = \partial$ has degree -1. The E^1-term is

$$E^1_{s,t} = H_{s+t}(X_s, X_{s-1})$$

and the d^1-differential is

$$d^1_{s,t} = j_* \circ \partial\colon H_{s+t}(X_s, X_{s-1}) \longrightarrow H_{s+t-1}(X_{s-1}, X_{s-2}) \,,$$

i.e., the connecting homomorphism in the long exact sequence in homology for the triple (X_s, X_{s-1}, X_{s-2}). Here $A_s = 0$ for $s < 0$, so we have a homological right half-plane spectral sequence, with exiting differentials. By Theorem 7.10, it converges strongly to

$$A_\infty = \operatorname*{colim}_s H_*(X_s) \cong H_*(X) \,.$$

In the special case when $X_s = X^{(s)}$ is the s-skeleton of a CW complex X, $E^1_{s,0} = H_s(X^{(s)}, X^{(s-1)}) = C_s(X)$ and $E^1_{s,t} = 0$ for $t \neq 0$, so (E^1, d^1) equals the cellular chain complex of X, concentrated on the horizontal axis. The E^2-term equals the cellular homology, and the spectral sequence collapses at this stage. These observations give a spectral sequence proof of the fact that cellular homology is isomorphic to singular homology for CW complexes.

8.2. **Cohomology of sequences of cofibrations.** Applying cohomology to the same sequence of spaces, we get another unrolled exact couple

$$\begin{tikzcd}
\cdots \arrow[r] & H^*(X_{s+1}) \arrow[r, "i^*"] & H^*(X_s) \arrow[r, "i^*"] \arrow[d, "\delta"] & H^*(X_{s-1}) \arrow[r, "i^*"] \arrow[d, "\delta"] & H^*(X_{s-2}) \arrow[r, "i^*"] \arrow[d, "\delta"] & \cdots \\
& \cdots & H^*(X_{s+1}, X_s) \arrow[ul, "j^*"] & H^*(X_s, X_{s-1}) \arrow[ul, "j^*"] & H^*(X_{s-1}, X_{s-2}) \arrow[ul, "j^*"] & \cdots ,
\end{tikzcd}$$

now with $A^s = A_{-s} = H^*(X_{s-1})$ and $E^s = E_{-s} = H^*(X_s, X_{s-1})$. In this case $i = i^*$ and $k = j^*$ preserve degrees, and $j = \delta$ has degree $+1$. The associated spectral sequence is a left half-plane spectral sequence with entering differentials, and converges conditionally to the limit

$$A_{-\infty} = \lim_s H^*(X_s)$$

since $A_\infty = \operatorname{colim}_s H^*(X_s) = 0$. By Theorem 7.12, the spectral sequence converges strongly to this limit if $RE^\infty = 0$. In general, the homomorphism $H^*(X) \to \lim_s H^*(X_s)$ is not an isomorphism, so this spectral sequence is not always useful for the computation of $H^*(X)$.

Instead, one can consider the sequence of pairs of spaces

$$(X, \emptyset) = (X, X_{-1}) \subset (X, X_0) \subset \cdots \subset (X, X_{s-1}) \subset (X, X_s) \subset \cdots \subset (X, X)$$

and apply relative cohomology. The result is an unrolled exact couple

$$\begin{tikzcd}
\cdots \arrow[r] & H^*(X, X_{s+1}) \arrow[r, "j^*"] & H^*(X, X_s) \arrow[r, "j^*"] \arrow[d, "i^*"] & H^*(X, X_{s-1}) \arrow[r, "j^*"] \arrow[d, "i^*"] & H^*(X, X_{s-2}) \arrow[r, "j^*"] \arrow[d, "i^*"] & \cdots \\
& \cdots & H^*(X_{s+1}, X_s) \arrow[ul, "\delta"] & H^*(X_s, X_{s-1}) \arrow[ul, "\delta"] & H^*(X_{s-1}, X_{s-2}) \arrow[ul, "\delta"] & \cdots ,
\end{tikzcd}$$

where $A^s = A_{-s} = H^*(X, X_{s-1})$ and $E^s = E_{-s} = H^*(X_s, X_{s-1})$. In this case $i = j^*$ and $j = i^*$ preserve degrees, and $k = \delta$ has degree $+1$. The associated spectral sequence has

$$E_1^{s,t} = E^1_{-s,-t} = H^{s+t}(X_s, X_{s-1})$$

and $d^1 = i^* \circ \delta$. In homological indexing it is concentrated in the left half-plane, hence has entering differentials, and converges conditionally to the colimit

$$A_\infty = \operatorname*{colim}_s H^*(X, X_s) \cong H^*(X)$$

whenever $A_{-\infty} = \lim_s H^*(X, X_s) = 0$ and $RA_{-\infty} = \operatorname{Rlim}_s H^*(X, X_s) = 0$. In view of the Milnor lim-Rlim short exact sequence

$$0 \to \operatorname*{Rlim}_s H^{*-1}(X, X_s) \longrightarrow H^*(X, \operatorname*{hocolim}_s X_s) \longrightarrow \lim_s H^*(X, X_s) \to 0\,,$$

where we use the equivalence $X \simeq \operatorname{hocolim}_s X_s$, these conditions are always satisfied. By Theorem 7.12, the spectral sequence therefore converges strongly to $H^*(X)$ whenever $RE^\infty = 0$, e.g., if the spectral sequence collapses at a finite stage.

8.3. **The Atiyah–Hirzebruch spectral sequence.** Replacing singular homology with a generalized homology theory E_*, such as stable homotopy, topological K-homology or complex bordism, we instead obtain an unrolled exact couple

$$\begin{tikzcd}
\dots \arrow[r] & E_*(X_{s-2}) \arrow[r, "i"] & E_*(X_{s-1}) \arrow[r, "i"] \arrow[d, "j"] & E_*(X_s) \arrow[r, "i"] \arrow[d, "j"] & E_*(X_{s+1}) \arrow[r, "i"] \arrow[d, "j"] & \dots \\
\dots & & E_*(X_{s-1}, X_{s-2}) \arrow[ul, "\partial"] & E_*(X_s, X_{s-1}) \arrow[ul, "\partial"] & E_*(X_{s+1}, X_s) \arrow[ul, "\partial"] & \dots
\end{tikzcd}$$

with associated spectral sequence having $A_s = E_*(X_s)$ and $E_s = E_*(X_s, X_{s-1})$. The E^1-term is

$$E^1_{s,t} = E_{s+t}(X_s, X_{s-1})$$

and the d^1-differential is $j\partial$, as before. This is now the connecting homomorphism in the long exact sequence in E_*-theory, for the triple (X_s, X_{s-1}, X_{s-2}). Again this is a right half-plane spectral sequence, converging strongly to the colimit

$$A_\infty = \operatorname*{colim}_s E_*(X_s) \cong E_*(X)\,.$$

In this generality the special case $X_s = X^{(s)}$ is interesting, since

$$E^1_{s,t} = E_{s+t}(X^{(s)}, X^{(s-1)}) = C_s(X; E_t)$$

is the group of cellular s-chains of X with coefficients in the coefficient group $E_t = E_t(*) = \pi_t(E)$ of the generalized homology theory E_*. The d^1-differential is the boundary homomorphism in the cellular chain complex $C_*(X; E_t)$, so the E^2-term

$$E^2_{s,t} = H_s(X; E_t)$$

is the s-th cellular homology group of X with coefficients in E_t. This example is the E_*-theory Atiyah–Hirzebruch spectral sequence

$$E^2_{s,t} = H_s(X; E_t) \Longrightarrow_s E_{s+t}(X)$$

converging strongly to $E_*(X)$. The target group is filtered by the images

$$F_s E_*(X) = \operatorname{im}(E_*(X_s) \longrightarrow E_*(X))$$

and there are isomorphisms $F_s E_*(X)/F_{s-1}E_*(X) \cong (E^\infty_s)_*$, for all integers s. If $H_*(X)$ and $E_* = E_*(*)$ are concentrated in even degrees (meaning that $H_s(X) = 0$ for s odd and $E_t = 0$ for t odd), and at least one of these graded groups are torsion-free, then

$$E^2_{*,*} = H_*(X; E_*) \cong H_*(X) \otimes_{\mathbb{Z}} E_*$$

is concentrated in bidegrees (s, t) with both s and t even. It follows that each differential

$$d^r_{s,t} \colon E^r_{s,t} \longrightarrow E^r_{s-r,t+r-1}$$

must be zero for bidegree reasons, so that the spectral sequence collapses at the E^2-term, with $E^2 = E^\infty$. This happens frequently enough to be worthy of note, for instance if $E = KU$ or MU represents complex K-theory or complex (co-)bordism.

The Atiyah–Hirzebruch spectral sequence for stable homotopy theory

$$E^2_{s,t} = H_s(X; \pi^S_t) \Longrightarrow_s \pi^S_{s+t}(X)$$

is sometimes useful in conjunction with the Adams spectral sequence.

The cohomological version of the Atiyah–Hirzebruch spectral sequence is the spectral sequence

$$E_2^{s,t} = H^s(X; E^t) \Longrightarrow_s E^{s+t}(X)$$

with entering differentials, where $E^t = E^t(*) = \pi_{-t}(E)$, associated to the unrolled exact couple with

$$A^{s,t} = E^{s+t}(X, X^{(s-1)})$$

and

$$E^{s,t} = E^{s+t}(X^{(s)}, X^{(s-1)}) = C^s(X; E^t)\,.$$

It converges conditionally to the colimit, and converges strongly if $RE_\infty = 0$.

The original paper of Atiyah and Hirzebruch (1961) concerned the generalized cohomology theory given by topological K-theory, with $K^t = K_{-t} = \mathbb{Z}$ for t even and $K^t = K_{-t} = 0$ for t odd, so the K-cohomology Atiyah–Hirzebruch spectral sequence

$$E_2^{s,t} = H^s(X; K^t) \Longrightarrow K^{s+t}(X)$$

collapses at the E_2-term for each space X whose cohomology $H^*(X)$ is concentrated in even degrees. [[Describe d_3-differential in terms of cohomology operations.]]

8.4. **The Serre spectral sequence.** Consider a Serre fibration $p\colon E \to B$, with B path-connected. Suppose that the base space B is a CW complex, with skeleton filtration $\{B^{(s)}\}_s$. Define a filtration

$$\emptyset = E_{-1} \subset E_0 \subset \cdots \subset E_{s-1} \subset E_s \subset \cdots \subset E$$

of the total space E by taking the preimages of this skeleton filtration:

$$E_s = p^{-1}(B^{(s)})\,.$$

We get an unrolled exact couple with $A_{s,t} = H_{s+t}(E_s)$ and $E_{s,t} = H_{s+t}(E_s, E_{s-1})$, and an associated spectral sequence

$$E^1_{s,t} = H_{s+t}(E_s, E_{s-1})$$

converging strongly to $H_{s+t}(E)$. We use the hypothesis that $p\colon E \to B$ is a Serre fibration to rewrite the E^1-term in terms of the cellular chains on B. Let

$$\Phi = \coprod_\alpha \Phi_\alpha \colon \coprod_\alpha D^s \longrightarrow B^{(s)}$$

be the combined characteristic maps of the s-cells of B, and let $\phi_\alpha\colon \partial D^s \to B^{(s-1)}$ be the attaching map of the α-indexed s-cell, i.e., the restriction of Φ_α to $\partial D^s \subset D^s$, viewed as a map into $B^{(s-1)} \subset B^{(s)}$.

Let $\Phi^*_\alpha E = D^s \times_B E$ be the pullback of E along Φ_α, and let $\phi^*_\alpha E = \partial D^s \times_B E$ be its restriction to ∂D^s.

$$\begin{tikzcd}
\phi^*_\alpha E \arrow[rr] \arrow[dr] \arrow[dd] & & \Phi^*_\alpha E \arrow[dr] \arrow[dd] & & \\
& E_{s-1} \arrow[rr] \arrow[dd] & & E_s \arrow[r] \arrow[dd] & E \arrow[dd, "p"] \\
\partial D^s \arrow[rr] \arrow[dr, "\phi_\alpha"'] & & D^s \arrow[dr, "\Phi_\alpha"] & & \\
& B^{(s-1)} \arrow[rr] & & B^{(s)} \arrow[r] & B
\end{tikzcd}$$

By excision, the sum of homomorphisms

$$\bigoplus_\alpha H_*(\Phi^*_\alpha E, \phi^*_\alpha E) \xrightarrow{\cong} H_*(E_s, E_{s-1})$$

is an isomorphism. For each α, the map

$$(\Phi^*_\alpha E, \phi^*_\alpha E) \longrightarrow (D^s \times \Phi^*_\alpha E, \partial D^s \times \Phi^*_\alpha E)$$

is a homotopy equivalence of pairs, since D^s is contractible. For any fixed choice of base point $d_0 \in D^s$, mapping to $b_\alpha = \Phi_\alpha(d_0) \in B$, the inclusion

$$F_{b_\alpha} = p^{-1}(b_\alpha) = \{b_\alpha\} \times_B E \cong \{d_0\} \times_{D^s} \Phi_\alpha^* E \subset \Phi_\alpha^* E$$

is a (weak) homotopy equivalence, in view of the long exact sequence in homotopy for the Serre fibration $\Phi_\alpha^* E \to D^s$, again using that D^s is contractible. Hence there are preferred isomorphisms

$$H_*(\Phi_\alpha^* E, \phi_\alpha^* E) \cong H_*(D^s \times \Phi_\alpha^* E, \partial D^s \times \Phi_\alpha^* E) \cong H_*(D^s, \partial D^s) \otimes H_*(\Phi_\alpha^* E) \cong H_*(D^s, \partial D^s) \otimes H_*(F_{b_\alpha}) .$$

Thus

$$E^1_{s,t} \cong \bigoplus_\alpha H_t(F_{b_\alpha})$$

with $b_\alpha = \Phi_\alpha(d_0)$, varying with α. By definition this is the group of cellular s-chains $C_s(B; \mathscr{H}_t(F))$ of B with local coefficients in the system $\mathscr{H}_t(F)$, taking $b \in B$ to $H_t(F_b)$.

A *local coefficient system* on B can be defined as a functor from the fundamental groupoid $\Pi_1(B)$ of B to the category of abelian groups. The objects of $\Pi_1(B)$ are the points of B, and a morphism from b_0 to b_1 is a homotopy class $[f]$, relative to the endpoints, of paths $f \colon I \to B$ from b_1 to b_0. With this convention, the composite of $[f]$ and the class $[g]$ of a path $g \colon I \to B$ from b_2 to b_1 is the class $[g] \circ [f] = [g * f]$ of the path $g * f$ from b_2 to b_0. When B is path connected, all objects of $\Pi_1(B)$ are isomorphic, and for any choice of base point $b_0 \in B$, the inclusion $\pi_1(B, b_0) \subset \Pi_1(B)$ of the fundamental group of B based at b_0, viewed as a groupoid with one object, is an equivalence of categories.

The local coefficient system $\mathscr{H}_t(F)$ takes $b \in B$ to $H_t(F_b)$, where $F_b = p^{-1}(b)$ is the fiber of $p \colon E \to B$ over b. To the homotopy class $[f]$ of a path f from b_1 to b_0, as above, we associate the composite isomorphism

$$[f]_* \colon H_t(F_{b_1}) \xrightarrow{\cong} H_t(I \times_B E) \xleftarrow{\cong} H_t(F_{b_0}) .$$

Here each inclusion $F_{b_t} \to I \times_B E$ is a (weak) homotopy equivalence, since $I \times_B E \to I$ is a Serre fibration, and the interval I is contractible. Exercise: Prove that if $H \colon I \times I \to B$ is a homotopy, relative to the endpoints, from f to $f' \colon I \to B$, then $[f]_* = [f']_*$.

A boundary homomorphism

$$\partial \colon C_s(B; \mathscr{H}_t(F)) \longrightarrow C_{s-1}(B; \mathscr{H}_t(F))$$

can be defined so as to agree with $d^1_{s,t}$ under the identifications above. [[ETC]]

In particular, $(C_*(B; \mathscr{H}_t(F), \partial)$ is a chain complex, and its homology $H_*(B; \mathscr{H}_t(F))$ is the cellular homology of B with local coefficients in $\mathscr{H}_t(F)$. This then computes the E^2-term of the homological Serre spectral sequence

$$E^2_{s,t} = H_s(B; \mathscr{H}_t(F)) \Longrightarrow_s H_{s+t}(E) .$$

If B is simply-connected, then $\mathscr{H}_t(F)$ is isomorphic (as a coefficient system) to the constant system at $H_t(F_{b_0})$, for any fixed choice of base point $b_0 \in B$, so in this case we can write the E^2-term as $H_s(B; H_t(F))$, with ordinary coefficients.

[[Relate to π-equivariant homology for the universal covering space $\tilde{B}$, with $\pi = \pi_1(B, b_0)$.]]

The cohomological version of the Serre spectral sequence is associated to the unrolled exact couple with

$$A^{s,t} = H^{s+t}(E, E_{s-1})$$

and

$$E^{s,t} = H^{s+t}(E_s, E_{s-1}) .$$

It has

$$E_1^{s,t} = C^s(B; \mathscr{H}^t(F))$$

and

$$E_2^{s,t} = H^s(B; \mathscr{H}^t(F)) \Longrightarrow_s H^{s+t}(E) .$$

It is concentrated in the first quadrant, in the cohomological indexing, and converges strongly to the colimit $H^*(E)$.

[[There are many examples of calculations with Serre spectral sequences in the literature, e.g., for loop-path fibrations $\Omega X \to PX \to X$, or for homogeneous spaces $H \to G \to G/H$ or $G/H \to BH \to BG$.]]

8.5. **Homotopy of towers of fibrations.** Turning in a different direction, consider a tower of spaces

$$Y \to \dots \to Y^s \to Y^{s-1} \to \dots \to Y^0 \to Y^{-1} = *$$

where each map $p\colon Y^s \to Y^{s-1}$ is a Serre fibration, and $Y = \lim_s Y^s \simeq \operatorname{holim}_s Y^s$.

We assume that Y is not empty, so that we can choose a base point $y_0 \in Y$, and take its image y_s under $Y \to Y^s$ as the base point for Y^s, for each integer s. Let

$$F^s = p^{-1}(y_{s-1}) = \{y_{s-1}\} \times_{Y^{s-1}} Y^s$$

be the fiber of $p\colon Y^s \to Y^{s-1}$ at y_{s-1}, based at y_s, so that there is a long exact sequence of homotopy groups

$$\dots \to \pi_n(F^s, y_s) \longrightarrow \pi_n(Y^s, y_s) \longrightarrow \pi_n(Y^{s-1}, y_{s-1}) \xrightarrow{\partial} \pi_{n-1}(F^s, y_s) \to \dots .$$

We would like to link these together to an unrolled exact couple, but note that in general the end

$$\dots \to \pi_1(F^s, y_s) \longrightarrow \pi_1(Y^s, y_s) \longrightarrow \pi_1(Y^{s-1}, y_{s-1})$$
$$\xrightarrow{\partial} \pi_0(F^s, y_s) \longrightarrow \pi_0(Y^s, y_s) \longrightarrow \pi_0(Y^{s-1}, y_{s-1})$$

of this sequence is not a diagram of abelian groups, and we might not be able to extend the sequence to the right with trivial groups.

Bousfield–Kan (1972, Section IX.4) address this problem by considering "extended" spectral sequences, which consist of possibly non-abelian groups and pointed sets near the edge.

Another solution is to assume that each Y^s is a homotopy commutative H-space, with y_s as neutral element, and that each map $p\colon Y^s \to Y^{s-1}$ is strictly compatible with this H-space structure. Then each fiber F^s is also a homotopy commutative H-space, and the diagram above is one of abelian groups and group homomorphisms. It is still not necessarily exact at $\pi_0(Y^{s-1}, y_{s-1})$, since $\pi_0(p)$ does not need to be surjective. We must therefore make this additional assumption. It is satisfied, for instance, if each space Y^s is path-connected.

Under these additional hypotheses, we get an unrolled exact couple

$$\begin{tikzcd}
\dots \arrow[r] & \pi_*(Y^{s+1}, y_{s+1}) \arrow[r, "p_*"] & \pi_*(Y^s, y_s) \arrow[r, "p_*"] \arrow[d, "\partial"] & \pi_*(Y^{s-1}, y_{s-1}) \arrow[r, "p_*"] \arrow[d, "\partial"] & \pi_*(Y^{s-2}, y_{s-2}) \arrow[r] \arrow[d, "\partial"] & \dots \\
\dots & & \pi_*(F^{s+1}, y_{s+1}) \arrow[ul] & \pi_*(F^s, y_s) \arrow[ul] & \pi_*(F^{s-1}, y_{s-1}) \arrow[ul] &
\end{tikzcd}$$

with $i = p_*$ of degree 0, $j = \partial$ of degree -1, and k of degree 0. The associated spectral sequence

$$E_1^{s,*} = \pi_*(F^s, y_s) \Longrightarrow_s \pi_*(Y, y_0)$$

has entering differentials and converges conditionally to the limit $\lim_s \pi_*(Y^s, y_s)$. [[Claim: If $RE_\infty = 0$, then $\operatorname{Rlim}_s \pi_*(Y^s, y_s) = 0$ and the spectral sequence converges strongly to $\pi_*(Y, y_0)$.]]

8.6. **Homotopy of towers of spectra.** The difficulty with the lack of abelian group structures, and lack of surjectivity at π_0, is not present when we consider towers of spectra. Consider a diagram of spectra

$$\dots \to Y^s \xrightarrow{i} Y^{s-1} \to \dots \to Y^0 = Y$$

and let $Y^\infty = \operatorname{holim}_s Y^s$, so that there is a Milnor lim-Rlim short exact sequence

$$0 \to \operatorname*{Rlim}_s \pi_{n+1}(Y^s) \longrightarrow \pi_n(Y^\infty) \longrightarrow \lim_s \pi_n(Y^s) \to 0 .$$

Let K^s be the homotopy cofiber of the map $i\colon Y^{s+1} \to Y^s$, so that there is a Puppe cofiber sequence

$$Y^{s+1} \xrightarrow{i} Y^s \xrightarrow{j} K^s \xrightarrow{\partial} \Sigma Y^{s+1} .$$

We let $Y^s = Y$ and $K^s = *$ for all $s < 0$. Applying homotopy to these spectra, we get an unrolled exact couple of graded abelian groups

$$\begin{tikzcd}
\dots \arrow[r] & \pi_*(Y^{s+2}) \arrow[r, "i"] & \pi_*(Y^{s+1}) \arrow[r, "i"] \arrow[d, "j"] & \pi_*(Y^s) \arrow[r, "i"] \arrow[d, "j"] & \pi_*(Y^{s-1}) \arrow[r] \arrow[d, "j"] & \dots \\
& \dots & \pi_*(K^{s+1}) \arrow[ul, "k"] & \pi_*(K^s) \arrow[ul, "k"] & \pi_*(K^{s-1}) \arrow[ul, "k"] &
\end{tikzcd}$$

with $i = i_*$ and $j = j_*$ of degree 0, and $k = \partial_*$ of degree -1.

In homological indexing, we would write $A_{s,t} = \pi_{s+t}(Y^{-s})$ and $E_{s,t} = \pi_{s+t}(K^{-s})$, for $s \le 0$, but we switch to Adams indexing $A^{s,t} = A_{-s,t}$ and $E^{s,t} = E_{-s,t}$ so that

$$A^{s,t} = \pi_{t-s}(Y^s)$$
$$E^{s,t} = \pi_{t-s}(K^s)\,.$$

The associated spectral sequence

$$E_1^{s,t} = \pi_{t-s}(K^s) \Longrightarrow_s \pi_{t-s}(Y)$$

has entering differentials. By definition, it converges conditionally to the colimit $A_\infty (= A^{-\infty}) = \pi_*(Y)$ if the two groups

$$A_{-\infty}(= A^\infty) = \lim_s \pi_*(Y^s) \qquad \text{and} \qquad RA_{-\infty}(= RA^\infty) = \operatorname{Rlim}_s \pi_*(Y^s)$$

both vanish. By the lim-Rlim exact sequence recalled above, this is equivalent to the condition that $\pi_*(Y^\infty) = 0$, i.e., that $\operatorname{holim}_s Y^s \simeq *$.

Proposition 8.1. *The spectral sequence*

$$E_1^{s,t} = \pi_{t-s}(K^s) \Longrightarrow_s \pi_{t-s}(Y)$$

associated to the tower $\cdots \to Y^s \to Y^{s-1} \to \cdots \to Y^0 = Y$ *converges conditionally to the colimit* $\pi_*(Y)$ *if (and only if)* $\operatorname{holim}_s Y^s \simeq *$. *If* $RE_\infty = 0$ *then the spectral sequence converges strongly to that colimit, equipped with the descending filtration by the image subgroups* $F^s = \operatorname{im}(\pi_*(Y^s) \to \pi_*(Y))$.

The mod p Adams spectral sequence converging to $\pi_*(Y)^\wedge_p$ will be constructed as a special case of this spectral sequence, where we make special assumptions about the Puppe cofiber sequence displayed above, so as to be able to express the E_2-term of the spectral sequence in purely algebraic terms.

9. The Steenrod algebra

9.1. Steenrod's reduced squares and powers.

Theorem 9.1. *There are natural transformations*

$$Sq^i \colon \tilde{H}^n(X; \mathbb{F}_2) \longrightarrow \tilde{H}^{n+i}(X; \mathbb{F}_2)$$

for $i \ge 0$, *of contravariant functors from based spaces to abelian groups, called Steenrod's reduced squares. These satisfy* $Sq^0(x) = x$, $Sq^1(x) = \beta(x)$ *(the Bockstein homomorphism associated to the extension* $\mathbb{F}_2 \to \mathbb{Z}/4 \to \mathbb{F}_2$*),* $Sq^i(x) = x^2$ *for* $i = |x|$, *and* $Sq^i(x) = 0$ *for* $i > |x|$. *They also satisfy the internal Cartan formula*

$$Sq^k(xy) = \sum_{i+j=k} Sq^i(x) Sq^j(y)$$

and the Adem relations

$$Sq^a Sq^b = \sum_{j=0}^{[a/2]} \binom{b-1-j}{a-2j} Sq^{a+b-j} Sq^j$$

for $0 < a < 2b$.

Proofs can be found in Steenrod and Epstein (1962).

By naturality, the internal Cartan formula for the cup product $xy = x \cup y$ is equivalent to an external Cartan formula for the smash product $x \wedge y$. See Figure 9 for the Adem relations in degrees ≤ 11.

Example 9.2. The squaring operations for $X = \mathbb{R}P^\infty_+$ can be calculated as follows: Consider the total squaring operation $Sq(x) = \sum_{i \ge 0} Sq^i(x)$. Then $Sq(xy) = Sq(x)Sq(y)$. In $\tilde{H}^*(X; \mathbb{F}_2) = H^*(\mathbb{R}P^\infty; \mathbb{F}_2) = \mathbb{F}_2[u]$ with $|u| = 1$ we have $Sq(u) = u + u^2$, so $Sq(u^n) = (u + u^2)^n = \sum_{i=0}^n \binom{n}{i} u^{n+i}$. Hence $Sq^i(u^n) = \binom{n}{i} u^{n+i}$.

We outline one possible construction of the squaring operations. Let $H_n = K(\mathbb{F}_2, n)$ be an Eilenberg–Mac Lane complex of type $(\mathbb{F}_2, n)$, i.e., a space with $\pi_i(H_n) = \mathbb{F}_2$ for $i = n$ and 0 otherwise. For $n = 0$ we may take $H_0 = \mathbb{F}_2$. For $n \ge 1$ we may construct H_n from the Moore space $S^n \cup_2 e^{n+1}$ by the method of killing homotopy groups. Note that $H_1 \simeq \mathbb{R}P^\infty$.

There is a universal class $\iota_n \in \tilde{H}^n(H_n; \mathbb{F}_2)$ that corresponds to the identity homomorphism under the isomorphisms $\tilde{H}^n(H_n; \mathbb{F}_2) \cong \operatorname{Hom}(H_n(\mathbb{F}_2), \mathbb{F}_2) \cong \operatorname{Hom}(\mathbb{F}_2, \mathbb{F}_2)$. By a theorem of Eilenberg and Mac Lane, there is a natural isomorphism

$$[X, H_n] \simeq H^n(X; \mathbb{F}_2)$$

$$
\begin{array}{ll}
Sq^1Sq^1 = 0 & Sq^1Sq^2 = Sq^3 \\
Sq^1Sq^3 = 0 & Sq^2Sq^2 = Sq^3Sq^1 \\
Sq^1Sq^4 = Sq^5 & Sq^2Sq^3 = Sq^5 + Sq^4Sq^1 \\
Sq^3Sq^2 = 0 & Sq^1Sq^5 = 0 \\
Sq^2Sq^4 = Sq^6 + Sq^5Sq^1 & Sq^3Sq^3 = Sq^5Sq^1 \\
Sq^1Sq^6 = Sq^7 & Sq^2Sq^5 = Sq^6Sq^1 \\
Sq^3Sq^4 = Sq^7 & Sq^4Sq^3 = Sq^5Sq^2 \\
Sq^1Sq^7 = 0 & Sq^2Sq^6 = Sq^7Sq^1 \\
Sq^3Sq^5 = Sq^7Sq^1 & Sq^4Sq^4 = Sq^7Sq^1 + Sq^6Sq^2 \\
Sq^5Sq^3 = 0 & Sq^1Sq^8 = Sq^9 \\
Sq^2Sq^7 = Sq^9 + Sq^8Sq^1 & Sq^3Sq^6 = 0 \\
Sq^4Sq^5 = Sq^9 + Sq^8Sq^1 + Sq^7Sq^2 & Sq^5Sq^4 = Sq^7Sq^2 \\
Sq^1Sq^9 = 0 & Sq^2Sq^8 = Sq^{10} + Sq^9Sq^1 \\
Sq^3Sq^7 = Sq^9Sq^1 & Sq^4Sq^6 = Sq^{10} + Sq^8Sq^2 \\
Sq^5Sq^5 = Sq^9Sq^1 & Sq^6Sq^4 = Sq^7Sq^3 \\
Sq^1Sq^{10} = Sq^{11} & Sq^2Sq^9 = Sq^{10}Sq^1 \\
Sq^3Sq^8 = Sq^{11} & Sq^4Sq^7 = Sq^{11} + Sq^9Sq^2 \\
Sq^5Sq^6 = Sq^{11} + Sq^9Sq^2 & Sq^6Sq^5 = Sq^9Sq^2 + Sq^8Sq^3 \\
Sq^7Sq^4 = 0 &
\end{array}
$$

FIGURE 9. The Adem relations at $p = 2$ in degrees ≤ 11

that maps the homotopy class of $f\colon X \to H_n$ to $f^*(\iota_n)$. See Hatcher (2002, Theorem 4.57).

The smash product $\iota_n \wedge \iota_n \in \tilde{H}^{2n}(H_n \wedge H_n; \mathbb{F}_2)$ is represented by a map

$$\phi\colon H_n \wedge H_n \longrightarrow H_{2n}\,.$$

The composite $\phi\gamma$, where $\gamma\colon H_n \wedge H_n \to H_n \wedge H_n$ denotes the twist map, represents the same cohomology class, hence there is a homotopy $I_+ \wedge H_n \wedge H_n \longrightarrow H_{2n}$ from ϕ to $\phi\gamma$. We identify the interval I with the upper semicircle in S^1, and reinterpret this homotopy as a C_2-equivariant map $S^1_+ \wedge H_n \wedge H_n \longrightarrow H_{2n}$ where the generator of C_2 takes (s, x, y) to $(-s, y, x)$, and acts trivially on the target. Equivalently, it provides a map

$$\phi_1\colon S^1_+ \wedge_{C_2} H_n \wedge H_n \longrightarrow H_{2n}\,,$$

which expresses the homotopy commutativity of the cup product ϕ. There exists unique extensions, up to homotopy, $\phi_k\colon S^k_+ \wedge_{C_2} H_n \wedge H_n \longrightarrow H_{2n}$ of this map, for all $k \geq 2$, where C_2 acts antipodally on S^k. In the limit, these define a homotopy class of maps

$$\Phi\colon S^\infty_+ \wedge_{C_2} H_n \wedge H_n \longrightarrow H_{2n}\,,$$

where S^∞ is a contractible space with free C_2-action. We call $D_2(H_n) = S^\infty_+ \wedge_{C_2} H_n \wedge H_n$ the *quadratic construction* on H_n. The structure map $\Phi\colon D_2(H_n) \to H_{2n}$ is part of the E_∞ ring spectrum structure on the Eilenberg–Mac Lane spectrum $H = \{n \mapsto H_n\}$. Let

$$\nabla\colon \mathbb{R}P^\infty_+ \wedge H_n \longrightarrow S^\infty_+ \wedge_{C_2} H_n \wedge H_n$$

be given by $([s], x) = [s, x, x]$, for $s \in S^\infty$ with image $[s] \in \mathbb{R}P^\infty = S^\infty/C_2$. The composite map $\Phi\nabla$ induces a homomorphism

$$(\Phi\nabla)^*\colon \tilde{H}^*(H_{2n}\ \mathbb{F}_2) \longrightarrow H^*(\mathbb{R}P^\infty; \mathbb{F}_2) \otimes \tilde{H}^*(H_n; \mathbb{F}_2)\,.$$

Here $H^*(\mathbb{R}P^\infty; \mathbb{F}_2) = \mathbb{F}_2[u]$ with $|u| = 1$, so we can write $(\Phi\nabla)^*(\iota_{2n})$ in a unique way as a sum

$$(\Phi\nabla)^*(\iota_{2n}) = \sum_{i=0}^{n} u^{n-i} \otimes Sq^i(\iota_n)\,,$$

for some well-defined classes $Sq^i(\iota_n) \in \tilde{H}^{n+i}(H_n; \mathbb{F}_2)$, with $0 \le i \le n$. We define $Sq^i(\iota_n) = 0$ for $i < 0$ and for $i > n$. For a general class $x \in \tilde{H}^n(X; \mathbb{F}_2)$, write $x = f^*(\iota_n)$ for a map $f\colon X \to H_n$, and define

$$Sq^i(x) = f^*(Sq^i(\iota_n)) \in \tilde{H}^{n+i}(X; \mathbb{F}_2) .$$

This defines an operation

$$Sq^i \colon \tilde{H}^n(X; \mathbb{F}_2) \longrightarrow \tilde{H}^{n+i}(X; \mathbb{F}_2)$$

which is obviously natural in X.

It is easy to see that $Sq^n(x) = \phi^*(x \wedge x) = x^2$ for $|x| = n$, while checking that $Sq^0(x) = x$ and $Sq^1(x) = \beta(x)$ requires more work. [[Relate $Sq^{n-1}(x)$ to $x \cup_1 x$ derived from the commuting homotopy.]]

The situation at an odd prime p is similar.

Theorem 9.3. *There are natural transformations*

$$P^i \colon \tilde{H}^n(X; \mathbb{F}_p) \longrightarrow \tilde{H}^{n+2i(p-1)}(X; \mathbb{F}_p)$$

for $i \ge 0$, of contravariant functors from based spaces to abelian groups, called Steenrod's reduced powers. These satisfy $P^0(x) = x$, $P^i(x) = x^p$ for $2i = |x|$, and $P^i(x) = 0$ for $2i > |x|$. They also satisfy the Cartan formula

$$P^k(xy) = \sum_{i+j=k} P^i(x) P^j(y)$$

and the Adem relations

$$P^a P^b = \sum_{j=0}^{[a/p]} (-1)^{a+j} \binom{(p-1)(b-j)-1}{a-pj} P^{a+b-j} P^j$$

for $0 < a < pb$, and

$$P^a \beta P^b = \sum_{j=0}^{[a/p]} (-1)^{a+j} \binom{(p-1)(b-j)}{a-pj} \beta P^{a+b-j} P^j - \sum_{j=0}^{[(a-1)/p]} (-1)^{a+j} \binom{(p-1)(b-j)-1}{a-pj-1} P^{a+b-j} \beta P^j$$

for $0 < a \le pb$. Here $\beta\colon \tilde{H}^n(X; \mathbb{F}_p) \to \tilde{H}^{n+1}(X; \mathbb{F}_p)$ is the Bockstein homomorphism associated to the extension $\mathbb{F}_p \to \mathbb{Z}/p^2 \to \mathbb{F}_p$, which satisfies $\beta^2 = 0$.

The first few p-primary Adem relations (for $0 < a < p$ and $b = 1$) are

$$P^a P^1 = (-1)^a \binom{p-2}{a} P^{a+1}$$

and

$$P^a \beta P^1 = (-1)^a \binom{p-1}{a} \beta P^{a+1} - (-1)^a \binom{p-2}{a-1} P^{a+1} \beta .$$

They imply that $(P^1)^a$ is a unit in $\mathbb{F}_p$ times P^a, for all $0 < a < p$, that $(P^1)^p = 0$, and that $P^{p-1}\beta P^1 = \beta P^p - P^p \beta$.

9.2. **The Steenrod algebra.**

Definition 9.4. Let the mod 2 Steenrod algebra $\mathscr{A} = \mathscr{A}(2)$ be the graded (associative, unital) $\mathbb{F}_2$-algebra generated by the symbols Sq^i of degree i for $i \ge 0$, subject to the relations $Sq^0 = 1$ and $Sq^a Sq^b = \sum_j \binom{b-1-j}{a-2j} Sq^{a+b-j} Sq^j$ for $0 < a < 2b$.

For each finite sequence $I = (i_1, \dots, i_\ell)$ of non-negative integers we write

$$Sq^I = Sq^{i_1} \cdot \ldots \cdot Sq^{i_\ell}$$

for the product in $\mathscr{A}$. We say that I has length ℓ and degree $i_1 + \cdots + i_\ell$.

For any based space X, the reduced mod 2 cohomology $\tilde{H}^*(X; \mathbb{F}_2)$ is naturally a left $\mathscr{A}$-module, with the action given by

$$Sq^I(x) = Sq^{i_1}(\dots(Sq^{i_\ell}(x))\dots) .$$

We write

$$\lambda \colon \mathscr{A} \otimes \tilde{H}^*(X; \mathbb{F}_2) \longrightarrow \tilde{H}^*(X; \mathbb{F}_2)$$

for the left module action map.

n	admissible Sq^I of degree n
0	$Sq^{()}$
1	Sq^1
2	Sq^2
3	Sq^3, Sq^2Sq^1
4	Sq^4, Sq^3Sq^1
5	Sq^5, Sq^4Sq^1
6	Sq^6, Sq^5Sq^1, Sq^4Sq^2
7	$Sq^7, Sq^6Sq^1, Sq^5Sq^2, Sq^4Sq^2Sq^1$
8	$Sq^8, Sq^7Sq^1, Sq^6Sq^2, Sq^5Sq^2Sq^1$
9	$Sq^9, Sq^8Sq^1, Sq^7Sq^2, Sq^6Sq^3, Sq^6Sq^2Sq^1$
10	$Sq^{10}, Sq^9Sq^1, Sq^8Sq^2, Sq^7Sq^3, Sq^7Sq^2Sq^1, Sq^6Sq^3Sq^1$
11	$Sq^{11}, Sq^{10}Sq^1, Sq^9Sq^2, Sq^8Sq^3, Sq^8Sq^2Sq^1, Sq^7Sq^3Sq^1$

FIGURE 10. The admissible monomials at $p = 2$ in degrees ≤ 11

If $i_s < 2i_{s+1}$ for some $1 \le s < \ell$, then the product Sq^I can be rewritten in terms of other products Sq^J with lower moment $\sum_s sj_s < \sum_s si_s$. Likewise, if some $i_s = 0$, then the product Sq^I can be rewritten as a Sq^J of shorter length. Hence only the monomials Sq^I with I admissible, in the sense of the following definition, are needed to generate $\mathscr{A}$ additively.

Definition 9.5. $I = (i_1, \dots, i_\ell)$ is *admissible* if $i_s \ge 2i_{s+1}$ for all $1 \le s < \ell$, and if each $i_s \ge 1$. The empty sequence $I = ()$ is admissible of length 0, and $Sq^{()} = 1$.

See Figure 10 for the admissible monomials in degrees ≤ 11.

Theorem 9.6. *The admissible monomials Sq^I are linearly independent, hence form a vector space basis for the Steenrod algebra:*

$$\mathscr{A} = \mathbb{F}_2\{Sq^I \mid I \text{ admissible}\} .$$

This can be proved by evaluating the action of the Sq^I on the cohomology of products $X = (\mathbb{R}P^\infty)^n$ of many copies of $\mathbb{R}P^\infty$, see Steenrod and Epstein (1972, Theorem I.3.1).

Definition 9.7. For each odd prime p, let the mod p Steenrod algebra $\mathscr{A} = \mathscr{A}(p)$ be the graded $\mathbb{F}_p$-algebra generated by the symbols P^i of degree $2i(p-1)$ for $i \ge 0$ and β of degree 1, subject to the relations $P^0 = 1$, the Adem relation for P^aP^b when $0 < a < pb$, the Adem relation for $P^a\beta P^b$ when $0 < a \le pb$, and $\beta^2 = 0$.

For each finite sequence $I = (\epsilon_0, i_1, \epsilon_1, \dots, i_\ell, \epsilon_\ell)$, with $i_s \ge 0$ and $\epsilon_s \in \{0, 1\}$ for each s, we write

$$P^I = \beta^{\epsilon_0} P^{i_1} \beta^{\epsilon_1} \cdot \ldots \cdot P^{i_\ell} \beta^{\epsilon_\ell}$$

for the product in $\mathscr{A}$. Here $\beta^0 = 1$. The degree of I is $\epsilon_0 + 2i_1(p-1) + \epsilon_1 + \cdots + 2i_\ell(p-1) + \epsilon_\ell$.

For any based space X, the reduced mod p cohomology $\tilde{H}^*(X; \mathbb{F}_p)$ is naturally a left $\mathscr{A}$-module, with the action given by

$$P^I(x) = \beta^{\epsilon_0}(P^{i_1}(\beta^{\epsilon_1}(\dots(P^{i_\ell}(\beta^{\epsilon_\ell}(x)))\dots))) .$$

We write

$$\lambda \colon \mathscr{A} \otimes \tilde{H}^*(X; \mathbb{F}_p) \longrightarrow \tilde{H}^*(X; \mathbb{F}_p)$$

for the left module action map.

Definition 9.8. $I = (\epsilon_0, i_1, \epsilon_1 \dots, i_\ell, \epsilon_\ell)$ is *admissible* if $i_s \ge \epsilon_s + pi_{s+1}$ for all $1 \le s < \ell$, and if each $i_s \ge 1$. The empty sequence $I = ()$ is admissible of length 0, and $P^{()} = 1$.

See Figure 11 for the admissible monomials for $p = 3$ in degrees ≤ 19.

Theorem 9.9. *The admissible monomials P^I are linearly independent, hence form a vector space basis for the Steenrod algebra:*

$$\mathscr{A} = \mathbb{F}_p\{P^I \mid I \text{ admissible}\} .$$

See Steenrod and Epstein (1972, Theorem VI.2.5).

n	admissible P^I of degree n
0	$P^{()}$
1	β
4	P^1
5	$\beta P^1, P^1\beta$
6	$\beta P^1\beta$
8	P^2
9	$\beta P^2, P^2\beta$
10	$\beta P^2\beta$
12	P^p
13	$\beta P^p, P^p\beta$
14	$\beta P^p\beta$
16	P^{p+1}, P^pP^1
17	$\beta P^{p+1}, P^{p+1}\beta, \beta P^pP^1, P^pP^1\beta$
18	$\beta P^{p+1}\beta, \beta P^pP^1\beta$

FIGURE 11. The admissible monomials at $p = 3$ in degrees ≤ 19

9.3. Indecomposables and subalgebras.

Definition 9.10. For each prime p, let $\phi\colon \mathscr{A}\otimes\mathscr{A} \to \mathscr{A}$ be the algebra multiplication map, let $\eta\colon \mathbb{F}_p \to \mathscr{A}$ be the unit map, and let $\epsilon\colon \mathscr{A} \to \mathbb{F}_p$ be the counit map, so that $\epsilon\eta = 1$. Let $I(\mathscr{A}) = \ker(\epsilon)$ be the *augmentation ideal*. It equals the ideal of elements of positive degree in $\mathscr{A}$. The decomposable part of $\mathscr{A}$ is the image

$$I(\mathscr{A})^2 = \mathrm{im}(\phi\colon I(\mathscr{A}) \otimes I(\mathscr{A}) \longrightarrow I(\mathscr{A}))$$

and the *indecomposable* part of $\mathscr{A}$ is the $\mathbb{F}_p$-vector space

$$Q(\mathscr{A}) = I(\mathscr{A})/I(\mathscr{A})^2 .$$

Theorem 9.11. *The squaring operation Sq^k is decomposable if and only if $k = 2^i$ for some $i \ge 0$, so*

$$Q(\mathscr{A}) = \mathbb{F}_2\{Sq^{2^i} \mid i \ge 0\} .$$

Hence the elements Sq^{2^i} for $i \ge 0$ form a minimal set of algebra generators for $\mathscr{A} = \mathscr{A}(2)$.

Proof. To prove that Sq^{2^i} is indecomposable, consider its action on u^{2^i} in $H^*(\mathbb{R}P^\infty;\mathbb{F}_2)$. We have $Sq^j(u^{2^i}) = \binom{2^i}{j}u^{2^i+j}$, which is 0 for $0 < j < 2^i$ and equals 2^{i+1} for $j = 2^i$. It follows that Sq^{2^i} cannot be written as a sum of products of positive-degree operations.

The Adem relation for Sq^aSq^b with $0 < a < 2b$ shows that Sq^{a+b} is decomposable if $\binom{b-1}{a} \not\equiv 0 \mod 2$. If k is not a power of 2, then $k = a + b$ with $0 < a < b$ and $b = 2^i$, for some i. Then $\binom{b-1}{a} \equiv 1 \mod 2$ by the case $p = 2$ of the following lemma, since $b - 1 = 1 + 2 + \cdots + 2^{i-1}$ and $\binom{1}{0} = \binom{1}{1} = 1$. $\square$

Lemma 9.12. *Let $n = n_0 + n_1p + \cdots + n_\ell p^\ell$ and $k = k_0 + k_1p + \cdots + k_\ell p^\ell$, with $0 \le n_s, k_s < p$ for all s. Then*

$$\binom{n}{k} \equiv \prod_{s=0}^{\ell} \binom{n_s}{k_s} \mod p .$$

Proof. The coefficient of $x^k = \prod_s x^{k_sp^s}$ in

$$(1 + x)^n = \prod_s (1 + x)^{n_sp^s} \equiv \prod_s (1 + x^{p^s})^{n_s} \mod p$$

is the product over s of the coefficient of $x^{k_sp^s}$ in $(1 + x^{p^s})^{n_s}$. $\square$

Theorem 9.13. *The power operation P^k is decomposable if and only if $k = p^i$ for some $i \ge 0$, so*

$$Q(\mathscr{A}) = \mathbb{F}_p\{\beta, P^{p^i} \mid i \ge 0\} .$$

Hence the elements β and P^{p^i} for $i \ge 0$ form a minimal set of algebra generators for $\mathscr{A} = \mathscr{A}(p)$.

Example 9.14. For $p = 2$, the subalgebra of $\mathscr{A}$ generated by Sq^1 is the exterior algebra

$$A(0) = E(0) = \mathbb{F}_2\{1, Sq^1\}\,.$$

The subalgebra of $\mathscr{A}$ generated by Sq^1 and Sq^2 is the 8-dimensional algebra

$$A(1) = \mathbb{F}_2\{1, Sq^1, Sq^2, Sq^3, Sq^2Sq^1, Sq^3Sq^1, Sq^5 + Sq^4Sq^1, Sq^5Sq^1\}\,.$$

It can be displayed as follows, where for typographical reasons we write Sq^2Sq^3 for $Sq^5 + Sq^4Sq^1$.

$$\begin{tikzcd}
 & & & Sq^3 \arrow[drr] & & & \\
1 \arrow[r] \arrow[rr, bend left] & Sq^1 \arrow[drr] & Sq^2 \arrow[ur] \arrow[rr] & & Sq^3Sq^1 \arrow[rr, bend right] & Sq^2Sq^3 \arrow[r] & Sq^5Sq^1 \\
 & & & Sq^2Sq^1 \arrow[ur] & & &
\end{tikzcd}$$

The arrows of length 1 connect θ to $Sq^1\theta$, and the arrows of length 2 connect θ to $Sq^2\theta$, for $\theta \in A(1) \subset \mathscr{A}$.

[[Define $A(n)$ for general n?]]

Example 9.15. For p odd, the subalgebra of $\mathscr{A}$ generated by β is the exterior algebra

$$A(0) = E(0) = \mathbb{F}_p\{1, \beta\}\,.$$

The subalgebra of $\mathscr{A}$ generated by β and P^1 is the $4p$-dimensional algebra

$$A(1) = \mathbb{F}_p\{1, \beta, P^1, \beta P^1, P^1\beta, \beta P^1\beta, \dots, P^{p-1}, \beta P^{p-1}, P^{p-1}\beta, \beta P^{p-1}\beta, P^p\beta - \beta P^p, \beta P^p\beta\}\,.$$

For $p = 3$ it can be displayed as follows, where we use the notation $[P^p, \beta] = P^p\beta - \beta P^p$.

$$\begin{tikzcd}
 & & & \beta P^1 \arrow[drr] & \beta P^2 \arrow[drrr] & & & & \\
1 \arrow[r] \arrow[rr, bend left] & \beta \arrow[drr] & P^1 \arrow[ur] \arrow[rr, bend left, "2"] & \beta P^1\beta \arrow[rrr, bend right] & P^2 \arrow[u] & + & \beta P^2\beta \arrow[rr, bend right, "2"'] & {[P^p, \beta]} \arrow[r] & \beta P^p\beta \\
 & & & P^1\beta \arrow[u] \arrow[r, "2"] & P^2\beta \arrow[urr] & & & &
\end{tikzcd}$$

The arrows of length 1 connect θ to $\beta\theta$, and the arrows of length 4 connect θ to $P^1\theta$, except the arrows labeled '2', which connect θ to $2P^1\theta = -P^1\theta$. The arrow from βP^1 to the symbol '+' is meant to express that P^1 applied to βP^1 is the sum $\beta P^2 + P^2\beta$.

[[Define $A(n)$ for general n?]]

9.4. **Eilenberg–Mac Lane spectra.**

Definition 9.16. Let $H = H\mathbb{F}_p$ denote the mod p Eilenberg–Mac Lane spectrum, with n-th space H_n an Eilenberg–Mac Lane complex of type $(\mathbb{F}_p, n)$, for each $n \geq 0$. The structure map $\sigma\colon \Sigma H_n \to H_{n+1}$ is left adjoint to a homotopy equivalence $\tilde{\sigma}\colon H_n \xrightarrow{\simeq} \Omega H_{n+1}$, for each $n \geq 0$.

There are maps $\eta_n\colon S^n \to H_n$ and pairings $\phi_{m,n}\colon H_m \wedge H_n \to H_{m+n}$, suitably compatible with the spectrum structure maps, which define a unit map $\eta\colon S \to H$ and a pairing $\phi\colon H \wedge H \to H$ that make H a homotopy commutative ring spectrum. In particular, $\phi(\eta \wedge 1) \simeq 1 \simeq \phi(1 \wedge \eta)$ and $\phi(\phi \wedge 1) \simeq \phi(1 \wedge \phi)$.

[[This multiplication can be refined to that of an E_∞ ring spectrum, or a commutative structured ring spectrum.]]

Proposition 9.17 (Whitehead)**.** *There are natural isomorphisms*

$$H_n(Y; \mathbb{F}_p) \cong \pi_n(H \wedge Y) = [S^n, H \wedge Y]$$

and

$$H^n(Y; \mathbb{F}_p) \cong \pi_{-n}F(Y, H) = [Y, \Sigma^n H]$$

for all spectra Y and integers n.

The unit map η induces the mod p Hurewicz homomorphism

$$h_1 = \eta_*\colon \pi_n(Y) \longrightarrow H_n(Y; \mathbb{F}_p)\,.$$

The multiplication ϕ induces the smash product pairing

$$\wedge = \phi_* \colon H^m(X;\mathbb{F}_p) \otimes H^n(Y;\mathbb{F}_p) \longrightarrow H^{m+n}(X \wedge Y;\mathbb{F}_p)\,.$$

Using the Serre spectral sequence for the loop-path fibration over H_n, Serre and Cartan were able to calculate $H^*(H_n;\mathbb{F}_p)$ for $p = 2$ and for p odd, respectively. Recall the fundamental class $\iota_n \in \tilde{H}^n(H_n;\mathbb{F}_p)$.

Proposition 9.18 (Serre (1953), Cartan (1954))**.** *The homomorphism*

$$\Sigma^n \mathscr{A} \longrightarrow \tilde{H}^*(H_n;\mathbb{F}_p)\,,$$

mapping $\Sigma^n\theta$ to $\theta(\iota_n)$ for each $\theta \in \mathscr{A}$, induces an isomorphism in degrees $* \le 2n$. Hence there is an isomorphism

$$\mathscr{A} \xrightarrow{\cong} H^*(H;\mathbb{F}_p) = [H,H]_{-*}$$

of graded algebras, taking each class $\theta \in \mathscr{A}$ to its representing homotopy class of maps $H \to \Sigma^i H$, where $i = |\theta|$.

The second claim follows from the first, because of the exact sequence

$$0 \to \operatorname*{Rlim}_n H^{n+i-1}(H_n;\mathbb{F}_p) \longrightarrow H^i(H;\mathbb{F}_p) \longrightarrow \lim_n H^{n+i}(H_n;\mathbb{F}_p) \to 0\,.$$

The limit system stabilizes for $n \ge i$, so the derived limit is zero.

We collect a few lemmas relating maps of spectra to homomorphisms of cohomology groups.

Lemma 9.19. *Let*

$$K = \bigvee_u \Sigma^{n_u} H$$

be a wedge sum of suspensions of H, and suppose that K is bounded below and of finite type. Then the canonical map

$$K \xrightarrow{\simeq} \prod_u \Sigma^{n_u} H$$

is an equivalence, and the d-invariant

$$d\colon [X,K] \xrightarrow{\cong} \operatorname{Hom}_{\mathscr{A}}(H^*(K;\mathbb{F}_p), H^*(X;\mathbb{F}_p))$$

is an isomorphism, for any spectrum X. In particular,

$$d\colon \pi_t(K) \xrightarrow{\cong} \operatorname{Hom}^t_{\mathscr{A}}(H^*(K;\mathbb{F}_p),\mathbb{F}_p)\,.$$

We discussed this earlier, in subsection 6.3.

Lemma 9.20. *Let $\pi_*(Y)$ be bounded below, with $H_*(Y;\mathbb{F}_p) = \mathbb{F}_p\{\alpha_u\}_u$ of finite type. Let $\{a_u\}_u$ be the dual basis for $H^*(Y;\mathbb{F}_p)$, with $|a_u| = |\alpha_u| = n_u$. Let $\alpha_u \colon S^{n_u} \to H \wedge Y$ and $a_u \colon Y \to \Sigma^{n_u} H$ also denote the representing homotopy classes of maps. Then the sum of the composites*

$$\Sigma^{n_u} H = H \wedge S^{n_u} \xrightarrow{1\wedge\alpha_u} H \wedge H \wedge Y \xrightarrow{\phi\wedge 1} H \wedge Y$$

is an equivalence

$$\bigvee_u \Sigma^{n_u} H \xrightarrow{\simeq} H \wedge Y$$

and the product of the composites

$$H \wedge Y \xrightarrow{1\wedge a_u} H \wedge \Sigma^{n_u} H \xrightarrow{\phi\wedge 1} H \wedge S^{n_u} = \Sigma^{n_u} H$$

is an equivalence

$$H \wedge Y \xrightarrow{\simeq} \prod_n \Sigma^{n_u} H\,.$$

Proof. The two maps induce the isomorphisms

$$\bigoplus_u \Sigma^{n_u}\mathbb{F}_p \xrightarrow{\cong} H_*(Y;\mathbb{F}_p) \xrightarrow{\cong} \prod_u \Sigma^{n_u}\mathbb{F}_p$$

at the level of homotopy groups. □

Lemma 9.21. *Let $j \colon Y \to K$ be a map of spectra, with $K \simeq \bigvee_u \Sigma^{n_u} H$ bounded below and of finite type, and suppose that $j^* \colon H^*(K;\mathbb{F}_p) \to H^*(Y;\mathbb{F}_p)$ is surjective. Then a map $f \colon X \to Y$ induces the zero homomorphism $f^* \colon H^*(Y;\mathbb{F}_p) \to H^*(X;\mathbb{F}_p)$ if and only if the composite $jf \colon X \to K$ is null-homotopic.*

Proof. By Lemma 9.19,
$$[X, K] \cong \mathrm{Hom}_{\mathscr{A}}(H^*(K; \mathbb{F}_p), H^*(X; \mathbb{F}_p))$$
is an isomorphism. Hence jf is null-homotopic if and only if f^*j^* is zero. By assumption j^* is surjective, so this holds if and only if f^* is zero. □

Corollary 9.22. *Let Y be bounded below, with $H_*(Y; \mathbb{F}_p)$ of finite type. Then a map $f: X \to Y$ induces the zero homomorphism $f^*: H^*(Y; \mathbb{F}_p) \to H^*(X; \mathbb{F}_p)$ if and only if the composite*
$$X \xrightarrow{f} Y \xrightarrow{j} H \wedge Y$$
is null-homotopic, where
$$j = \eta \wedge 1: Y \cong S \wedge Y \longrightarrow H \wedge Y$$
induces the mod p Hurewicz homomorphism.

Proof. We only need to verify that j^* is surjective. It is dual to the homomorphism $j_*: H_*(Y; \mathbb{F}_p) \longrightarrow H_*(H \wedge Y; \mathbb{F}_p)$ induced by the map
$$1 \wedge \eta \wedge 1: H \wedge Y \cong H \wedge S \wedge Y \longrightarrow H \wedge H \wedge Y\,.$$
The ring spectrum multiplication
$$\phi \wedge 1: H \wedge H \wedge Y \longrightarrow H \wedge Y$$
induces a right inverse $H_*(H \wedge Y; \mathbb{F}_p) \to H_*(Y; \mathbb{F}_p)$ to j_*, showing that j_* is (split) injective and j^* is (split) surjective. □

10. The Adams spectral sequence

10.1. **Adams resolutions.**

Definition 10.1. A spectrum Y is *bounded below* if there exists an integer N such that $\pi_*(Y) = 0$ for all $* < N$. It is of *finite type* if $\pi_*(Y)$ is finitely generated in each degree. If Y is bounded below, then it is of finite type if and only if $H_*(Y; \mathbb{Z})$ is finitely generated in each degree [[explain this?]], and we say that it is *of finite type mod p* if $H_*(Y; \mathbb{F}_p)$ is finitely generated in each degree. This is equivalent to asking that $H_*(Y; \mathbb{F}_p)$ is finite in each degree.

Hereafter we fix a prime p, and briefly write $H_*Y = H_*(Y; \mathbb{F}_p)$ and $H^*Y = H^*(Y; \mathbb{F}_p)$ for mod p homology and cohomology.

Definition 10.2. An *mod p Adams resolution* of a spectrum Y is a diagram of spectra
$$\begin{tikzcd}
\cdots \arrow[r] & Y^2 \arrow[r, "i"] \arrow[d, "j"] & Y^1 \arrow[r, "i"] \arrow[d, "j"] & Y^0 \arrow[r, equal] \arrow[d, "j"] & Y \\
& K^2 \arrow[ul, dashed] & K^1 \arrow[ul, dashed, "\partial"] & K^0 \arrow[ul, dashed, "\partial"] &
\end{tikzcd}$$
where $\partial: K^s \to \Sigma Y^{s+1}$ for each $s \geq 0$, such that (a) each diagram
$$Y^{s+1} \xrightarrow{i} Y^s \xrightarrow{j} K^s \xrightarrow{\partial} \Sigma Y^{s+1}$$
is a homotopy cofiber sequence, (b) each spectrum K^s is a wedge sum of suspension of mod p Eilenberg–Mac Lane spectra, that is bounded below and of finite type, and (c) each map $j: Y^s \to K^s$ induces a surjection $j^*: H^*K^s \to H^*Y^s$ in mod p cohomology.

Writing $K^s = \bigvee_u \Sigma^{n_u} H$, the finiteness condition in (b) is equivalent to asking that $\{u \mid n_u \leq N\}$ is finite for each integer N. By induction on s it implies that each Y^s is bounded below and of finite type mod p. In view of the long exact sequence
$$\cdots \to H^*(\Sigma Y^{s+1}) \xrightarrow{\partial^*} H^*K^s \xrightarrow{j^*} H^*Y^s \xrightarrow{i^*} H^*Y^{s+1} \to \dots\,,$$
condition (c) is equivalent to asking that $i^*: H^*Y^s \to H^*Y^{s+1}$ is zero, or equivalently, that $\partial^*: H^*(\Sigma Y^{s+1}) \to H^*K^s$ is injective, for each $s \geq 0$. By the universal coefficient theorem, these conditions are also equivalent to asking that $i_*: H_*Y^{s+1} \to H_*Y^s$ is zero, that $j_*: H_*Y^s \to H_*K^s$ is injective, or that $\partial_*: H_*K_s \to H_*(\Sigma Y^{s+1})$ is surjective, for each $s \geq 0$.

Lemma 10.3. *If Y is bounded below and of finite type mod p, then it admits an Adams resolution.*

Proof. Starting with $Y^0 = Y$, suppose that Y^s has been constructed, for some $s \geq 0$, as a bounded below spectrum of finite type mod p. Let $K^s = H \wedge Y^s$, and let $j = \eta \wedge 1$ be the map

$$Y^s = S \wedge Y^s \xrightarrow{\eta\wedge 1} H \wedge Y^s = K^s \,.$$

By Lemma 9.20, $K^s \simeq \bigvee_u \Sigma^{n_u} H$ is a wedge sum of suspensions of H. Here

$$\pi_* K^s = H_* Y^s = \bigoplus_u \Sigma^{n_u} \mathbb{F}_p$$

is bounded below, and

$$H_* K^s \cong H_* H \otimes H_* Y^s$$

is a tensor product of bounded below $\mathbb{F}_p$-vector spaces of finite type, hence is again bounded below and of finite type. The map j induces a surjection $j^* \colon H^* K^s \to H^* Y^s$ by the proof of Corollary 9.22: It suffices to prove that $j_* \colon H_* Y^s \to H_* K^s$ is injective, but this is the homomorphism induced on homotopy by the map

$$1 \wedge j = 1 \wedge \eta \wedge 1 \colon H \wedge Y^s = H \wedge S \wedge Y^s \longrightarrow H \wedge H \wedge Y^s = H \wedge K^s$$

which is split by the map

$$\phi \wedge 1 \colon H \wedge H \wedge Y^s \longrightarrow H \wedge Y^s \,.$$

Let Y^{s+1} be the homotopy fiber of $j \colon Y^s \to K^s$, and let $i \colon Y^{s+1} \to Y^s$ be the canonical map from the homotopy fiber. Then there is a homotopy (co)fiber sequence

$$Y^{s+1} \xrightarrow{i} Y^s \xrightarrow{j} K^s \xrightarrow{\partial} \Sigma K^{s+1} \,,$$

which identifies the homotopy cofiber of j with the suspension of the homotopy fiber of j. By the long exact sequences in homotopy and mod p homology, it follows that Y^{s+1} is bounded below and of finite type mod p. Now continue the construction by induction. □

Let $\bar{H}$ be the homotopy cofiber of the unit map $\eta \colon S \to H$, so that there is a homotopy cofiber sequence

$$\Sigma^{-1}\bar{H} \longrightarrow S \xrightarrow{\eta} H \longrightarrow \bar{H} \,. \tag{4}$$

The homotopy cofiber constructed in the proof above can then be written as

$$\Sigma^{-1}\bar{H} \wedge Y^s \longrightarrow S \wedge Y^s \xrightarrow{\eta\wedge 1} H \wedge Y^s \longrightarrow \bar{H} \wedge Y^s \,.$$

By induction, we therefore have

$$Y^s = (\Sigma^{-1}\bar{H})^{\wedge s} \wedge Y$$
$$K^s = H \wedge (\Sigma^{-1}\bar{H})^{\wedge s} \wedge Y$$

for all $s \geq 0$, with $j = \eta \wedge 1$.

Definition 10.4. The *canonical Adams resolution* of Y is the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & (\Sigma^{-1}\bar{H})^{\wedge 2} \wedge Y \arrow[r, "i"] \arrow[d, "j"] & \Sigma^{-1}\bar{H} \wedge Y \arrow[r, "i"] \arrow[d, "j"] & Y \arrow[r, equal] \arrow[d, "j"] & Y \\
& H \wedge (\Sigma^{-1}\bar{H})^{\wedge 2} \wedge Y \arrow[ul, dashed] & H \wedge \Sigma^{-1}\bar{H} \wedge Y \arrow[ul, dashed, "\partial"] & H \wedge Y \arrow[ul, dashed, "\partial"] &
\end{tikzcd}$$

constructed in the proof above.

By the Künneth theorem,

$$H^* Y^s \cong H^*(\Sigma^{-1}\bar{H})^{\otimes s} \otimes H^* Y = (\Sigma^{-1} I(\mathscr{A}))^{\otimes s} \otimes H^* Y$$
$$H^* K^s \cong H^* H \otimes H^*(\Sigma^{-1}\bar{H})^{\otimes s} \otimes H^* Y = \mathscr{A} \otimes (\Sigma^{-1} I(\mathscr{A}))^{\otimes s} \otimes H^* Y$$

for each $s \geq 0$, with

$$j^* \colon H^* K^s \cong \mathscr{A} \otimes H^* Y^s \xrightarrow{\epsilon\otimes 1} \mathbb{F}_p \otimes H^* Y^s \cong H^* Y^s$$

induced by the augmentation $\epsilon \colon \mathscr{A} \to \mathbb{F}_p$ of the Steenrod algebra, and

$$\partial^* \colon H^* \Sigma Y^{s+1} = I(\mathscr{A}) \otimes H^* Y^s \longrightarrow \mathscr{A} \otimes H^* Y^s = H^* K^s$$

is induced by the inclusion $I(\mathscr{A}) \subset \mathscr{A}$. Note also that the canonical Adams resolution in natural in the spectrum Y.

The added generality of arbitrary Adams resolutions, as opposed to the canonical ones, will be useful when we consider convergence questions and multiplicative structure.

Lemma 10.5. *For any Adams resolution of Y, let*

$$\begin{aligned} P_s &= H^*(\Sigma^s K^s) \\ \partial_s &= \partial^* j^* \colon H^*(\Sigma^s K^s) \to H^*(\Sigma^{s-1} K^{s-1}) \end{aligned}$$

and $\epsilon = j^* \colon H^*K^0 \to H^*Y$. Then

$$\cdots \to P_s \xrightarrow{\partial_s} P_{s-1} \to \cdots \to P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\epsilon} H^*Y \to 0$$

is a resolution of H^*Y by free $\mathscr{A}$-modules, each of which is bounded below and of finite type.

With this indexing, the homomorphisms ∂_s and ϵ all preserve the cohomological grading of P_s and H^*Y, which we call the *internal degree* and usually denote by t.

Proof. By assumption (b), $K^s \simeq \bigvee_u \Sigma^{n_u} H$ with $\{u \mid n_u \le N\}$ finite for each N, so

$$H^*(K^s) \cong \prod_u \Sigma^{n_u} H^*(H) = \prod_u \Sigma^{n_u} \mathscr{A} = \bigoplus_u \Sigma^{n_u} \mathscr{A}$$

is a bounded below free $\mathscr{A}$-module of finite type, for each $s \ge 0$. Hence each $P_s = H^*(\Sigma^s K^s)$ is a bounded below free $\mathscr{A}$-module of finite type.

By assumption (c), the long exact sequence in cohomology of each cofiber sequence

$$Y^{s+1} \xrightarrow{i} Y^s \xrightarrow{j} K^s \xrightarrow{\partial} \Sigma Y^{s+1}$$

breaks up into short exact sequences of $\mathscr{A}$-modules

$$0 \to H^*(\Sigma Y^{s+1}) \xrightarrow{\partial^*} H^*(K^s) \xrightarrow{j^*} H^*(Y^s) \to 0 \,.$$

These splice together to a long exact sequence

$$\begin{tikzcd} H^*(\Sigma^3 Y^3) \arrow[dr, tail, "\partial^*"] & H^*(\Sigma^2 Y^2) \arrow[dr, tail, "\partial^*"] & H^*(\Sigma Y^1) \arrow[dr, tail, "\partial^*"] & H^*Y \\ \cdots \arrow[u, two heads] \arrow[r] & H^*(\Sigma^2 K^2) \arrow[u, two heads, "j^*"] \arrow[r, "\partial_2"'] & H^*(\Sigma K^1) \arrow[u, two heads, "j^*"] \arrow[r, "\partial_1"'] & H^*K^0 \arrow[u, two heads, "j^*"'] \end{tikzcd}$$

along the lower edge of this commutative diagram of graded $\mathscr{A}$-modules and degree-preserving homomorphisms. Alternatively, this diagram may be displayed as follows:

$$\begin{tikzcd} \cdots \arrow[dr] & & H^*(\Sigma^2 Y^2) \arrow[dr, tail, "\partial^*"] & & H^*(\Sigma Y^1) \arrow[dr, tail, "\partial^*"] & & H^*Y \\ \cdots \arrow[r] & H^*(\Sigma^2 K^2) \arrow[ur, two heads, "j^*"] \arrow[rr, "\partial_2"'] & & H^*(\Sigma K^1) \arrow[ur, two heads, "j^*"] \arrow[rr, "\partial_1"'] & & H^*K^0 \arrow[ur, two heads, "j^*"'] & \end{tikzcd}$$

Here $\operatorname{im}(\partial_{s+1}) = \operatorname{im}(\partial^*) = \ker(j^*) = \ker(\partial_s)$ as subgroups of $H^*(\Sigma^s K^s)$, for $s \ge 1$, since j^* is surjective and ∂^* is injective, $\operatorname{im}(\partial_1) = \operatorname{im}(\partial^*) = \ker(j^*)$ as subgroups of H^*K^0, since j^* is surjective, and $j^* \colon H^*K^0 \longrightarrow H^*Y$ is already known to be surjective. Hence $\epsilon \colon P_* \to H^*Y$ is a free $\mathscr{A}$-module resolution of H^*Y. □

We say that the Adams resolution $\{Y^{s+1} \to Y^s \to K^s \to \Sigma Y^{s+1}\}_s$ of Y is a *realization* of the free $\mathscr{A}$-module resolution $P_* = (P_s, \partial_s)$ of H^*Y. It is induced by passage to cohomology from the diagram

$$\cdots \leftarrow \Sigma^2 K^2 \xleftarrow{j\partial} \Sigma K^1 \xleftarrow{j\partial} K^0 \xleftarrow{j} Y \,,$$

where each composite of two maps is null-homotopic. In the case of the canonical resolution, this diagram appears as follows:

$$\cdots \leftarrow H \wedge (\bar{H})^{\wedge 2} \wedge Y \xleftarrow{j\partial} H \wedge \bar{H} \wedge Y \xleftarrow{j\partial} H \wedge Y \xleftarrow{j} Y \,.$$

The associated free resolution has the form

$$\cdots \to \mathscr{A} \otimes I(\mathscr{A})^{\otimes 2} \otimes H^*Y \xrightarrow{\partial_2} \mathscr{A} \otimes I(\mathscr{A}) \otimes H^*Y \xrightarrow{\partial_1} \mathscr{A} \otimes H^*Y \xrightarrow{\epsilon} H^*Y \to 0 \,.$$

Here

$$\epsilon(\theta_0 \otimes y) = \epsilon(\theta_0)y$$
$$\partial_1(\theta_0 \otimes \theta_1 \otimes y) = \epsilon(\theta_0)\theta_1 \otimes y$$
$$\partial_2(\theta_0 \otimes \theta_1 \otimes \theta_2 \otimes y) = \epsilon(\theta_0)\theta_1 \otimes \theta_2 \otimes y$$

and so on, where $\theta_0 \in \mathscr{A}$, $\theta_1, \dots, \theta_s \in I(\mathscr{A})$, $y \in H^*Y$, and $\epsilon(\theta_0)\theta_1$ is viewed as an element of $\mathscr{A}$.

We shall return to this complex later, in the context of the bar resolution of the $\mathscr{A}$-module H^*Y. The complex above is isomorphic to the bar resolution, but not equal to it. Note that each term $\mathscr{A} \otimes I(\mathscr{A})^{\otimes} \otimes H^*Y$ has the "diagonal" $\mathscr{A}$-module structure, prescribed by the Künneth theorem, which is not the same as the "induced" $\mathscr{A}$-module structure where $\mathscr{A}$ only acts on the leftmost tensor factor. Nonetheless, each term is free as an $\mathscr{A}$-module, by the argument given in Lemma 9.20 for the existence of a wedge sum decomposition of $K^s = H \wedge Y^s$.

10.2. **The Adams E_2-term.** We follows Adams (1958), using the spectrum level reformulation that appears in Moss (1968).

Let Y be a spectrum such that $\pi_*(Y)$ is bounded below and H_*Y is of finite type. Consider any Adams resolution

$$\begin{tikzcd} \cdots \arrow[r] & Y^2 \arrow[r, "i"] \arrow[d, "j"] & Y^1 \arrow[r, "i"] \arrow[d, "j"] & Y^0 \arrow[r, equal] \arrow[d, "j"] & Y \\ & K^2 \arrow[ul, dashed] & K^1 \arrow[ul, dashed, "\partial"] & K^0 \arrow[ul, dashed, "\partial"] & \end{tikzcd}$$

of Y. Applying homotopy groups, we get an unrolled exact couple of Adams type

$$\begin{tikzcd} \cdots \arrow[r] & \pi_*(Y^2) \arrow[r, "i_*"] \arrow[d, "j_*"] & \pi_*(Y^1) \arrow[r, "i_*"] \arrow[d, "j_*"] & \pi_*(Y^0) \arrow[r, equal] \arrow[d, "j_*"] & \pi_*(Y) \\ & \pi_*(K^2) \arrow[ul] & \pi_*(K^1) \arrow[ul, "\partial_*"] & \pi_*(K^0) \arrow[ul, "\partial_*"] & \end{tikzcd}$$

where

$$A^s = \pi_*(Y^s)$$
$$E^s = \pi_*(K^s)$$

are graded abelian groups, for each filtration degree $s \geq 0$, with components

$$A^{s,t} = \pi_{t-s}(Y^s)$$
$$E^{s,t} = \pi_{t-s}(K^s)$$

in each internal degree t. By convention, $A^s = A^0$ and $E^s = 0$ for $s < 0$. The homomorphisms $i = i_*$ and $j = j_*$ have degree 0, and $k = \partial_*$ has degree -1. There is an associated spectral sequence $(E_r, d_r)_r = (E_r^{*,*}, d_r^{*,*})_r$ of Adams type, with

$$E_1^{s,t} = \pi_{t-s}(K^s)$$

and

$$d_1^{s,t} = (j\partial)_* \colon \pi_{t-s}(K^s) \longrightarrow \pi_{t-s}(\Sigma K^{s+1}) = \pi_{t-s-1}(K^{s+1})$$

for $s \geq 0$. The d_r-differentials have bidegree $(r, r-1)$: If $x \in \pi_{t-s}(K^s) = E_1^{s,t}$ is such that $\partial_*(x) = i_*^{r-1}(y)$ for some $y \in \pi_{t-s-1}(Y^{s+r})$, then $d_r([x]) = [j_*(y)]$ is the class of $j_*(y) \in \pi_{t-s-1}(K^{s+r}) = E_1^{s+r,t+r-1}$.

This is the *Adams spectral sequence for* Y, sometimes denoted $E_r(Y) = E_r^{*,*}(Y)$. We shall be interested in the possible convergence of this spectral sequence to the achieved colimit

$$G = \pi_*(Y) = \operatorname*{colim}_s \pi_*(Y^s)\,,$$

filtered by the image groups

$$F^s = F^s\pi_*(Y) = \operatorname{im}(i_*^s \colon \pi_*(Y^s) \to \pi_*(Y))\,.$$

This is an exhaustive and descending filtration:

$$\cdots \subset F^{s+1} \subset F^s \subset \cdots \subset F^1 \subset F^0 = \pi_*(Y)\,.$$

We recall that, by definition, the spectral sequence is conditionally convergent to $\pi_*(Y)$ if $\lim_s \pi_*(Y^s) = 0$ and $\operatorname{Rlim}_s \pi_*(Y^s) = 0$.

Definition 10.6. An element in $E_r^{s,t}$ is said to be of *filtration* s, *total degree* $t-s$ and *internal degree* t. An element in $F^s \subset \pi_*(Y)$ is said to be of *Adams filtration* $\geq s$.

[[EDIT FROM HERE]]

A class in $\pi_*(Y)$ has Adams filtration 0 if it is detected by the d-invariant in $\pi_*(K^0)$, i.e., if it has non-zero mod 2 Hurewicz image. If the Hurewicz image is zero, then the class lifts to $\pi_*(Y^1)$. Then it has Adams filtration 1 if the lift is detected in $\pi_*(K^1)$, i.e., if the lift has non-zero mod 2 Hurewicz image. If also that Hurewicz image is zero, then the class lifts to $\pi_*(Y^2)$. And so on.

Theorem 10.7. *The E_2-term of the Adams spectral sequence of Y is*

$$E_2^{s,t} = \mathrm{Ext}_{\mathscr{A}}^{s,t}(H^*(Y), \mathbb{F}_2) .$$

In particular, it is independent of the choice of Adams resolution.

Proof. The Adams E_1-term and d_1-differential is the complex

$$\dots \longleftarrow \pi_*(\Sigma^2 K^2) \xleftarrow{(j\partial)_*} \pi_*(\Sigma K^1) \xleftarrow{(j\partial)_*} \pi_*(K^0) \longleftarrow 0$$

of graded abelian groups. It maps isomorphically, under the d-invariant $\pi_*(K) \to \mathrm{Hom}_{\mathscr{A}}(H^*(K), \mathbb{F}_2)$, to the complex

$$\dots \longleftarrow \mathrm{Hom}_{\mathscr{A}}(H^*(\Sigma^2 K^2), \mathbb{F}_2) \xleftarrow{((j\partial)^*)^*} \mathrm{Hom}_{\mathscr{A}}(H^*(\Sigma K^1), \mathbb{F}_2) \xleftarrow{((j\partial)^*)^*} \mathrm{Hom}_{\mathscr{A}}(H^*(K^0), \mathbb{F}_2) \longleftarrow 0$$

where $((j\partial)^*)^* = \mathrm{Hom}_{\mathscr{A}}((j\partial)^*, 1)$. With the notation of the previous subsection, this complex can be rewritten as

$$\dots \longleftarrow \mathrm{Hom}_{\mathscr{A}}(P_2, \mathbb{F}_2) \xleftarrow{\partial_2^*} \mathrm{Hom}_{\mathscr{A}}(P_1, \mathbb{F}_2) \xleftarrow{\partial_1^*} \mathrm{Hom}_{\mathscr{A}}(P_0, \mathbb{F}_2) \longleftarrow 0 .$$

This is the complex $\mathrm{Hom}_{\mathscr{A}}(P_*, \mathbb{F}_2)$ obtained by applying the functor $\mathrm{Hom}_{\mathscr{A}}(-, \mathbb{F}_2)$ to the resolution $\epsilon \colon P_* \to H^*(Y)$ of $H^*(Y)$ by free $\mathscr{A}$-modules. Its cohomology groups are by definition, the Ext-groups

$$\mathrm{Ext}_{\mathscr{A}}^s(H^*(Y), \mathbb{F}_2) = H^s(\mathrm{Hom}_{\mathscr{A}}(P_*, \mathbb{F}_2)) .$$

At the same time, the cohomology of the E_1-term of a spectral sequence is the E_2-term. Hence

$$E_2^s \cong \mathrm{Ext}_{\mathscr{A}}^s(H^*(Y), \mathbb{F}_2) .$$

As regards the internal grading, $E_1^{s,t} = \pi_{t-s}(K^s) \cong \pi_t(\Sigma^s K^s)$ corresponds to the $\mathscr{A}$-module homomorphisms $H^*(\Sigma^s K^s) \to \Sigma^t \mathbb{F}_2$. This is the same as the $\mathscr{A}$-module homomorphisms $H^*(\Sigma^s K^s) \to \mathbb{F}_2$ that lower the cohomological degrees by t. We denote the group of these homomorphisms by $\mathrm{Hom}_{\mathscr{A}}^t(H^*(\Sigma^s K^s), \mathbb{F}_2) = \mathrm{Hom}_{\mathscr{A}}^t(P_s, \mathbb{F}_2)$, and similarly for the derived functors. With these conventions, $E_2^{s,t} \cong \mathrm{Ext}_{\mathscr{A}}^{s,t}(H^*(Y), \mathbb{F}_2)$, as asserted. □

We are particularly interested in the special case $Y = S$, with $H^*(S) = \mathbb{F}_2$ and $\pi_*(S) = \pi_*^S$ equal to the stable homotopy groups of spheres.

Theorem 10.8. *The Adams spectral sequence for S has E_2-term*

$$E_2^{s,t} = \mathrm{Ext}_{\mathscr{A}}^{s,t}(\mathbb{F}_2, \mathbb{F}_2) .$$

On the other hand, we can also generalize (following Brinkmann (1968)). Let X be any spectrum and apply the functor $[X, -]_*$ to an Adams resolution of Y. This yields an unrolled exact couple

$$\begin{tikzcd}
\dots \arrow[r] & {[X,Y^2]_*} \arrow[r, "i_*"] \arrow[d, "j_*"] & {[X,Y^1]_*} \arrow[r, "i_*"] \arrow[d, "j_*"] & {[X,Y^0]_*} \arrow[r, equal] \arrow[d, "j_*"] & {[X,Y]_*} \\
& {[X,K^2]_*} \arrow[ul, "\partial_*"] & {[X,K^1]_*} \arrow[ul, "\partial_*"] & {[X,K^0]_*} \arrow[ul, "\partial_*"] &
\end{tikzcd}$$

where $A^s = [X, Y^s]_*$, $E^s = [X, K^s]_*$ are graded abelian groups, i_* and j_* have degree 0, and ∂_* has degree -1. There is an associated spectral sequence with

$$E_1^{s,t} = [X, K^s]_{t-s}$$

and

$$d_1^{s,t} = (j\partial)_* \colon [X, K^s]_{t-s} \to [X, K^{s+1}]_{t-s-1} .$$

The d_r-differentials have bidegree $(r, r-1)$. The expected abutment is the graded abelian group $G = [X, Y]_*$, filtered by the image groups $F^s = \mathrm{im}(i_*^s \colon [X, Y^s]_* \to [X, Y]_*)$.

Theorem 10.9. *The Adams spectral sequence $\{E_r(X,Y) = E_r^{*,*}(X,Y)\}_r$ of maps $X \to Y$, with expected abutment $[X,Y]_*$, has E_2-term*

$$E_2^{s,t} = \mathrm{Ext}_{\mathscr{A}}^{s,t}(H^*(Y), H^*(X)) .$$

The proof is the same as for $X = S$, replacing $\mathbb{F}_2$ by $H^*(X)$ in the right hand argument of all $\mathrm{Hom}_{\mathscr{A}}$- and $\mathrm{Ext}^s_{\mathscr{A}}$-groups. [[ETC]]

[[EDIT TO HERE]]

10.3. **A minimal resolution at** $p = 2$**.** To compute the Adams E_2-term for the sphere spectrum, at $p = 2$, we need to compute

$$\mathrm{Ext}_{\mathscr{A}}^{*,*}(\mathbb{F}_2, \mathbb{F}_2) = H^{*,*}(\mathrm{Hom}_{\mathscr{A}}(P_*, \mathbb{F}_2), \delta)$$

for any free $\mathscr{A}$-module resolution

$$\dots \longrightarrow P_s \xrightarrow{\partial_s} P_{s-1} \to \dots \to P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\epsilon} \mathbb{F}_2 \to 0$$

of $\mathbb{F}_2$, where $\delta = \mathrm{Hom}_{\mathscr{A}}(\partial, 1)$. We now construct such a free resolution P_* by hand, in a small range of degrees. We start in filtration degree $s = 0$, and calculate up to some internal degree t. Then we proceed with filtration degree $s = 1$, calculate up to the same internal degree t, and then repeat for larger s.

We need a surjection $\epsilon \colon P_0 \to \mathbb{F}_2$, so we let

$$P_0 = \mathscr{A}\{g_{0,0}\} = \mathscr{A}$$

be the free $\mathscr{A}$-module on a generator $g_{0,0} = 1$ in internal degree 0.

In this filtration a single generator suffices, but in higher filtrations, infinitely many generators will be needed. We will denote the generators in filtration s by $g_{s,0}$, $g_{s,1}$, $g_{s,2}$ and so on, in order of increasing (or more precisely, non-decreasing) internal degrees.

10.3.1. *Filtration* $s = 1$. Next, we need a surjection $\partial_1 \colon P_1 \to \ker(\epsilon)$, where $\ker(\epsilon) = I(\mathscr{A})$. An additive basis for $\ker(\epsilon)$ is given by the classes Sq^I for I admissible of length ≥ 1. We listed these monomials in internal degree $t \le 11$ in Figure 10.

Starting in the lowest degree, we first need a generator $g_{1,0} = [Sq^1]$ in internal degree 1 that is mapped by ∂_1 to Sq^1. The free summand $\mathscr{A}\{g_{1,0}\}$ of P_1 will then map by ∂_1 to all sums of classes of the form $Sq^I \circ Sq^1$ with I admissible. In view of the Adem relation $Sq^1 \circ Sq^1 = 0$, this image consists of all sums of classes of the form Sq^J, where $J = (j_1, \dots, j_\ell)$ is admissible and $j_\ell = 1$. See the left hand column of Figure 12.

The lowest-degree class in $\ker(\epsilon)$ that is not in the image from $\mathscr{A}\{g_{1,0}\}$ is Sq^2, in internal degree 2, so we must add a second generator $g_{1,1} = [Sq^2]$ to P_1, mapping under ∂_1 to Sq^2. Using the Adem relations, we can compute the image $Sq^I \circ Sq^2$ of each basis element $Sq^I g_{1,1}$ of $\mathscr{A}\{g_{1,1}\}$. See the right hand column of Figure 12.

The images of $Sq^2 g_{1,0}$ and $Sq^1 g_{1,1}$, namely $Sq^2 Sq^1$ and Sq^3, generate $\ker(\epsilon)$ in internal degree 3, but Sq^4 is not in the image from $\mathscr{A}\{g_{1,0}, g_{1,1}\}$, so we must add a third generator $g_{1,2} = [Sq^4]$ to P_1, mapping to $Sq^4 g_{0,0}$ under ∂_1. See the left hand column of Figure 13.

All classes in degree $t \le 7$ of $\ker(\epsilon)$ are then hit by ∂_1 on $\mathscr{A}\{g_{1,0}, g_{1,1}, g_{1,2}\}$, but $Sq^8 g_{0,0}$ is not in that image. We must therefore add a fourth generator $g_{1,3} = [Sq^8]$ to P_1, mapping to Sq^8. See the right hand column of Figure 13.

In general, we must add enough $\mathscr{A}$-module generators $g_{1,i}$ to P_1 so that their images $\partial_1(g_{1,i})$ generate the $\mathbb{F}_2$-vector space $Q(\mathscr{A}) = I(\mathscr{A})/I(\mathscr{A})^2$ of algebra indecomposables in the augmented algebra $\mathscr{A}$. This is necessary, since if $\partial_1 \colon P_1 \to \ker(\epsilon)$ is surjective, then so is its composite with the canonical map $\ker(\epsilon) = I(\mathscr{A}) \to Q(\mathscr{A})$. It is also a sufficient condition, because of $\partial_1 \colon P_1 \to \ker(\epsilon)$ is surjective below degree t and $P_1 \to Q(\mathscr{A})$ is surjective in degree t, then any chosen class in $I(\mathscr{A})$ of degree t is congruent modulo $I(\mathscr{A})^2$ to a class in the image of ∂_1. Any class in $I(\mathscr{A})^2$ is a sum of products of classes of degree less than t, hence is also in the image of ∂_1, by the assumption that ∂_1 is surjective below degree t. Thus the chosen class in $I(\mathscr{A})$ is also in the image of ∂_1. [[State this as a lemma?]]

By Theorem 9.11, the Sq^{2^i} for $i \ge 0$ give a basis for $Q(\mathscr{A})$, hence the minimal choice of a free $\mathscr{A}$-module mapping onto $\ker(\epsilon)$ is

$$P_1 = \mathscr{A}\{g_{1,0}, g_{1,1}, g_{1,2}, g_{1,3}, \dots\}$$

with $g_{1,i} = [Sq^{2^i}]$ in internal degree 2^i, for each $i \ge 0$. Here $g_{1,4}$ is in degree 16, hence the first four generators suffice in our smaller range of degrees.

[[Comment on how $\partial_1([\theta]) = \theta$, and how P_* relates to the bar resolution.]]

$$
\begin{aligned}
g_{1,0} = [Sq^1] &\xmapsto{\partial_1} Sq^1 \\
Sq^1[Sq^1] &\longmapsto 0 \\
Sq^2[Sq^1] &\longmapsto Sq^2Sq^1 \\
Sq^3[Sq^1] &\longmapsto Sq^3Sq^1 \\
Sq^2Sq^1[Sq^1] &\longmapsto 0 \\
Sq^4[Sq^1] &\longmapsto Sq^4Sq^1 \\
Sq^3Sq^1[Sq^1] &\longmapsto 0 \\
Sq^5[Sq^1] &\longmapsto Sq^5Sq^1 \\
Sq^4Sq^1[Sq^1] &\longmapsto 0 \\
Sq^6[Sq^1] &\longmapsto Sq^6Sq^1 \\
Sq^5Sq^1[Sq^1] &\longmapsto 0 \\
Sq^4Sq^2[Sq^1] &\longmapsto Sq^4Sq^2Sq^1 \\
Sq^7[Sq^1] &\longmapsto Sq^7Sq^1 \\
Sq^6Sq^1[Sq^1] &\longmapsto 0 \\
Sq^5Sq^2[Sq^1] &\longmapsto Sq^5Sq^2Sq^1 \\
Sq^4Sq^2Sq^1[Sq^1] &\longmapsto 0 \\
Sq^8[Sq^1] &\longmapsto Sq^8Sq^1 \\
Sq^7Sq^1[Sq^1] &\longmapsto 0 \\
Sq^6Sq^2[Sq^1] &\longmapsto Sq^6Sq^2Sq^1 \\
Sq^5Sq^2Sq^1[Sq^1] &\longmapsto 0 \\
Sq^9[Sq^1] &\longmapsto Sq^9Sq^1 \\
Sq^8Sq^1[Sq^1] &\longmapsto 0 \\
Sq^7Sq^2[Sq^1] &\longmapsto Sq^7Sq^2Sq^1 \\
Sq^6Sq^3[Sq^1] &\longmapsto Sq^6Sq^3Sq^1 \\
Sq^6Sq^2Sq^1[Sq^1] &\longmapsto 0 \\
Sq^{10}[Sq^1] &\longmapsto Sq^{10}Sq^1 \\
Sq^9Sq^1[Sq^1] &\longmapsto 0 \\
Sq^8Sq^2[Sq^1] &\longmapsto Sq^8Sq^2Sq^1 \\
Sq^7Sq^3[Sq^1] &\longmapsto Sq^7Sq^3Sq^1 \\
Sq^7Sq^2Sq^1[Sq^1] &\longmapsto 0
\end{aligned}
$$

$$
\begin{aligned}
g_{1,1} = [Sq^2] &\xmapsto{\partial_1} Sq^2 \\
Sq^1[Sq^2] &\longmapsto Sq^3 \\
Sq^2[Sq^2] &\longmapsto Sq^3Sq^1 \\
Sq^3[Sq^2] &\longmapsto 0 \\
Sq^2Sq^1[Sq^2] &\longmapsto Sq^5 + Sq^4Sq^1 \\
Sq^4[Sq^2] &\longmapsto Sq^4Sq^2 \\
Sq^3Sq^1[Sq^2] &\longmapsto Sq^5Sq^1 \\
Sq^5[Sq^2] &\longmapsto Sq^5Sq^2 \\
Sq^4Sq^1[Sq^2] &\longmapsto Sq^5Sq^2 \\
Sq^6[Sq^2] &\longmapsto Sq^6Sq^2 \\
Sq^5Sq^1[Sq^2] &\longmapsto 0 \\
Sq^4Sq^2[Sq^2] &\longmapsto Sq^5Sq^2Sq^1 \\
Sq^7[Sq^2] &\longmapsto Sq^7Sq^2 \\
Sq^6Sq^1[Sq^2] &\longmapsto Sq^6Sq^3 \\
Sq^5Sq^2[Sq^2] &\longmapsto 0 \\
Sq^4Sq^2Sq^1[Sq^2] &\longmapsto Sq^9 + Sq^8Sq^1 + Sq^7Sq^2 + Sq^6Sq^2Sq^1 \\
Sq^8[Sq^2] &\longmapsto Sq^8Sq^2 \\
Sq^7Sq^1[Sq^2] &\longmapsto Sq^7Sq^3 \\
Sq^6Sq^2[Sq^2] &\longmapsto Sq^6Sq^3Sq^1 \\
Sq^5Sq^2Sq^1[Sq^2] &\longmapsto Sq^9Sq^1 + Sq^7Sq^2Sq^1 \\
Sq^9[Sq^2] &\longmapsto Sq^9Sq^2 \\
Sq^8Sq^1[Sq^2] &\longmapsto Sq^8Sq^3 \\
Sq^7Sq^2[Sq^2] &\longmapsto Sq^7Sq^3Sq^1 \\
Sq^6Sq^3[Sq^2] &\longmapsto 0 \\
Sq^6Sq^2Sq^1[Sq^2] &\longmapsto Sq^9Sq^2 + Sq^8Sq^3 + Sq^7Sq^3Sq^1
\end{aligned}
$$

FIGURE 12. ∂_1 on $\mathscr{A}\{g_{1,0}, g_{1,1}\} \subset P_1$

10.3.2. *Filtration $s = 2$.* To continue, we need a surjection $\partial_2 \colon P_2 \to \ker(\partial_1)$. First we go through the linear algebra exercise of computing an additive basis for $\ker(\partial_1)$. The result is shown in Figure 14.

The class in lowest degree in $\ker(\partial_1)$ is $Sq^1[Sq^1]$, which corresponds to the Adem relation $Sq^1Sq^1 = 0$. We put a first generator $g_{2,0}$ of degree 2 in P_2, with $\partial_2(g_{2,0}) = Sq^1[Sq^1]$. See the left hand column of Figure 15.

The first class in $\ker(\partial_1)$ that is not in the image of ∂_2 on $\mathscr{A}\{g_{2,0}\}$ is $Sq^3[Sq^1] + Sq^2[Sq^2]$, which corresponds to the Adem relation $Sq^2Sq^2 = Sq^3Sq^1$. We add a second generator $g_{2,1}$ to P_2, in degree 4, with $\partial_2(g_{2,1}) = Sq^3[Sq^1] + Sq^2[Sq^2]$, and compute the value of $\partial_2(Sq^I g_{2,1}) = Sq^I(Sq^3[Sq^1] + Sq^2[Sq^2])$ in $\ker(\partial_1) \subset P_1$ for each admissible I, using the Adem relations. See the right hand column of Figure 15.

The lowest degree class not in the image of ∂_2 on $\mathscr{A}\{g_{2,0}, g_{2,1}\} \subset P_2$ is $Sq^4[Sq^1] + Sq^2Sq^1[Sq^2] + Sq^1[Sq^4]$, in degree 5. It corresponds to the Adem relation $Sq^2Sq^3 = Sq^5 + Sq^4Sq^1$, in view of the

$$
\begin{aligned}
g_{1,2} = [Sq^4] &\xmapsto{\partial_1} Sq^4 \\
Sq^1[Sq^4] &\longmapsto Sq^5 \\
Sq^2[Sq^4] &\longmapsto Sq^6 + Sq^5Sq^1 \\
Sq^3[Sq^4] &\longmapsto Sq^7 \\
Sq^2Sq^1[Sq^4] &\longmapsto Sq^6Sq^1 \\
Sq^4[Sq^4] &\longmapsto Sq^7Sq^1 + Sq^6Sq^2 \\
Sq^3Sq^1[Sq^4] &\longmapsto Sq^7Sq^1 \\
Sq^5[Sq^4] &\longmapsto Sq^7Sq^2 \\
Sq^4Sq^1[Sq^4] &\longmapsto Sq^9 + Sq^8Sq^1 + Sq^7Sq^2 \\
Sq^6[Sq^4] &\longmapsto Sq^7Sq^3 \\
Sq^5Sq^1[Sq^4] &\longmapsto Sq^9Sq^1 \\
Sq^4Sq^2[Sq^4] &\longmapsto Sq^{10} + Sq^9Sq^1 + Sq^8Sq^2 + Sq^7Sq^2Sq^1 \\
Sq^7[Sq^4] &\longmapsto 0 \\
Sq^6Sq^1[Sq^4] &\longmapsto Sq^9Sq^2 + Sq^8Sq^3 \\
Sq^5Sq^2[Sq^4] &\longmapsto Sq^{11} + Sq^9Sq^2 \\
Sq^4Sq^2Sq^1[Sq^4] &\longmapsto Sq^{10}Sq^1 + Sq^8Sq^2Sq^1
\end{aligned}
\qquad
\begin{aligned}
g_{1,3} = [Sq^8] &\xmapsto{\partial_1} Sq^8 \\
Sq^1[Sq^8] &\longmapsto Sq^9 \\
Sq^2[Sq^8] &\longmapsto Sq^{10} + Sq^9Sq^1 \\
Sq^3[Sq^8] &\longmapsto Sq^{11} \\
Sq^2Sq^1[Sq^8] &\longmapsto Sq^{10}Sq^1
\end{aligned}
$$

FIGURE 13. ∂_1 on $\mathscr{A}\{g_{1,2}, g_{1,3}\} \subset P_1$

identities $Sq^1Sq^2 = Sq^3$ and $Sq^1Sq^4 = Sq^5$. We add a third generator $g_{2,2}$ to P_2, with $\partial_2(g_{2,2}) = Sq^4[Sq^1] + Sq^2Sq^1[Sq^2] + Sq^1[Sq^4]$, and compute $\partial_2(Sq^I g_{2,2})$, as before. See Figure 16. [[At this point we deviate from the minimal resolution chosen by `ext`, which replaces $Sq^2Sq^1[Sq^2]$ with $Sq^{(0,1)}[Sq^2] = (Sq^3 + Sq^2Sq^1)[Sq^2]$.]]

The first class in $\ker(\partial_1)$ not in the image of ∂_2 on $\mathscr{A}\{g_{2,0}, g_{2,1}, g_{2,2}\}$ is $Sq^7[Sq^1]+Sq^6[Sq^2]+Sq^4[Sq^4]$. We add a fourth generator $g_{2,3}$ to P_2 in degree 8, corresponding to the Adem relation $Sq^4Sq^4 = Sq^7Sq^1 + Sq^6Sq^2$, and let $\partial_2(g_{2,3}) = Sq^7[Sq^1] + Sq^6[Sq^2] + Sq^4[Sq^4]$.

$$
\begin{aligned}
g_{2,3} &\xmapsto{\partial_2} Sq^7[Sq^1] + Sq^6[Sq^2] + Sq^4[Sq^4] \\
Sq^1 g_{2,3} &\longmapsto Sq^7[Sq^2] + Sq^5[Sq^4] \\
Sq^2 g_{2,3} &\longmapsto (Sq^9 + Sq^8Sq^1)[Sq^1] + Sq^7Sq^1[Sq^2] + (Sq^6 + Sq^5Sq^1)[Sq^4] \\
Sq^3 g_{2,3} &\longmapsto Sq^9Sq^1[Sq^1] + Sq^7[Sq^4] \\
Sq^2Sq^1 g_{2,3} &\longmapsto (Sq^9 + Sq^8Sq^1)[Sq^2] + Sq^6Sq^1[Sq^4]
\end{aligned}
$$

This still leaves $Sq^8[Sq^1] + Sq^7[Sq^2] + Sq^4Sq^1[Sq^4] + Sq^1[Sq^8]$ not in the image of ∂_2, so we add a fifth generator $g_{2,4}$ in degree 9, corresponding to the Adem relation $Sq^4Sq^5 = Sq^9 + Sq^8Sq^1 + Sq^7Sq^2$, and let $\partial_2(g_{2,4}) = Sq^8[Sq^1] + Sq^7[Sq^2] + Sq^4Sq^1[Sq^4] + Sq^1[Sq^8]$.

$$
\begin{aligned}
g_{2,4} &\xmapsto{\partial_2} Sq^8[Sq^1] + Sq^7[Sq^2] + Sq^4Sq^1[Sq^4] + Sq^1[Sq^8] \\
Sq^1 g_{2,4} &\longmapsto Sq^9[Sq^1] + Sq^5Sq^1[Sq^4] \\
Sq^2 g_{2,4} &\longmapsto (Sq^{10} + Sq^9Sq^1)[Sq^1] + (Sq^9 + Sq^8Sq^1)[Sq^2] + Sq^6Sq^1[Sq^4] + Sq^2Sq^1[Sq^8]
\end{aligned}
$$

Finally we need a sixth generator, $g_{2,5}$ in degree 10, mapping to $Sq^7Sq^2[Sq^1]+Sq^8[Sq^2]+Sq^4Sq^2[Sq^4]+Sq^2[Sq^8]$. It derives from the Adem relations for Sq^2Sq^8 and for Sq^4Sq^6, using the Adem relation for Sq^2Sq^4. [[Can we pick a different generator that corresponds to just a single Adem relation?]]

$$
\begin{aligned}
g_{2,5} &\xmapsto{\partial_2} Sq^7Sq^2[Sq^1] + Sq^8[Sq^2] + Sq^4Sq^2[Sq^4] + Sq^2[Sq^8] \\
Sq^1 g_{2,5} &\longmapsto Sq^9[Sq^2] + Sq^5Sq^2[Sq^4] + Sq^3[Sq^8]
\end{aligned}
$$

Now $\partial_2 \colon \mathscr{A}\{g_{2,0}, \dots, g_{2,5}\} \to \ker(\partial_1)$ is surjective in degrees $t \le 11$. In fact, it is surjective for $t \le 15$.

$Sq^1[Sq^1]$
$Sq^2Sq^1[Sq^1]$
$Sq^3[Sq^1] + Sq^2[Sq^2]$
$Sq^3Sq^1[Sq^1]$
$Sq^3[Sq^2]$
$Sq^4[Sq^1] + Sq^2Sq^1[Sq^2] + Sq^1[Sq^4]$
$Sq^4Sq^1[Sq^1]$
$Sq^5[Sq^1] + Sq^3Sq^1[Sq^2]$
$Sq^5Sq^1[Sq^1]$
$(Sq^5 + Sq^4Sq^1)[Sq^2]$
$Sq^6[Sq^1] + Sq^2Sq^1[Sq^4]$
$Sq^6Sq^1[Sq^1]$
$Sq^4Sq^2Sq^1[Sq^1]$
$Sq^5Sq^1[Sq^2]$
$Sq^5Sq^2[Sq^1] + Sq^4Sq^2[Sq^2]$
$Sq^7[Sq^1] + Sq^6[Sq^2] + Sq^4[Sq^4]$
$Sq^7[Sq^1] + Sq^3Sq^1[Sq^4]$
$Sq^7Sq^1[Sq^1]$
$Sq^5Sq^2Sq^1[Sq^1]$
$Sq^5Sq^2[Sq^2]$
$Sq^7[Sq^2] + Sq^5[Sq^4]$
$Sq^6Sq^2[Sq^1] + Sq^4Sq^2Sq^1[Sq^2] + Sq^4Sq^1[Sq^4]$
$Sq^8[Sq^1] + Sq^7[Sq^2] + Sq^4Sq^1[Sq^4] + Sq^1[Sq^8]$
$Sq^8Sq^1[Sq^1]$
$Sq^6Sq^2Sq^1[Sq^1]$
$Sq^6Sq^3[Sq^1] + Sq^6Sq^2[Sq^2]$
$(Sq^9 + Sq^7Sq^2)[Sq^1] + Sq^5Sq^2Sq^1[Sq^2]$
$Sq^7Sq^1[Sq^2] + Sq^6[Sq^4]$
$Sq^9[Sq^1] + Sq^5Sq^1[Sq^4]$
$Sq^7Sq^2[Sq^1] + Sq^8[Sq^2] + Sq^4Sq^2[Sq^4] + Sq^2[Sq^8]$
$Sq^7Sq^2[Sq^1] + Sq^8[Sq^2] + Sq^4Sq^2[Sq^4] + Sq^2[Sq^8]$
$Sq^9Sq^1[Sq^1]$
$Sq^7Sq^2Sq^1[Sq^1]$
$Sq^6Sq^3Sq^1[Sq^1]$
$Sq^7Sq^3[Sq^1] + Sq^7Sq^2[Sq^2]$
$Sq^6Sq^3[Sq^2]$
$Sq^7Sq^3[Sq^1] + (Sq^9 + Sq^8Sq^1 + Sq^6Sq^2Sq^1)[Sq^2]$
$Sq^7[Sq^4]$
$(Sq^9 + Sq^8Sq^1)[Sq^2] + Sq^6Sq^1[Sq^4]$
$(Sq^{10} + Sq^8Sq^2)[Sq^1] + Sq^4Sq^2Sq^1[Sq^4]$
$Sq^9[Sq^2] + Sq^5Sq^2[Sq^4] + Sq^3[Sq^8]$
$Sq^{10}[Sq^1] + Sq^2Sq^1[Sq^8]$

FIGURE 14. A basis for $\ker(\partial_1)$ in degrees ≤ 11

10.3.3. *Filtration* $s = 3$. We carry on to filtration degree $s = 3$, looking for a surjection $\partial_3 : P_3 \to \ker(\partial_2)$. First we must compute a basis for $\ker(\partial_2) \subset P_2$, in our range of degrees. The result is displayed in Figure 17.

As usual, the lowest degree class is $Sq^1 g_{2,0}$, so we first put a generator $g_{3,0}$ of degree 3 in P_3 with $\partial_3(g_{3,0}) = Sq^1 g_{2,0}$. The extension to $\mathscr{A}\{g_{3,0}\}$ is given in the left hand column of Figure 18.

The lowest class not in the image of this extension is $\partial_3(g_{3,1}) = Sq^4 g_{2,0} + Sq^2 g_{2,1} + Sq^1 g_{2,2}$ in degree 6. See the right hand column of Figure 18.

After this, the next class not in the image of ∂_3 on $\mathscr{A}\{g_{3,0}, g_{3,1}\}$ is $\partial_3(g_{3,2}) = Sq^8 g_{2,0} + (Sq^5 + Sq^4Sq^1)g_{2,2} + Sq^1 g_{2,4}$ in degree 10:

$$\begin{aligned} g_{3,2} &\xmapsto{\partial_3} Sq^8 g_{2,0} + (Sq^5 + Sq^4Sq^1)g_{2,2} + Sq^1 g_{2,4} \\ Sq^1 g_{3,2} &\longmapsto Sq^9 g_{2,0} + Sq^5Sq^1 g_{2,2} \end{aligned}$$

Finally, we need a fourth generator, $g_{3,3}$ in degree 11, with

$$g_{3,3} \xmapsto{\partial_3} Sq^4Sq^2Sq^1 g_{2,0} + Sq^6 g_{2,2} + Sq^2Sq^1 g_{2,3}\,.$$

(This generator will be particularly interesting when we get to the multiplicative structure in the Adams E_2-term, since it is dual to the indecomposable class c_0 in $\mathrm{Ext}^{3,11}_{\mathscr{A}}(\mathbb{F}_2, \mathbb{F}_2)$.) Then $\partial_3 : \mathscr{A}\{g_{3,0}, \dots, g_{3,3}\} \to \ker(\partial_2)$ is surjective in degrees $t \leq 11$.

$$
\begin{aligned}
g_{2,0} &\overset{\partial_2}{\longmapsto} Sq^1[Sq^1] \\
Sq^1 g_{2,0} &\longmapsto 0 \\
Sq^2 g_{2,0} &\longmapsto Sq^2Sq^1[Sq^1] \\
Sq^3 g_{2,0} &\longmapsto Sq^3Sq^1[Sq^1] \\
Sq^2Sq^1 g_{2,0} &\longmapsto 0 \\
Sq^4 g_{2,0} &\longmapsto Sq^4Sq^1[Sq^1] \\
Sq^3Sq^1 g_{2,0} &\longmapsto 0 \\
Sq^5 g_{2,0} &\longmapsto Sq^5Sq^1[Sq^1] \\
Sq^4Sq^1 g_{2,0} &\longmapsto 0 \\
Sq^6 g_{2,0} &\longmapsto Sq^6Sq^1[Sq^1] \\
Sq^5Sq^1 g_{2,0} &\longmapsto 0 \\
Sq^4Sq^2 g_{2,0} &\longmapsto Sq^4Sq^2Sq^1[Sq^1] \\
Sq^7 g_{2,0} &\longmapsto Sq^7Sq^1[Sq^1] \\
Sq^6Sq^1 g_{2,0} &\longmapsto 0 \\
Sq^5Sq^2 g_{2,0} &\longmapsto Sq^5Sq^2Sq^1[Sq^1] \\
Sq^4Sq^2Sq^1 g_{2,0} &\longmapsto 0 \\
Sq^8 g_{2,0} &\longmapsto Sq^8Sq^1[Sq^1] \\
Sq^7Sq^1 g_{2,0} &\longmapsto 0 \\
Sq^6Sq^2 g_{2,0} &\longmapsto Sq^6Sq^2Sq^1[Sq^1] \\
Sq^5Sq^2Sq^1 g_{2,0} &\longmapsto 0 \\
Sq^9 g_{2,0} &\longmapsto Sq^9Sq^1[Sq^1] \\
Sq^8Sq^1 g_{2,0} &\longmapsto 0 \\
Sq^7Sq^2 g_{2,0} &\longmapsto Sq^7Sq^2Sq^1[Sq^1] \\
Sq^6Sq^3 g_{2,0} &\longmapsto Sq^6Sq^3Sq^1[Sq^1] \\
Sq^6Sq^2Sq^1 g_{2,0} &\longmapsto 0
\end{aligned}
$$

$$
\begin{aligned}
g_{2,1} &\overset{\partial_2}{\longmapsto} Sq^3[Sq^1] + Sq^2[Sq^2] \\
Sq^1 g_{2,1} &\longmapsto Sq^3[Sq^2] \\
Sq^2 g_{2,1} &\longmapsto (Sq^5 + Sq^4Sq^1)[Sq^1] + Sq^3Sq^1[Sq^2] \\
Sq^3 g_{2,1} &\longmapsto Sq^5Sq^1[Sq^1] \\
Sq^2Sq^1 g_{2,1} &\longmapsto (Sq^5 + Sq^4Sq^1)[Sq^2] \\
Sq^4 g_{2,1} &\longmapsto Sq^5Sq^2[Sq^1] + Sq^4Sq^2[Sq^2] \\
Sq^3Sq^1 g_{2,1} &\longmapsto Sq^5Sq^1[Sq^2] \\
Sq^5 g_{2,1} &\longmapsto Sq^5Sq^2[Sq^2] \\
Sq^4Sq^1 g_{2,1} &\longmapsto Sq^5Sq^2[Sq^2] \\
Sq^6 g_{2,1} &\longmapsto Sq^6Sq^3[Sq^1] + Sq^6Sq^2[Sq^2] \\
Sq^5Sq^1 g_{2,1} &\longmapsto 0 \\
Sq^4Sq^2 g_{2,1} &\longmapsto (Sq^9 + Sq^8Sq^1 + Sq^7Sq^2 + Sq^6Sq^2Sq^1)[Sq^1]+ \\
&\qquad + Sq^5Sq^2Sq^1[Sq^2] \\
Sq^7 g_{2,1} &\longmapsto Sq^7Sq^3[Sq^1] + Sq^7Sq^2[Sq^2] \\
Sq^6Sq^1 g_{2,1} &\longmapsto Sq^6Sq^3[Sq^2] \\
Sq^5Sq^2 g_{2,1} &\longmapsto (Sq^9Sq^1 + Sq^7Sq^2Sq^1)[Sq^1] \\
Sq^4Sq^2Sq^1 g_{2,1} &\longmapsto (Sq^9 + Sq^8Sq^1 + Sq^7Sq^2 + Sq^6Sq^2Sq^1)[Sq^2]
\end{aligned}
$$

FIGURE 15. ∂_2 on $\mathscr{A}\{g_{2,0}, g_{2,1}\} \subset P_2$

10.3.4. *Filtration $s = 4$.* In degrees ≤ 11 we have an additive basis

$$
\begin{array}{ll}
Sq^1 g_{3,0} & Sq^6Sq^1 g_{3,0} \\
Sq^2Sq^1 g_{3,0} & Sq^4Sq^2Sq^1 g_{3,0} \\
Sq^3Sq^1 g_{3,0} & Sq^7Sq^1 g_{3,0} \\
Sq^4Sq^1 g_{3,0} & Sq^5Sq^2Sq^1 g_{3,0} \\
Sq^5Sq^1 g_{3,0} & Sq^8 g_{3,0} + (Sq^5 + Sq^4Sq^1) g_{3,1} + Sq^1 g_{3,2}
\end{array}
$$

for $\ker(\partial_3)$, and a surjection $\partial_4 \colon P_4 = \mathscr{A}\{g_{4,0}, g_{4,1}\} \to \ker(\partial_3)$ where

$$\partial_4(g_{4,0}) = Sq^1 g_{3,0}$$

in degree 4, and

$$\partial_4(g_{4,1}) = Sq^8 g_{3,0} + (Sq^5 + Sq^4Sq^1) g_{3,1} + Sq^1 g_{3,2}$$

in degree 11.

$$
\begin{aligned}
g_{2,2} &\xmapsto{\partial_2} Sq^4[Sq^1] + Sq^2Sq^1[Sq^2] + Sq^1[Sq^4] \\
Sq^1 g_{2,2} &\longmapsto Sq^5[Sq^1] + Sq^3Sq^1[Sq^2] \\
Sq^2 g_{2,2} &\longmapsto (Sq^6 + Sq^5Sq^1)[Sq^1] + Sq^2Sq^1[Sq^4] \\
Sq^3 g_{2,2} &\longmapsto Sq^7[Sq^1] + Sq^3Sq^1[Sq^4] \\
Sq^2Sq^1 g_{2,2} &\longmapsto Sq^6Sq^1[Sq^1] + Sq^5Sq^1[Sq^2] \\
Sq^4 g_{2,2} &\longmapsto (Sq^7Sq^1 + Sq^6Sq^2)[Sq^1] + Sq^4Sq^2Sq^1[Sq^2] + Sq^4Sq^1[Sq^4] \\
Sq^3Sq^1 g_{2,2} &\longmapsto Sq^7Sq^1[Sq^1] \\
Sq^5 g_{2,2} &\longmapsto Sq^7Sq^2[Sq^1] + Sq^5Sq^2Sq^1[Sq^2] + Sq^5Sq^1[Sq^4] \\
Sq^4Sq^1 g_{2,2} &\longmapsto (Sq^9 + Sq^8Sq^1 + Sq^7Sq^2)[Sq^1] + Sq^5Sq^2Sq^1[Sq^2] \\
Sq^6 g_{2,2} &\longmapsto Sq^7Sq^3[Sq^1] + Sq^6Sq^2Sq^1[Sq^2] + Sq^6Sq^1[Sq^4] \\
Sq^5Sq^1 g_{2,2} &\longmapsto Sq^9Sq^1[Sq^1] \\
Sq^4Sq^2 g_{2,2} &\longmapsto (Sq^{10} + Sq^9Sq^1 + Sq^8Sq^2 + Sq^7Sq^2Sq^1)[Sq^1] + Sq^4Sq^2Sq^1[Sq^4]
\end{aligned}
$$

FIGURE 16. ∂_2 on $\mathscr{A}\{g_{2,2}\} \subset P_2$

$$
\begin{array}{ll}
Sq^1 g_{2,0} & Sq^7Sq^1 g_{2,0} \\
Sq^2Sq^1 g_{2,0} & Sq^5Sq^2Sq^1 g_{2,0} \\
Sq^3Sq^1 g_{2,0} & Sq^5Sq^1 g_{2,1} \\
Sq^4 g_{2,0} + Sq^2 g_{2,1} + Sq^1 g_{2,2} & Sq^6Sq^2 g_{2,0} + Sq^4Sq^2 g_{2,1} + Sq^4Sq^1 g_{2,2} \\
Sq^4Sq^1 g_{2,0} & Sq^8 g_{2,0} + (Sq^5 + Sq^4Sq^1) g_{2,2} + Sq^1 g_{2,4} \\
Sq^5 g_{2,0} + Sq^3 g_{2,1} & Sq^8Sq^1 g_{2,0} \\
Sq^5Sq^1 g_{2,0} & Sq^6Sq^2Sq^1 g_{2,0} \\
Sq^6 g_{2,0} + Sq^3Sq^1 g_{2,1} + Sq^2Sq^1 g_{2,2} & (Sq^9 + Sq^7Sq^2) g_{2,0} + Sq^5Sq^2 g_{2,1} \\
Sq^6Sq^1 g_{2,0} & Sq^9 g_{2,0} + Sq^5Sq^1 g_{2,2} \\
Sq^4Sq^2Sq^1 g_{2,0} & Sq^4Sq^2Sq^1 g_{2,0} + Sq^6 g_{2,2} + Sq^2Sq^1 g_{2,3} \\
(Sq^5 + Sq^4Sq^1) g_{2,1} & \\
Sq^7 g_{2,0} + Sq^3Sq^1 g_{2,2} &
\end{array}
$$

FIGURE 17. A basis for $\ker(\partial_2)$ in degrees ≤ 11

10.3.5. *Filtration* $s \geq 5$. Things become quite simple from filtration degree $s = 5$ and onwards. In degrees ≤ 11 we have an additive basis

$$
\begin{array}{ll}
Sq^1 g_{4,0} & Sq^5Sq^1 g_{4,0} \\
Sq^2Sq^1 g_{4,0} & Sq^6Sq^1 g_{4,0} \\
Sq^3Sq^1 g_{4,0} & Sq^4Sq^2Sq^1 g_{4,0} \\
Sq^4Sq^1 g_{4,0} &
\end{array}
$$

for $\ker(\partial_4)$, and a surjection $\partial_5 \colon P_5 = \mathscr{A}\{g_{5,0}\} \to \ker(\partial_4)$ where $\partial_5(g_{5,0}) = Sq^1 g_{4,0}$ in degree 5. Continuing, we have a surjection $\partial_s \colon P_s = \mathscr{A}\{g_{s,0}\} \to \ker(\partial_{s-1})$ in degrees ≤ 11, where $\partial_s(g_{s,0}) = Sq^1 g_{s-1,0}$ in degree s, for all $5 \leq s \leq 11$.

Definition 10.10. We say that P_* is a *minimal resolution* when $\operatorname{im}(\partial_{s+1}) \subset I(\mathscr{A}) \cdot P_s$ for all $s \geq 0$. Then $1 \otimes \partial_{s+1} \colon \mathbb{F}_2 \otimes_{\mathscr{A}} P_{s+1} \to \mathbb{F}_2 \otimes_{\mathscr{A}} P_s$ and $\operatorname{Hom}(\partial_{s+1}, 1) \colon \operatorname{Hom}_{\mathscr{A}}(P_s, \mathbb{F}_2) \to \operatorname{Hom}_{\mathscr{A}}(P_{s+1}, \mathbb{F}_2)$ are the zero homomorphisms, so that

$$
\operatorname{Tor}_s^{\mathscr{A}}(\mathbb{F}_2, \mathbb{F}_2) = \mathbb{F}_2 \otimes_{\mathscr{A}} P_s = \mathbb{F}_2\{g_{s,i}\}_i
$$

$$
\begin{aligned}
g_{3,0} &\xmapsto{\partial_3} Sq^1 g_{2,0} \\
Sq^1 g_{3,0} &\longmapsto 0 \\
Sq^2 g_{3,0} &\longmapsto Sq^2 Sq^1 g_{2,0} \\
Sq^3 g_{3,0} &\longmapsto Sq^3 Sq^1 g_{2,0} \\
Sq^2 Sq^1 g_{3,0} &\longmapsto 0 \\
Sq^4 g_{3,0} &\longmapsto Sq^4 Sq^1 g_{2,0} \\
Sq^3 Sq^1 g_{3,0} &\longmapsto 0 \\
Sq^5 g_{3,0} &\longmapsto Sq^5 Sq^1 g_{2,0} \\
Sq^4 Sq^1 g_{3,0} &\longmapsto 0 \\
Sq^6 g_{3,0} &\longmapsto Sq^6 Sq^1 g_{2,0} \\
Sq^5 Sq^1 g_{3,0} &\longmapsto 0 \\
Sq^4 Sq^2 g_{3,0} &\longmapsto Sq^4 Sq^2 Sq^1 g_{2,0} \\
Sq^7 g_{3,0} &\longmapsto Sq^7 Sq^1 g_{2,0} \\
Sq^6 Sq^1 g_{3,0} &\longmapsto 0 \\
Sq^5 Sq^2 g_{3,0} &\longmapsto Sq^5 Sq^2 Sq^1 g_{2,0} \\
Sq^4 Sq^2 Sq^1 g_{3,0} &\longmapsto 0 \\
Sq^8 g_{3,0} &\longmapsto Sq^8 Sq^1 g_{2,0} \\
Sq^7 Sq^1 g_{3,0} &\longmapsto 0 \\
Sq^6 Sq^2 g_{3,0} &\longmapsto Sq^6 Sq^2 Sq^1 g_{2,0} \\
Sq^5 Sq^2 Sq^1 g_{3,0} &\longmapsto 0
\end{aligned}
$$

$$
\begin{aligned}
g_{3,1} &\xmapsto{\partial_3} Sq^4 g_{2,0} + Sq^2 g_{2,1} + Sq^1 g_{2,2} \\
Sq^1 g_{3,1} &\longmapsto Sq^5 g_{2,0} + Sq^3 g_{2,1} \\
Sq^2 g_{3,1} &\longmapsto (Sq^6 + Sq^5 Sq^1) g_{2,0} + Sq^3 Sq^1 g_{2,1} + Sq^2 Sq^1 g_{2,2} \\
Sq^3 g_{3,1} &\longmapsto Sq^7 g_{2,0} + Sq^3 Sq^1 g_{2,2} \\
Sq^2 Sq^1 g_{3,1} &\longmapsto Sq^6 Sq^1 g_{2,0} + (Sq^5 + Sq^4 Sq^1) g_{2,1} \\
Sq^4 g_{3,1} &\longmapsto (Sq^7 Sq^1 + Sq^6 Sq^2) g_{2,0} + Sq^4 Sq^2 g_{2,1} + Sq^4 Sq^1 g_{2,2} \\
Sq^3 Sq^1 g_{3,1} &\longmapsto Sq^7 Sq^1 g_{2,0} + Sq^5 Sq^1 g_{2,1} \\
Sq^5 g_{3,1} &\longmapsto Sq^7 Sq^2 g_{2,0} + Sq^5 Sq^2 g_{2,1} + Sq^5 Sq^1 g_{2,2} \\
Sq^4 Sq^1 g_{3,1} &\longmapsto (Sq^9 + Sq^8 Sq^1 + Sq^7 Sq^2) g_{2,0} + Sq^5 Sq^2 g_{2,1}
\end{aligned}
$$

FIGURE 18. ∂_3 on $\mathscr{A}\{g_{3,0}, g_{3,1}\} \subset P_3$

and

$$\mathrm{Ext}^s_{\mathscr{A}}(\mathbb{F}_2, \mathbb{F}_2) = \mathrm{Hom}_{\mathscr{A}}(P_s, \mathbb{F}_2) = \mathbb{F}_2\{g_{s,i}\}_i^*$$

for each $s \geq 0$, where $P_s = \mathscr{A}\{g_{s,i}\}_i$. Equivalently, the number of generators of P_s is minimal in each internal degree. (This number is finite, since $\mathscr{A}$ is of finite type.)

Theorem 10.11. *There is a minimal resolution $\epsilon\colon P_* \to \mathbb{F}_2$ with $P_0 = \mathscr{A}\{g_{0,0}\}$ and $P_s = \mathscr{A}\{g_{s,i} \mid i \geq 0\}$, where $\partial_s\colon P_s \to P_{s-1}$ is given in internal degrees $t \leq 11$ by*

$$
\begin{aligned}
\partial_1(g_{1,0}) &= Sq^1 g_{0,0} \\
\partial_1(g_{1,1}) &= Sq^2 g_{0,0} \\
\partial_1(g_{1,2}) &= Sq^4 g_{0,0} \\
\partial_1(g_{1,3}) &= Sq^8 g_{0,0} \\
\partial_2(g_{2,0}) &= Sq^1 g_{1,0} \\
\partial_2(g_{2,1}) &= Sq^3 g_{1,0} + Sq^2 g_{1,1} \\
\partial_2(g_{2,2}) &= Sq^4 g_{1,0} + Sq^2 Sq^1 g_{1,1} + Sq^1 g_{1,2} \\
\partial_2(g_{2,3}) &= Sq^7 g_{1,0} + Sq^6 g_{1,1} + Sq^4 g_{1,2} \\
\partial_2(g_{2,4}) &= Sq^8 g_{1,0} + Sq^7 g_{1,1} + Sq^4 Sq^1 g_{1,2} + Sq^1 g_{1,3} \\
\partial_2(g_{2,5}) &= Sq^7 Sq^2 g_{1,0} + Sq^8 g_{1,1} + Sq^4 Sq^2 g_{1,2} + Sq^2 g_{1,3} \\
\partial_3(g_{3,0}) &= Sq^1 g_{2,0} \\
\partial_3(g_{3,1}) &= Sq^4 g_{2,0} + Sq^2 g_{2,1} + Sq^1 g_{2,2} \\
\partial_3(g_{3,2}) &= Sq^8 g_{2,0} + (Sq^5 + Sq^4 Sq^1) g_{2,2} + Sq^1 g_{2,4} \\
\partial_3(g_{3,3}) &= (Sq^7 + Sq^4 Sq^2 Sq^1) g_{2,1} + Sq^6 g_{2,2} + Sq^2 Sq^1 g_{2,3} \\
\partial_4(g_{4,0}) &= Sq^1 g_{3,0} \\
\partial_4(g_{4,1}) &= Sq^8 g_{3,0} + (Sq^5 + Sq^4 Sq^1) g_{3,1} + Sq^1 g_{3,2} \\
\partial_5(g_{5,0}) &= Sq^1 g_{4,0} \\
&\dots \\
\partial_{11}(g_{11,0}) &= Sq^1 g_{10,0}\,.
\end{aligned}
$$

Proof. This summarizes the calculations above. The resolution is minimal, since we only added generators $g_{s,i}$ with $\partial_s(g_{s,i}) \in I(\mathscr{A}) \cdot P_{s-1} = I(\mathscr{A})\{g_{s-1,j}\}_j$. It should be clear that we can continue that way, since $\mathscr{A}$ is connected. If any sum involving $1 \cdot g_{s,i}$ occurs in $\ker(\partial_s)$, then $g_{s,i}$ could be omitted from the basis for P_s and $\partial_s\colon P_s \to \ker(\partial_{s-1})$ would still be surjective. $\square$

Theorem 10.12. $\mathrm{Ext}^{s,*}_{\mathscr{A}}(\mathbb{F}_2, \mathbb{F}_2) \cong \mathbb{F}_2\{\gamma_{s,i}\}_i$ *where $\gamma_{s,i}\colon P_s \to \mathbb{F}_2$ is the $\mathscr{A}$-module homomorphism dual to $g_{s,i}$, for each $s \geq 0$. The bidegrees of the generators in internal degrees $t \leq 11$ are as displayed in the following chart. The horizontal coordinate is the topological degree $t - s$, the vertical coordinate is the* *cohomological degree s, and the sum of these coordinates is the internal degree t.*

	$\gamma_{11,0}$	·	·	·	·	·	·	·	·	·	·	·
10	$\gamma_{10,0}$		·	·	·	·	·	·	·	·	·	·
	$\gamma_{9,0}$			·	·	·	·	·	·	·	·	·
8	$\gamma_{8,0}$				·	·	·	·	·	·	·	·
	$\gamma_{7,0}$					·	·	·	·	·	·	?
6	$\gamma_{6,0}$						·	·	·	·	?	?
	$\gamma_{5,0}$							·	·	?	?	?
4	$\gamma_{4,0}$							$\gamma_{4,1}$	?	?	?	?
	$\gamma_{3,0}$			$\gamma_{3,1}$				$\gamma_{3,2}$	$\gamma_{3,3}$	?	?	?
2	$\gamma_{2,0}$		$\gamma_{2,1}$	$\gamma_{2,2}$			$\gamma_{2,3}$	$\gamma_{2,4}$	$\gamma_{2,5}$		?	?
	$\gamma_{1,0}$	$\gamma_{1,1}$		$\gamma_{1,2}$				$\gamma_{1,3}$				?
0	$\gamma_{0,0}$											
	0		2		4		6		8		10	

We have not yet computed the groups labeled · or ?, but we will prove below that the groups labeled · are 0. (This is the Adams (1966) vanishing theorem.) In fact, many of the groups labeled ? are also zero.

Proof. For each $s \ge 0$ we have $\mathrm{Hom}_{\mathscr{A}}(P_s, \mathbb{F}_2) \cong \mathrm{Hom}_{\mathscr{A}}(\mathscr{A}\{g_{s,i}\}_i, \mathbb{F}_2) \cong \prod_i \mathbb{F}_2\{\gamma_{s,i}\}$, where $\gamma_{s,i}(g_{s,j}) = \delta_{i,j}$ is 1 if $i = j$ and 0 otherwise. It will be clear later that there are at most finitely many $g_{s,i}$ in a given bidegree, so this product is finite in each degree. Then $\gamma_{s,i} \circ \partial_{s+1} = 0$, so the cocomplex $\mathrm{Hom}_{\mathscr{A}}(P_*, \mathbb{F}_2)$ has trivial coboundary. Hence $\mathrm{Ext}^s_{\mathscr{A}}(\mathbb{F}_2, \mathbb{F}_2) \cong \mathrm{Hom}_{\mathscr{A}}(P_s, \mathbb{F}_2) \cong \mathbb{F}_2\{\gamma_{s,i}\}_i$, as claimed. □

Lemma 10.13. *Let $\epsilon \colon P_* \to \mathbb{F}_2$ be a free $\mathscr{A}$-module resolution. Then $\mathrm{Hom}_{\mathscr{A}}(P_s, \mathbb{F}_2) \cong \mathrm{Hom}(\mathbb{F}_2 \otimes_{\mathscr{A}} P_s, \mathbb{F}_2)$, so there is an isomorphism $\mathrm{Ext}^{s,t}_{\mathscr{A}}(\mathbb{F}_2, \mathbb{F}_2) \cong \mathrm{Hom}(\mathrm{Tor}^{\mathscr{A}}_{s,t}(\mathbb{F}_2, \mathbb{F}_2), \mathbb{F}_2)$.*

10.4. **A minimal resolution at $p = 3$.** Now consider the case of an odd prime p. The mod p Adams E_2-term for the sphere spectrum is

$$E_2^{*,*} = \mathrm{Ext}^{*,*}_{\mathscr{A}}(\mathbb{F}_p, \mathbb{F}_p) = H^{*,*}(\mathrm{Hom}_{\mathscr{A}}(P_*, \mathbb{F}_p), \delta)\,,$$

where

$$\dots \longrightarrow P_s \xrightarrow{\partial_s} P_{s-1} \to \dots \to P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\epsilon} \mathbb{F}_p \to 0$$

is any free $\mathscr{A}$-module resolution of $\mathbb{F}_p$ and $\delta = \mathrm{Hom}_{\mathscr{A}}(\partial, 1)$.

We calculate a minimal such resolution for $p = 3$ in internal degrees $t < 2p^2 - 2 = 16$. To begin, let $P_0 = \mathscr{A}\{g_{0,0}\} \cong \mathscr{A}$, with $g_{0,0}$ in degree 0 and $\epsilon(g_{0,0}) = 1$. The admissible monomials

$$\beta, P^1, \beta P^1, P^1\beta, \beta P^1\beta, P^2, \beta P^2, P^2\beta, \beta P^2\beta, P^p, \beta P^p, P^p\beta, \beta P^p\beta$$

form a basis for $\ker(\epsilon) = I(\mathscr{A})$ in degrees $t < 16$.

10.4.1. *Filtration $s = 1$.* To define a surjection $\partial_1 \colon P_1 \to \ker(\epsilon)$, it suffices to add generators to P_1 that map to a basis for the algebra indecomposables

$$Q(\mathscr{A}) = I(\mathscr{A})/I(\mathscr{A})^2 = \mathbb{F}_p\{\beta, P^1, P^p, \dots\}\,.$$

Let

$$P_1 = \mathscr{A}\{g_{1,0}, g_{1,1}, g_{1,2}, \dots\}$$

be generated by $g_{1,0}$ in degree $t = 1$ with $\partial_1(g_{1,0}) = \beta g_{0,0}$, $g_{1,1}$ in degree $t = 2p - 2 = 4$ with $\partial_1(g_{1,1}) = P^1 g_{0,0}$, $g_{1,2}$ in degree $t = 2p^2 - 2p = 12$ with $\partial_1(g_{1,2}) = P^p g_{0,0}$, and so on. In general, $g_{1,i+1}$ in degree $t = 2p^i(p-1)$ maps to $P^{p^i} g_{0,0}$ for each $i \ge 0$. The boundary ∂_1 is given in Figure 19, in internal degrees $t \le 15$. A basis for its kernel is shown in Figure 20, in the same range of degrees.

$$
\begin{aligned}
g_{1,0} &\overset{\partial_1}{\longmapsto} \beta g_{0,0} \\
\beta g_{1,0} &\longmapsto 0 \\
g_{1,1} &\longmapsto P^1 g_{0,0} \\
P^1 g_{1,0} &\longmapsto P^1 \beta g_{0,0} \\
\beta g_{1,1} &\longmapsto \beta P^1 g_{0,0} \\
\beta P^1 g_{1,0} &\longmapsto \beta P^1 \beta g_{0,0} \\
P^1 \beta g_{1,0} &\longmapsto 0 \\
\beta P^1 \beta g_{1,0} &\longmapsto 0 \\
P^1 g_{1,1} &\longmapsto P^1 P^1 g_{0,0} = 2P^2 g_{0,0} \\
P^2 g_{1,0} &\longmapsto P^2 \beta g_{0,0} \\
\beta P^1 g_{1,1} &\longmapsto 2\beta P^2 g_{0,0} \\
P^1 \beta g_{1,1} &\longmapsto P^1 \beta P^1 g_{0,0} = (\beta P^2 + P^2 \beta) g_{0,0} \\
\beta P^2 g_{1,0} &\longmapsto \beta P^2 \beta g_{0,0} \\
P^2 \beta g_{1,0} &\longmapsto 0 \\
\beta P^1 \beta g_{1,1} &\longmapsto \beta P^2 \beta g_{0,0} \\
\beta P^2 \beta g_{1,0} &\longmapsto 0 \\
P^2 g_{1,1} &\longmapsto 0 \\
g_{1,2} &\longmapsto P^p g_{0,0} \\
P^p g_{1,0} &\longmapsto P^p \beta g_{0,0} \\
\beta P^2 g_{1,1} &\longmapsto 0 \\
P^2 \beta g_{1,1} &\longmapsto P^2 \beta P^1 g_{0,0} = (\beta P^p - P^p \beta) g_{0,0} \\
\beta g_{1,2} &\longmapsto \beta P^p g_{0,0} \\
\beta P^p g_{1,0} &\longmapsto \beta P^p \beta g_{0,0} \\
P^p \beta g_{1,0} &\longmapsto 0 \\
\beta P^2 \beta g_{1,1} &\longmapsto -\beta P^p \beta g_{0,0} \\
\beta P^p \beta g_{1,0} &\longmapsto 0
\end{aligned}
$$

FIGURE 19. $\partial_1 \colon P_1 \to P_0$ for $p = 3$

10.4.2. *Filtration* $s = 2$. Next we define a surjection $\partial_2 \colon P_2 \to \ker(\partial_1)$. Let

$$P_2 = \mathscr{A}\{g_{2,0}, g_{2,1}, g_{2,2}, g_{2,3}, \dots\}$$

be generated by $g_{2,0}$ in degree $t = 2$ with $\partial_2(g_{2,0}) = \beta g_{1,0}$, by $g_{2,1}$ in degree $t = 4p - 3 = 9$ with $\partial_2(g_{2,1}) = 2P^2 g_{1,0} + (\beta P^1 - 2P^1\beta) g_{1,1}$, by $g_{2,2}$ in degree $2p^2 - 2p = 12$ with $\partial_2(g_{2,2}) = P^{p-1} g_{1,1} = P^2 g_{1,1}$, by $g_{2,3}$ in degree $2p^2 - 2p + 1 = 13$ with $\partial_2(g_{2,3}) = P^p g_{1,0} + P^2 \beta g_{1,1} - \beta g_{1,2}$, and so on. Note how $g_{2,0}$ corresponds to the relation $\beta^2 = 0$, $g_{2,1}$ corresponds to the Adem relation $P^1 \beta P^1 = \beta P^2 + P^2 \beta$ (and $P^1 P^1 = 2P^2$), and $g_{2,2}$ corresponds to the Adem relation $P^{p-1} P^1 = 0$. [[Continue with $g_{2,3}$.]] The boundary ∂_2 is given in Figure 21, in degrees $t \le 15$. A basis for its kernel is shown in Figure 22, in the same degrees.

10.4.3. *Filtration* $s = 3$. We continue by defining a surjection $\partial_3 \colon P_3 \to \ker(\partial_2)$. Let

$$P_3 = \mathscr{A}\{g_{3,0}, g_{3,1}, g_{3,2}, \dots\}$$

be generated by $g_{3,0}$ in degree $t = 3$ with $\partial_3(g_{3,0}) = \beta g_{2,0}$, by $g_{3,1}$ in degree $t = (?) = 13$ with $\partial_3(g_{3,1}) = P^1 g_{2,1} - \beta g_{2,2}$, by $g_{3,2}$ in degree $t = (?) = 14$ with $\partial_3(g_{3,2}) = P^p g_{2,0} + P^1 \beta g_{2,1} - \beta g_{2,3}$, and so on. The boundary ∂_3 is given in Figure 23, in degrees $t \le 15$. A basis for its kernel is shown in Figure 24, in the same range.

$$\beta g_{1,0}$$
$$P^1\beta g_{1,0}$$
$$\beta P^1\beta g_{1,0}$$
$$2P^2 g_{1,0} + (\beta P^1 - 2P^1\beta) g_{1,1}$$
$$P^2\beta g_{1,0}$$
$$\beta P^2 g_{1,0} - \beta P^1\beta g_{1,1}$$
$$\beta P^2\beta g_{1,0}$$
$$P^2 g_{1,1}$$
$$P^p g_{1,0} + P^2\beta g_{1,1} - \beta g_{1,2}$$
$$\beta P^2 g_{1,1}$$
$$\beta P^p g_{1,0} + \beta P^2\beta g_{1,1}$$
$$P^p\beta g_{1,0}$$
$$\beta P^p\beta g_{1,0}$$

FIGURE 20. A basis for $\ker(\partial_1)$ at $p = 3$

$$\begin{aligned}
g_{2,0} &\stackrel{\partial_2}{\longmapsto} \beta g_{1,0}\\
\beta g_{2,0} &\longmapsto 0\\
P^1 g_{2,0} &\longmapsto P^1\beta g_{1,0}\\
\beta P^1 g_{2,0} &\longmapsto \beta P^1\beta g_{1,0}\\
P^1\beta g_{2,0} &\longmapsto 0\\
\beta P^1\beta g_{2,0} &\longmapsto 0\\
g_{2,1} &\longmapsto 2P^2 g_{1,0} + (\beta P^1 - 2P^1\beta) g_{1,1}\\
P^2 g_{2,0} &\longmapsto P^2\beta g_{1,0}\\
\beta g_{2,1} &\longmapsto 2\beta P^2 g_{1,0} - 2\beta P^1\beta g_{1,1}\\
\beta P^2 g_{2,0} &\longmapsto \beta P^2\beta g_{1,0}\\
P^2\beta g_{2,0} &\longmapsto 0\\
\beta P^2\beta g_{2,0} &\longmapsto 0\\
g_{2,2} &\longmapsto P^2 g_{1,1}\\
P^1 g_{2,1} &\longmapsto (\beta P^2 - 3P^2\beta) g_{1,1} = \beta P^2 g_{1,1}\\
\beta g_{2,2} &\longmapsto \beta P^2 g_{1,1}\\
g_{2,3} &\longmapsto P^p g_{1,0} + P^2\beta g_{1,1} - \beta g_{1,2}\\
P^p g_{2,0} &\longmapsto P^p\beta g_{1,0}\\
\beta P^1 g_{2,1} &\longmapsto 0\\
P^1\beta g_{2,1} &\longmapsto (\beta P^p - P^p\beta) g_{1,0} - 2\beta P^p\beta g_{1,1}\\
\beta g_{2,3} &\longmapsto \beta P^p g_{1,0} + \beta P^2\beta g_{1,1}\\
\beta P^p g_{2,0} &\longmapsto \beta P^p\beta g_{1,0}\\
P^p\beta g_{2,0} &\longmapsto 0\\
\beta P^1\beta g_{2,1} &\longmapsto -\beta P^p\beta g_{1,0}
\end{aligned}$$

FIGURE 21. $\partial_2 \colon P_2 \to P_1$ for $p = 3$

$$
\begin{aligned}
&\beta g_{2,0}\\
&P^1\beta g_{2,0}\\
&\beta P^1\beta g_{2,0}\\
&P^2\beta g_{2,0}\\
&\beta P^2\beta g_{2,0}\\
&P^1 g_{2,1}-\beta g_{2,2}\\
&P^p g_{2,0}+P^1\beta g_{2,1}-\beta g_{2,3}\\
&\beta P^1 g_{2,1}\\
&\beta P^p g_{2,0}+\beta P^1\beta g_{2,1}\\
&P^p\beta g_{2,0}
\end{aligned}
$$

FIGURE 22. A basis for $\ker(\partial_2)$ at $p=3$

$$
\begin{aligned}
g_{3,0} &\xmapsto{\partial_3} \beta g_{2,0}\\
\beta g_{3,0} &\longmapsto 0\\
P^1 g_{3,0} &\longmapsto P^1\beta g_{2,0}\\
\beta P^1 g_{3,0} &\longmapsto \beta P^1\beta g_{2,0}\\
P^1\beta g_{3,0} &\longmapsto 0\\
\beta P^1\beta g_{3,0} &\longmapsto 0\\
P^2 g_{3,0} &\longmapsto P^2\beta g_{2,0}\\
\beta P^2 g_{3,0} &\longmapsto \beta P^2\beta g_{2,0}\\
P^2\beta g_{3,0} &\longmapsto 0\\
\beta P^2\beta g_{3,0} &\longmapsto 0\\
g_{3,1} &\longmapsto P^1 g_{2,1}-\beta g_{2,2}\\
\beta g_{3,1} &\longmapsto \beta P^1 g_{2,1}\\
g_{3,2} &\longmapsto P^p g_{2,0}+P^1\beta g_{2,1}-\beta g_{2,3}\\
P^p g_{3,0} &\longmapsto P^p\beta g_{2,0}\\
\beta g_{3,2} &\longmapsto \beta P^p g_{2,0}+\beta P^1\beta g_{2,1}
\end{aligned}
$$

FIGURE 23. $\partial_3\colon P_3\to P_2$ for $p=3$

$$
\begin{aligned}
&\beta g_{3,0}\\
&P^1\beta g_{3,0}\\
&\beta P^1\beta g_{3,0}\\
&P^2\beta g_{3,0}\\
&\beta P^2\beta g_{3,0}
\end{aligned}
$$

FIGURE 24. A basis for $\ker(\partial_3)$ at $p=3$

10.4.4. *Filtrations $s \geq 4$.* From here on we get surjections $\partial_s \colon P_s \to \ker(\partial_{s-1})$ for $s \geq 4$ by letting

$$P_s = \mathscr{A}\{g_{s,0}, \dots\}$$

with $g_{s,0}$ in degree s, where $\partial_s(g_{s,0}) = \beta g_{s-1,0}$, and so on.

Theorem 10.14. *There is a minimal resolution $\epsilon \colon P_* \to \mathbb{F}_3$, with $P_0 = \mathscr{A}\{g_{0,0}\}$ and $P_s = \mathscr{A}\{g_{s,i} \mid i \geq 0\}$, where $\partial_s \colon P_s \to P_{s-1}$ is given in internal degree $t \leq 15$ by*

$$\begin{aligned}
\partial_1(g_{1,0}) &= \beta g_{0,0} \\
\partial_1(g_{1,1}) &= P^1 g_{0,0} \\
\partial_1(g_{1,2}) &= P^p g_{0,0} \\
\partial_2(g_{2,0}) &= \beta g_{1,0} \\
\partial_2(g_{2,1}) &= 2P^2 g_{1,0} + (\beta P^1 - 2P^1\beta) g_{1,1} \\
\partial_2(g_{2,2}) &= P^{p-1} g_{1,1} \\
\partial_2(g_{2,3}) &= P^p g_{1,0} + P^2 \beta g_{1,1} - \beta g_{1,2} \\
\partial_3(g_{3,0}) &= \beta g_{2,0} \\
\partial_3(g_{3,1}) &= P^1 g_{2,1} - \beta g_{2,2} \\
\partial_3(g_{3,2}) &= P^p g_{2,0} + P^1 \beta g_{2,1} - \beta g_{2,3} \\
\partial_4(g_{4,0}) &= \beta g_{3,0} \\
&\dots \\
\partial_{15}(g_{15,0}) &= \beta g_{14,0} \,.
\end{aligned}$$

Theorem 10.15. $\mathrm{Ext}_{\mathscr{A}}^{s,*}(\mathbb{F}_3, \mathbb{F}_3) \cong \mathbb{F}_3\{\gamma_{s,i}\}_i$*, where $\gamma_{s,i} \colon P_s \to \mathbb{F}_3$ is the $\mathscr{A}$-module homomorphism dual to $g_{s,i}$. The generators in internal degree $t \leq 15$ are displayed in the following figure.*

	0		2		4		6		8		10		12		14	
6	$\gamma_{6,0}$										$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
	$\gamma_{5,0}$											$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
4	$\gamma_{4,0}$												$\cdot$	$\cdot$	$\cdot$	?
	$\gamma_{3,0}$										$\gamma_{3,1}$	$\gamma_{3,2}$		?	?	?
2	$\gamma_{2,0}$							$\gamma_{2,1}$			$\gamma_{2,2}$	$\gamma_{2,3}$			?	?
	$\gamma_{1,0}$			$\gamma_{1,1}$								$\gamma_{1,2}$				?
0	$\gamma_{0,0}$															

We have not yet computed the groups labeled $\cdot$ or ?, but by the May vanishing theorem, see Ravenel (1986, Theorem 3.4.5(b)), the groups labeled $\cdot$ are 0. In fact, many of the groups labeled ? are also zero.

The first possible differential is $d_2^{1,12}$ on $\gamma_{1,2}$), which indeed equals $\gamma_{3,1}$. Once we have proved convergence and the visible vanishing line, it follows that $\pi_*(S)^\wedge_3$ begins as follows.

n	$\pi_n(S)^\wedge_3$	gen.	rep.
0	$\mathbb{Z}_3$	ι	$\gamma_{0,0}$
1	0		
2	0		
3	$\mathbb{Z}/3$	α_1	$\gamma_{1,1}$
4	0		
5	0		
6	0		
7	$\mathbb{Z}/3$	α_2	$\gamma_{2,1}$
8	0		
9	0		
10	$\mathbb{Z}/3$	β_1	$\gamma_{2,2}$
11	$\mathbb{Z}/9$	$\alpha_{3/1}$	$\gamma_{2,3}$
12	0		

The cyclic groups in degrees $2i(p-1)-1 = 4i-1$, generated by the α-classes, equal the image $\mathrm{im}(J)_*$ of the J-homomorphism $J_*\colon \pi_{4i-1}(O) \to \pi_{4i-1}(S)$. As becomes visible in degree $2p(p-1)-1 = 11$, the order of this cyclic group $\mathrm{im}(J)_{2i(p-1)-1}$ varies with i. It is $p^{j+1} = 3^{j+1}$ where $j = v_p(i)$ is the p-valuation of i, or equivalently, the p-component of pi. The element of order p in this image is denoted α_i, for $i \geq 1$, and $\alpha_i = p^j \alpha_{i/j}$, where $\alpha_{i/j}$ is a generator of this cyclic group. This pattern persists for all odd primes p, but the case $p = 2$ is more complicated.

The first element of $\pi_*(S)_p^\wedge$ that is not in the *image of* J, hence is in the *cokernel of* J, is β_1 in $\pi_{2p(p-1)-2}(S)_p^\wedge$, represented in Adams filtration 2 by $\gamma_{2,2}$.

11. Bruner's `ext`-program

11.1. Overview. Robert R. Bruner (1993) has developed a package of C-programs and shell scripts, usually called `ext`, which can calculate $\mathrm{Ext}_{\mathscr{A}}^{s,t}(M, \mathbb{F}_2)$ over the mod 2 Steenrod algebra $\mathscr{A}$ for many modules M, in a finite range of filtration degrees s and internal degrees t.

The strategy is to compute a minimal free resolution $\epsilon\colon P_* \to M$ of the $\mathscr{A}$-module M, one internal degree t at a time, starting from filtration degree $s = 0$ and moving upwards. The $\mathscr{A}$-module basis $\{g_{s,i}\}_i$ for P_s then also gives an $\mathbb{F}_2$-vector space basis for $\mathbb{F}_2 \otimes_{\mathscr{A}} P_s = \mathrm{Tor}_s^{\mathscr{A}}(\mathbb{F}_2, M)$. The dual basis $\{\gamma_{s,i}\}_i$, with $\gamma_{s,i}(g_{s,j}) = \delta_{i,j}$, is then an $\mathbb{F}_2$-vector space basis for $\mathrm{Ext}_{\mathscr{A}}^s(M, \mathbb{F}_2)$.

The program can compute induced homomorphisms, Yoneda products and some Massey products, and produces output in text, TEX, Postscript and PDF formats. It can also make similar calculations over the subalgebra $A(2) = \langle Sq^1, Sq^2, Sq^4\rangle$ of $\mathscr{A}$, but is not prepared to calculate at odd primes p.

See subsection 11.2 for a guide to how to install the current version of `ext`. Thereafter, see subsection 11.3 to see how to encode an $\mathscr{A}$-module M in a format that the `ext` program can use, and subsection 11.4 for where to store and process such module definition files. To resolve a module, first see subsection 11.5 for how to create the subdirectory where that calculation takes place, and then see subsection 11.6 for how to run the script that calculates the minimal resolution. [[ETC]]

11.2. Installation. At the time of writing, the most recent version of `ext` is `ext.1.8.7` from April 14th 2014. It can be downloaded via Bruner's home page at `http://www.math.wayne.edu/~rrb/papers/`, or directly from `http://www.math.wayne.edu/~rrb/papers/ext.1.8.7.tar.gz`, using a web browser. Save the file `ext.1.8.7.tar.gz` in a directory. In this guide we will assume that this directory is called `ext`. You may be offered to create such a directory when saving the file, or you can create one using `mkdir ext`.

Open a terminal window and move to the `ext` directory, using a command like `cd ext`. The file is a compressed (gzip'ed) tape archive (tar-file). First uncompress it using `gunzip ext.1.8.7.tar.gz`. This enlarges the file from about 2 MB to about 5 MB, and gives it the new name `ext.1.8.7.tar`. Then unpack the archive using `tar -xvf ext.1.8.7.tar`. To list the resulting files use `ls`, giving output like `A A2 copyright doc ext.1.8.7.tar NEW README START_HERE TODO`. The files `START_HERE` (up to date for version 1.8) and `README` (dating from versions 1.6, 1.65 and 1.66) explain the basic usage of the `ext` program. There is further documentation in the `doc` subdirectory, and an account of the changes made since version 1.66 is given in `NEW`. The subdirectory `A` will contain the code and data for making calculations over the mod 2 Steenrod algebra $\mathscr{A}$. The subdirectory `A2` will contain the corresponding code and date for calculations over the subalgebra $A(2)$.

To complete the installation, follow the instructions in section I of `START_HERE`, namely do `cd A` followed by `./Install`. This runs the shell script `Install` in `ext/A`. The script compiles several programs, and assumes that the GNU C-compiler `gcc` is already installed on the system. If not, you will need to install `gcc` first. There will be some warning messages regarding `storage.c` and `splitname.c`. Apparently it is difficult to avoid these on all different systems. It may be possible to write `Install` in place of `./Install`, but this depends on the settings of your system, i.e., whether the current directory (`.`) is in the search path variable `$PATH`. We will not assume that it is, and therefore use the explicit `./`-commands. Finally, move up to `ext` and down to the `A2` directory using `cd ../A2`, and then do `./Install` in that directory to compile the remaining programs. Again there will be some warning messages. Do `cd ..` to return to the main directory (the one we are assuming is called `ext`). This completes the installation.

11.3. The module definition format. In order to calculate $\mathrm{Ext}_{\mathscr{A}}^{*,*}(M, \mathbb{F}_2)$, we must first specify the $\mathscr{A}$-module M. Before version 1.5, the user was expected to provide a program (called `module.c`) that contained functions keeping track of a $\mathbb{F}_2$-vector space basis for M, and the action of elements in the Steenrod algebra on those basis elements. This is documented in `ext/doc/readme.1.0` and `ext/doc/module.doc`, but is now largely irrelevant, due to the new interface for module definitions introduced in version 1.5, partly written by Jeff Igo. It is documented in `ext/doc/modfmt.ascii` and `ext/doc/modfmt.html`, in addition to the following explanation.

The $\mathscr{A}$-module M, which may eventually have a completely different name, must be presented to the `ext` program as a finite dimensional $\mathbb{F}_2$-vector space with a chosen ordered basis $(v_0, v_1, \dots, v_{n-1})$. If there are n basis vectors, they will be numbered from 0 to $n-1$, inclusive. The $\mathscr{A}$-module action must be specified by listing the value $Sq^r(v_i)$ of each Steenrod squaring operation Sq^r on each basis vector v_i, for $r \geq 1$ and $0 \leq i < n$, except that operations that take the value 0 can be omitted. This ensured that only finitely many values need to be specified.

If one is really interested in an infinite-dimensional module M, such as $H^*\mathbb{R}P^\infty = \mathbb{F}_2[x]$, one must choose to truncate this module at some finite internal degree b, discarding all generators in internal degrees $t > b$. This will not affect $\mathrm{Ext}^{s,t}_{\mathscr{A}}(M, \mathbb{F}_2)$ for $t \leq b$, so a partial calculation in a finite range of internal degrees is possible, if M is bounded below and of finite type. If M is not bounded below, or has infinitely many generators in a single degree, then the `ext`-program will not be able to calculate with it.

The module definition file will be a text file with two parts. The first part specifies the internal grading of the vector space basis. The second part specifies the action by the Steenrod operations.

The first part has the format

```
n
t0 t1 ... t(n-1)
```

where n is the $\mathbb{F}_2$-vector space dimension of M, i.e., the number of basis vectors $v_0, v_1, \dots, v_{n-1}$, and $t_0 t_1 \dots t_{n-1}$ are the internal degrees of those basis vectors. Beware that these basis vectors are assumed to be ordered so that the sequence of internal degrees is non-decreasing. In other words, a basis vector cannot be followed by a basis element in strictly lower internal degree. (If your module definition file does not satisfy this condition, the program `ext/A/samples/sortDef` can reorder the basis as needed.)

For example, if $M = H^*(S) = \mathbb{F}_2$ has a single generator in internal degree 0, the module definition file would begin:

```
1
0
```

If $M = \tilde{H}^*(\mathbb{R}P^4)$ has four generators x, x^2, x^3 and x^4 in degrees 1, 2, 3 and 4, the module definition file would begin:

```
4
1 2 3 4
```

Note that the names of the generators are irrelevant for the program; it simply considers the basis as an ordered list of n elements, and keeps track of the individual basis elements by their index in that list, which is a number between 0 and $n-1$. (This index is typically different from the internal degree of that generator.) However, the ordering of the basis elements (within a given internal degree) will be of importance when the Steenrod operations are to be specified.

The second part consists of a list of lines, one for each nonzero operation $Sq^r(v_i)$ with $r \geq 1$. If $Sq^r(v_i) = v_{j_1} + v_{j_2} + \cdots + v_{j_k}$ is a sum of k different terms, then that line will appear as follows:

```
i r k j1 j2 ... jk
```

The first entry, i, tells us which basis vector, v_i, is being acted upon. The second entry, r, tells us which Steenrod operation, Sq^r, is acting nontrivially on that basis vector. The value of $Sq^r(v_i)$ is a homogeneous element in M, hence is a sum of one or more of the basis vectors in that internal degree. The third entry (k) tells us how many different terms there are in that sum. The remainder of the line contains k entries, and these are the indices $j_1, j_2, \dots, j_k$ of the basis vectors that occur in the sum $Sq^r(v_i) = v_{j_1} + v_{j_2} + \cdots + v_{j_k}$. [[Usually $j_1 < j_2 < \cdots < j_k$. Is this necessary? Duplications are not allowed, I believe.]]

For example, if $M = H^*(S) = \mathbb{F}_2$, there are no nonzero operations Sq^r, so the second part is empty; it consists of zero lines.

If $M = \tilde{H}^*(\mathbb{R}P^4)$, the Steenrod operations satisfy $Sq^r(x^i) = \binom{i}{r} x^{r+i}$. The nonzero operations are $Sq^1(x) = x^2$, $Sq^1(x^3) = x^4$ and $Sq^2(x^2) = x^4$. The operation $Sq^1(x) = x^2$ is specified by the line

```
0 1 1 1
```

where the first 1 means that we are acting on the generator numbered 0, i.e., x, the second 1 means that we are specifying the value of Sq^1 on that generator, the third 1 means that $Sq^1(x) = x^2$ is a sum of one term only, and the last 1 means that that one term is the generator numbered 1, i.e., x^2. The operation $Sq^1(x^3) = x^4$ is specified by the line

```
2 1 1 3
```

where the first 2 means that we are acting on the generator numbered 2, i.e., x^3, the second 1 means that we are specifying the value of Sq^1 on that generator, the third 1 means that $Sq^1(x^3) = x^4$ is a sum of one term only, and the last 3 means that that one term is the generator numbered 3, i.e., x^4. The operation $Sq^2(x^2) = x^4$ is specified by the line

```
1 2 1 3
```

where the first 1 means that we are acting on the generator numbered 1, i.e., x^2, the second 2 means that we are specifying the value of Sq^2 on that generator, the third 1 means that $Sq^2(x^2) = x^4$ is a sum of one term only, and the last 3 means that that one term is the generator numbered 3, i.e., x^4. The combined second part of the module definition file for this M is therefore:

```
0 1 1 1
2 1 1 3
1 2 1 3
```

The ordering of the lines does not matter. If preferred, we could also have used the following specification

```
0 1 1 1
1 2 1 3
2 1 1 3
```

in order of the basis elements v_i, followed by the order of the squaring operations $Sq^r(v_i)$ on those basis elements. With this ordering, the whole module definition file for $\tilde{H}^*(\mathbb{R}P^4)$ would appear as follows.

```
4
1 2 3 4

0 1 1 1
1 2 1 3
2 1 1 3
```

This file can be created in a text editor.

11.4. **The `samples` directory.** Module definitions for $\mathscr{A}$-modules can conveniently be stored in the directory `ext/A/samples`. The file name can be freely chosen, but it is convenient to let it specify the $\mathscr{A}$-module, or perhaps a spectrum whose cohomology realizes that $\mathscr{A}$-module. The module definition for $\mathbb{F}_2$ can thus be saved under one of the names `F2`, `F2.def`, `S` or `S.def` in `ext/A/samples`. That directory also contains some tools for working with module definitions. See the file `ext/A/samples/README` for some documentation. The programs `tensorDef`, `dualizeDef`, `collapse` and `truncate` let you build new module definition files from old ones.

For example, if `M.def` and `N.def` contain the definitions of two $\mathscr{A}$-modules M and N, then the command `./tensorDef M.def N.def MN.def` will produce a new module definition file `MN.def`, presenting the tensor product $M \otimes N$ (with the diagonal $\mathscr{A}$-action, more on that later). If $M = H^*X$ and $N = H^*Y$, then $M \otimes N = H^*(X \wedge Y)$. If the ordered basis for M is $(v_i)_i$ and the ordered basis for N is $(w_j)_j$, the basis chosen for $M \otimes N$ will consist of the set of tensors $\{v_i \otimes w_j\}_{i,j}$, but the ordering of these basis vectors may not be obvious. The program `tensorDef` therefore outputs a list of the pairs (i, j), in the order that is chosen for $M \otimes N$. A copy of this output may be saved, since it can become useful later.

For another example, if `M.def` contains the definition of an $\mathscr{A}$-module M, then `./dualizeDef M.def DM.def` will produce a new module definition file `DM.def`, presenting the dual $\mathscr{A}$-module $M^* = \mathrm{Hom}_{\mathbb{F}_2}(M, \mathbb{F}_2)$ (with the conjugated $\mathscr{A}$-action, more on that later). If $M = H^*X$, then $M^* = H^*(DX) = H_{-*}(X)$, where $DX = F(X, S)$ is the functional dual of X. For finite CW spectra X, this is the same as the Spanier–Whitehead dual of X. [[Is the basis $\{v_i^*\}_i$ for M^* ordered by reversing the order of the basis $(v_i)_i$ for M?]]

Calling these commands without an argument, as in `./collapse` or `./truncate`, gives short messages explaining their usage.

The `consistency` command, in its improved version called `newconsistency`, checks whether the Steenrod operations listed in a module definition file actually define an $\mathscr{A}$-module, i.e., if the operations satisfy the Adem relations. If all Adem relations are satisfied, it exits quietly. If they are not, it lists the Adem relations that are not satisfied, and the generator on which this failure takes place.

[[Can use `newconsistency` to complete a partial definition of an $\mathscr{A}$-module, where only the action of the algebra indecomposables Sq^{2^i} are given, to one where the action of all Sq^r are given. To do this, start by adding operations to correct the lowest degree error message from `newconsistency`, and continue.]]

11.5. **Creating a new module.** To make $\mathrm{Ext}_{\mathscr{A}}$-calculations with an $\mathscr{A}$-module M, defined by a module definition file `M.def` in `ext/A/samples`, use `cd ..` or something similar to go to `ext/A`. Then use the command `./newmodule M samples/M.def` to create a subdirectory `ext/A/M` that contains the data and code relevant for the calculations for M. In general, replace `M` with a more memorable name for the module in question. `newmodule` calls on `newconsistency` to check that the module definition file `M.def` actually defines an $\mathscr{A}$-module. If it does not, go back and correct it before calling `newmodule again`.

A copy of the module definition file will be stored as `Def` in `ext/A/M`.

The $\mathrm{Ext}_{\mathscr{A}}$-calculations for M will be carried out by finding a finite part of a minimal resolution $P_* \to M$, in a range of filtration degrees $0 \le s \le s_{max}$, where s_{max} is the number stored in the file `ext/A/M/MAXFILT`.

$$P_{s_{max}} \longrightarrow \dots \longrightarrow P_s \xrightarrow{\partial_s} P_{s-1} \longrightarrow \dots \longrightarrow P_0 \xrightarrow{\epsilon} M \to 0\,.$$

Usually a common s_{max}-value for all $\mathscr{A}$-modules is set in the file `ext/A/MAXFILT`, and `newmodule` will copy this value into `ext/A/M/MAXFILT` when creating the directory for M. [[It should not be changed after `newmodule` has completed creating the module.]]

The data specifying the minimal resolution will be stored in the files `Diff.0, Diff.1, ....` Here `Diff.s` will specify the internal degrees of the $\mathscr{A}$-module generators $g_{s,i}$ for P_s, and the values $\partial_s(g_{s,i})$ in P_{s-1} of the boundary homomorphism ∂_s on these generators. These values will be expressed as sums of elements in the free $\mathscr{A}$-module on the generators $g_{s-1,j}$ of P_{s-1}. [[What happens for $s = 0$?]]

The computation will be done one internal degree t at a time, assuming that the calculations for lower internal degrees have already been done. The first line of each `Diff.s` contains two numbers. The second is the internal degree t up to which the calculation of P_s and ∂_s has been completed, so far. The first is the number of generators that have been added to P_s, in internal degrees less than or equal to t. Both of these numbers are set to 0 at the outset, when the module is created with `newmodule`. [[Can this confuse the program if M starts in negative degrees, and `dims` is started at $t = 0$?]]

[[Explain format of `Diff`-files.]]

11.6. **Resolving a module.** To resolve a module M, created from a module definition file `M.def` in `ext/A/samples` using `./newmodule M samples/M.def` in `ext/A`, move into `ext/A/M` using `cd M`. (In general, replace `M` by the directory name chosen for the module.) Suppose that the module M is concentrated in internal degrees $t \ge 0$, and that we want to make the calculation up to internal degree $t = 60$. Then we use the script `dims`, which automatically starts a series of scripts `nextt`, each handling one t at a time. To calculate in the range just mentioned, use `./dims 0 60`.

In general, the command `./dims a b` in the directory `ext/A/M` will calculate the resolution P_* for $0 \le s \le s_{max}$ in the range of internal degrees $a \le t \le b$, under the assumption that the calculation is already finished for $t < a$, starting with $t = a$ and working its way up.

For each t, the calculation proceeds on s at a time, calculating the kernel of $\partial_{s-1} \colon P_{s-1} \to P_{s-2}$ in degree t, identifying the image of $\partial_s \colon P_s \to P_{s-1}$ when restricted to the generators of P_s in internal degrees less than t, and choosing an $\mathbb{F}_2$-basis for a complementary subspace. For each basis vector v_i, an $\mathscr{A}$-module generator $g_{s,i}$ is added to P_s in internal degree t, and $\partial_s(g_{s,i})$ is set equal to v_i. [[Is this a fair representation of how the program actually works?]]

The subdirectory `ext/A/M/logs` will contain log files, recording the progress made. Use `ls logs` in `ext/A/M` to get a quick look at the progress, or try `ls -lrt logs` for more detailed timing information.

After `dims` is finished, the calculation can be continued with another call to the same script, for instance by `./dims 61 100`.

[[Explain `report` and `display`.]]

[[ETC]]

12. Convergence of the Adams spectral sequence

12.1. **The Hopf–Steenrod invariant.** For $p = 2$, the standard notation for the class $\gamma_{1,i}$, dual to the indecomposable Sq^{2^i}, is h_i. See Adams (1958). The h is for Hopf, since these classes detect the stable maps of spheres with Hopf invariant one.

Lemma 12.1. $\mathrm{Tor}_1^{\mathscr{A}}(\mathbb{F}_2, \mathbb{F}_2) \cong I(\mathscr{A})/I(\mathscr{A})^2 = Q(\mathscr{A}) \cong \mathbb{F}_2\{Sq^{2^i} \mid i \ge 0\}$ *and* $\mathrm{Ext}^1_{\mathscr{A}}(\mathbb{F}_2, \mathbb{F}_2) \cong \mathrm{Hom}(\mathrm{Tor}_1^{\mathscr{A}}(\mathbb{F}_2, \mathbb{F}_2), \mathbb{F}_2) \cong \mathbb{F}_2\{h_i \mid i \ge 0\}$ *where* h_i *has bidegree* $(s,t) = (1, 2^i)$ *and is dual to* Sq^{2^i}*, for each* $i \ge 0$.

Proof. There exists a free resolution $\cdots \to P_1 \to P_0 \to \mathbb{F}_2 \to 0$ where $P_0 = \mathscr{A}$ and $P_1 = \mathscr{A}\{g_{1,i}\}_i$ with $\partial_1 \colon g_{1,i} \mapsto Sq^{2^i}$ for all $i \ge 0$. The resolution is exact at P_0 since the Sq^{2^i} generate the left ideal $I(\mathscr{A}) \subset \mathscr{A}$, and it is minimal there since $\partial_1(P_1) \subset I(\mathscr{A})P_0$. It is also minimal at P_1, since the surjection $P_1 \to I(\mathscr{A})$ induces an isomorphism $\mathbb{F}_2\{g_{1,i}\}_i = \mathbb{F}_2 \otimes_{\mathscr{A}} P_1 = P_1/I(\mathscr{A})P_1 \to I(\mathscr{A})/I(\mathscr{A})^2 = Q(\mathscr{A})$, so that $\partial_2(P_2) = \ker(\partial_1) \subset I(\mathscr{A})P_1$. Hence $\mathrm{Tor}_1^{\mathscr{A}}(\mathbb{F}_2, \mathbb{F}_2) \cong \mathbb{F}_2 \otimes_{\mathscr{A}} P_1 \cong Q(\mathscr{A})$ and $\mathrm{Ext}^1_{\mathscr{A}}(\mathbb{F}_2, \mathbb{F}_2) \cong \mathrm{Hom}_{\mathscr{A}}(P_1, \mathbb{F}_2) \cong \mathbb{F}_2\{h_i\}_i$, as claimed. ((Proof using bar complex?)) □

Lemma 12.2. *For p odd, $\mathrm{Tor}_1^{\mathscr{A}}(\mathbb{F}_p, \mathbb{F}_p) \cong \mathbb{F}_p\{\beta, P^{p^i} \mid i \ge 0\}$ and $\mathrm{Ext}^1_{\mathscr{A}}(\mathbb{F}_p, \mathbb{F}_p) \cong \mathbb{F}_p\{a_0, h_i \mid i \ge 0\}$, where a_0 has bidegree $(s,t) = (1,1)$ and is dual to β, and h_i has bidegree $(s,t) = (1, 2p^i(p-1))$ and is dual to P^{p^i}, for each $i \ge 0$.*

Proof. The proof is similar to the case $p = 2$, using a free resolution $\epsilon \colon P_* \to \mathbb{F}_p$, with $P_0 = \mathscr{A}$ and $P_1 = \mathscr{A}\{g_{1,0}, g_{1,i+1} \mid i \ge 0\}$, where $\partial_1(g_{1,0}) = \beta$ and $\partial_1(g_{1,i+1}) = P^{p^i}$ for each $i \ge 0$. □

We shall soon prove that the Adams spectral sequence

$$E_2^{s,t} = \mathrm{Ext}^{s,t}_{\mathscr{A}}(\mathbb{F}_2, \mathbb{F}_2) \Longrightarrow \pi_{t-s}(S)^\wedge_2$$

converges to the 2-adic completion of the stable homotopy groups of spheres. The chart in Theorem 10.12 above displays the E_2-term in the range $t \le 11$. [[EDIT FROM HERE TO TAKE INTO ACCOUNT THE ADAMS VANISHING LINE.]] We will see later that the pattern above the diagonal line, where $s > t-s$, continues. There is an isomorphism $\mathrm{Ext}^{s,s}_{\mathscr{A}}(\mathbb{F}_2, \mathbb{F}_2) \cong \mathbb{F}_2\{\gamma_{s,0}\}$ for all $s \ge 0$, while $\mathrm{Ext}^{s,t}_{\mathscr{A}}(\mathbb{F}_2, \mathbb{F}_2) = 0$ for $t - s < 0$ and for $0 < t-s < s$. Thus the groups labeled $\cdot$ in the chart are 0. Granting this, the only possible d_r-differentials starting in total degree $t - s \le 6$, for $r \ge 2$, are the ones starting on $\gamma_{1,1} = h_1$ and landing in the group generated by $\gamma_{r+1,0}$.

However, these differentials are all 0, as can be seen either by proving that $\gamma_{s,0}$ detected $2^s \in \pi_0(S)$, or that $\gamma_{1,1}$ detects $\eta \in \pi_1(S)$, or by appealing to multiplicative structure in the spectral sequence. Granting this, we can conclude that $E_2 = E_\infty$ in this range of degrees, so that the groups $\mathbb{F}_2\{\gamma_{s,i}\}$ in one topological degree $n = t-s$, for $s \ge 0$ and $n \le 5$ are the filtration quotients of a complete Hausdorff filtration $\{F^s\}_s$ that exhausts $\pi_n(S)^\wedge_2$.

For $n = 0$, we already know that $\pi_0(S) = \mathbb{Z}$ so $\pi_0(S)^\wedge_2 = \mathbb{Z}_2$. The only possible filtration is the 2-adic one, with $F^s = 2^s\mathbb{Z}_2 \subset \mathbb{Z}_2$ and $F^s/F^{s+1} \cong 2^s\mathbb{Z}_2/2^{s+1}\mathbb{Z}_2 \cong \mathbb{F}_2\{\gamma_{s,0}\}$ for all $s \ge 0$. For $n = 1$ we deduce that $\pi_1(S)^\wedge_2 \cong \mathbb{Z}/2\{\gamma_{1,1}\} = \mathbb{Z}/2\{h_1\}$. In fact $\pi_1(S) = \mathbb{Z}/2\{\eta\}$ is generated by the complex Hopf map $\eta \colon S^1 \to S$. For $n = 2$ we deduce that $\pi_2(S)^\wedge_2 \cong \mathbb{Z}/2\{\gamma_{2,1}\}$. We shall see later that $\pi_2(S) = \mathbb{Z}/2\{\eta^2\}$ is generated by the composite $\eta^2 = \eta \circ \Sigma\eta \colon S^2 \to S$. For $n = 3$ we deduce that $\pi_3(S)^\wedge_2$ is an abelian group of order 8. We shall see later that $\pi_3(S)^\wedge_2 \cong \mathbb{Z}/(8)$ is the 2-Sylow subgroup of $\pi_3(S) \cong \mathbb{Z}/24$, generated by the quaternionic Hopf map $\nu \colon S^3 \to S$. Finally, for now, we conclude that $\pi_4(S)^\wedge_2 = 0$ and $\pi_5(S)^\wedge_2 = 0$, and in fact $\pi_4(S) = \pi_5(S) = 0$. [[EDIT TO HERE.]]

Lemma 12.3. *(Hopf, Steenrod) For $p = 2$, let $f \colon S^n \to S$ be a map with $0 = f^* \colon H^*(S) \to H^*(S^n)$, and let $C_f = \mathrm{hocofib}(f) = S \cup_f e^{n+1}$ be its mapping cone. Suppose that $Sq^{n+1} \colon H^0(C_f) \to H^{n+1}(C_f)$ is nonzero. Then $n + 1 = 2^i$ for some $i \ge 0$ and $[f] \in \pi_n(S)$ is detected in the Adams spectral sequence by $h_i \in E_2^{1,2^i}$.*

Proof. Consider the canonical Adams tower for $Y = S$, with $Y^0 = S$, $K^0 = H$, $Y^1 = \Sigma^{-1}\bar{H}$ and $K^1 = H \wedge \Sigma^{-1}\bar{H}$. The composite $j \circ f$ is null-homotopic, since $d(f) = f^* = 0$, so we have a map of cofiber sequences:

$$\begin{tikzcd}
S^n \arrow[r, "f"] \arrow[d, "e"] & S \arrow[r] \arrow[d, equal] & C_f \arrow[r] \arrow[d, "d"] & S^{n+1} \arrow[d, "\Sigma e"] \\
\Sigma^{-1}\bar{H} \arrow[r, "i"] \arrow[d, "j"] & S \arrow[r, "j"] & H \arrow[r, "\partial"] & \bar{H} \\
H \wedge \Sigma^{-1}\bar{H} & & &
\end{tikzcd}$$

Here $d\colon C_f \to H$ and $e\colon S^n \to \Sigma^{-1}\bar{H}$ are determined by a null-homotopy of f. Applying cohomology to the right hand part of the diagram, we get a map of $\mathscr{A}$-module extensions:

$$\begin{tikzcd}
\mathbb{F}_2 & H^*(C_f) \arrow[l, two heads] & \Sigma^{n+1}\mathbb{F}_2 \arrow[l, tail] \\
\mathbb{F}_2 \arrow[u, equal] & \mathscr{A} \arrow[l, two heads, "j^*"] \arrow[u, "d^*"] & I(\mathscr{A}) \arrow[l, tail, "\partial^*"] \arrow[u, "\Sigma e^*"]
\end{tikzcd}$$

Here $d^*(1) = 1$, so by assumption $d^*(Sq^{n+1}) \neq 0$. Hence $\Sigma e^*(Sq^{n+1}) \neq 0$. This is impossible if Sq^{n+1} is decomposable, so we must have $n+1 = 2^i$ for some $i \geq 0$. Then $e^* \neq 0$, which implies that $j \circ e\colon S^n \to H \wedge \Sigma^{-1}\bar{H}$ is essential (= not null-homotopic).

This proves that $[f] \in \pi_n(S)$ lifts to $\pi_n(Y^1)$ but not to $\pi_n(Y^2)$, hence corresponds under the isomorphism $F^1/F^2 \cong E_\infty^{1,*}$ to a nonzero class in $E_\infty^{1,2^i} \subset E_2^{1,2^i} = \mathbb{F}_2\{h_i\}$. The only possibility is that $[f]$ is detected by h_i. □

Lemma 12.4. *(Hopf, Steenrod) For p odd, let $f\colon S^n \to S$ be a map with $0 = f^*\colon H^*(S) \to H^*(S^n)$, and let $C_f = \mathrm{hocofib}(f) = S \cup_f e^{n+1}$ be its mapping cone. Suppose that $P^k\colon H^0(C_f) \to H^{n+1}(C_f)$ is nonzero, with $n+1 = 2k(p-1)$. Then $k = p^i$ for some $i \geq 0$ and $[f] \in \pi_n(S)$ is detected in the Adams spectral sequence by $h_i \in E_2^{1,2p^i(p-1)}$. Alternatively, suppose that $\beta\colon H^0(C_f) \to H^1(C_f)$ is nonzero. Then $n = 0$ and $[f] \in \pi_0(S)$ is detected by $a_0 \in E_2^{1,1}$.*

Proof. The proof is similar to the 2-primary case. □

The class of $\Sigma e^* \circ \partial_1\colon P_1 \to \Sigma^{n+1}\mathbb{F}_p$ in $\mathrm{Ext}_{\mathscr{A}}^{1,n+1}(\mathbb{F}_p, \mathbb{F}_p) = \mathbb{F}_p\{h_i\}$ is called the Hopf–Steenrod invariant, or the cohomology e-invariant, of $[f]$. It is only defined for the $[f]$ with vanishing d-invariant. More generally, we have a diagram

$$\begin{tikzcd}
F^2 \arrow[r, tail] & F^1 \arrow[r, tail] \arrow[d, "e"] & F^0 = [X, Y]_n \arrow[d, "d"] \\
& \mathrm{Ext}_{\mathscr{A}}^{1,n+1}(H^*(X), H^*(Y)) & \mathrm{Hom}_{\mathscr{A}}^n(H^*(X), H^*(Y))
\end{tikzcd}$$

for each pair of spectra X and Y.

Theorem 12.5. *The Hopf maps $2\colon S \to S$, $\eta\colon S^1 \to S$, $\nu\colon S^3 \to S$ and $\sigma\colon S^7 \to S$ are detected in the Adams spectral sequence by the classes h_0, h_1, h_2 and h_3, respectively. These are infinite cycles in the spectral sequence.*

Proof. In each case, $f\colon S^n \to S$ is the stable form of a fibration $\Sigma^{n+1}f\colon S^{2n+1} \to S^{n+1}$, with mapping cone a projective plane P^2. Here $H^*(P^2) = P(x)/(x^3) = \mathbb{F}_2\{1, x, x^2\}$, where $|x| = n+1$, by Poincaré duality Hence $Sq^{n+1}(x) = x^2 \neq 0$, and the previous lemma applies. Quite explicitly, $\Sigma C_2 = \mathbb{R}P^2$ has a nonzero Sq^1, $\Sigma^2 C_\eta = \mathbb{C}P^2$ has a nonzero Sq^2, $\Sigma^4 C_\nu = \mathbb{H}P^2$ has a nonzero Sq^4 and $\Sigma^8 C_\sigma = \mathbb{O}P^2$ has a nonzero Sq^8. □

The names η, ν and σ for the Hopf maps detected by h_1, h_2 and h_3 are supposedly unrelated to the correspondence between the initial phonemes in the Greek letters "eta", "nu" and "sigma" and in the first three Japanese numerals "ichi", "ni" and "san". We shall see later that none of the classes h_i for $i \geq 4$ survive to the E_∞-term, so there are no maps $S^n \to S$ with nonzero Hopf–Steenrod invariant for $n \geq 8$.

Theorem 12.6. *Let p be odd. There are maps $p\colon S \to S$ and $\alpha_1\colon S^{2p-3} \to S$ that are detected in the Adams spectral sequence by the classes a_0 and h_0, respectively. These are infinite cycles in the spectral sequence.*

Proof. The Bockstein homomorphism β acts nontrivially in the cohomology $H^*(C_p)$ of the mapping cone $C_p = S \cup_p e^1$ of the degree p map $p\colon S \to S$, so $[p] \in \pi_0(S)$ is detected in the Adams spectral sequence by a_0.

The map $\alpha_1 \in \pi_{2p-3}(S)$ is the stable image of the generator of $\pi_{2p}(S^3)^\wedge_p \cong \mathbb{Z}/p$ that we discussed in Theorem 5.3. It can be constructed as the stable attaching map of the $2p$-cell to the 2-cell in $\mathbb{C}P^p$, after p-completion, but this requires proving that the attaching map $\phi\colon S^{2p-1} \to \mathbb{C}P^{p-1}$ compresses into $i\colon S^2 = \mathbb{C}P^1 \subset \mathbb{C}P^{p-1}$. For each $2 \leq k \leq p-1$ the obstruction to compressing a map $S^{2p-1} \to \mathbb{C}P^k$ into $\mathbb{C}P^{k-1}$ lies in $\pi_{2p-1}(S^{2k}) \cong \pi_{2(p-k)-1}(S)$, so if we assume that we know that this group is trivial, after p-completion, then ϕ compresses as $i \circ \alpha$ for a map $\alpha\colon S^{2p-1} \to S^2$. [[Another proof of this fact can be given using the action of roots of unity in $\mathbb{Z}_p$ on $(\mathbb{C}P^p)^\wedge_p$.]] Then i induces a map $j\colon C_\alpha = S^2 \cup_\alpha e^{2p} \to \mathbb{C}P^p$, and $j^*\colon H^*(\mathbb{C}P^p) = \mathbb{F}_p[y]/(y^{p+1}) \to H^*(C_\alpha)$ maps 1 and y^p to generators of $H^*(C_\alpha)$. Since $P^1(y) = y^p$ in $H^*(\mathbb{C}P^p)$, it follows that P^1 acts nontrivially in $H^*(C_\alpha)$, so the stable class α_1 of α is detected by h_0, as claimed. $\square$

12.2. **Naturality.** The essential uniqueness of free resolutions lifts to the level of spectral realizations. Consider diagrams

$$\cdots \to Y^{s+1} \xrightarrow{i} Y^s \to \cdots \to Y^0 = Y$$

and

$$\cdots \to Z^{s+1} \xrightarrow{i} Z^s \to \cdots \to Z^0 = Z$$

with cofibers $K^s = \operatorname{hocofib}(Y^{s+1} \to Y^s)$ and $L^s = \operatorname{hocofib}(Z^{s+1} \to Z^s)$ for all $s \geq 0$. There are associated chain complexes

$$\cdots \to P_2 \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\epsilon} H^*(Y) \to 0$$

and

$$\cdots \to Q_2 \xrightarrow{\partial_2} Q_1 \xrightarrow{\partial_1} Q_0 \xrightarrow{\epsilon} H^*(Z) \to 0$$

of $\mathscr{A}$-modules, where $P_s = H^*(\Sigma^s K^s)$, $Q_s = H^*(\Sigma^s L^s)$, $\partial_s = \partial^* j^*$ and $\epsilon = j^*$.

Theorem 12.7. *Suppose that (a) each cofiber L^s is a wedge sum of Eilenberg–Mac Lane spectra that is bounded below and of finite type, and (b) each map $i\colon Y^{s+1} \to Y^s$ induces the zero map on cohomology. (For instance, the diagrams $\{Y^s\}_s$ and $\{Z^s\}_s$ might be Adams resolutions.) Let $f\colon Y \to Z$ be any map.*

(1) *Each Q_s is a free $\mathscr{A}$-module, and the augmented chain complex $\epsilon\colon P_* \to H^*(Y) \to 0$ is exact.*
(2) *There exists a chain map $g_*\colon Q_* \to P_*$ lifting f^*, in the sense that the diagram*

$$\begin{tikzcd}
\cdots \arrow[r] & P_2 \arrow[r, "\partial_2"] & P_1 \arrow[r, "\partial_1"] & P_0 \arrow[r, "\epsilon"] & H^*(Y) \arrow[r] & 0 \\
\cdots \arrow[r] & Q_2 \arrow[r, "\partial_2"] \arrow[u, "g_2"] & Q_1 \arrow[r, "\partial_1"] \arrow[u, "g_1"] & Q_0 \arrow[r, "\epsilon"] \arrow[u, "g_0"] & H^*(Z) \arrow[r] \arrow[u, "f^*"] & 0
\end{tikzcd}$$

commutes. Furthermore, there is a map of diagrams $\{f^s\colon Y^s \to Z^s\}_s$ lifting f and realizing g_*, in the sense that there is a homotopy commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & Y^2 \arrow[r, "i"] \arrow[d, "f^2"] & Y^1 \arrow[r, "i"] \arrow[d, "f^1"] & Y \arrow[d, "f"] \\
\cdots \arrow[r] & Z^2 \arrow[r, "i"] & Z^1 \arrow[r, "i"] & Z\,,
\end{tikzcd}$$

and given any choice of commuting homotopies, the induced map of homotopy cofibers $g^s\colon K^s \to L^s$ induces $g_s = (\Sigma^s g^s)^*\colon Q_s \to P_s$, for each $s \geq 0$.
(3) *If $\bar{g}_*\colon Q_* \to P_*$ is a second chain map lifting f^*, and $\{\bar{f}^s\}_s$ is a map of diagrams lifting f and realizing $\bar{g}_*$, then g_* and $\bar{g}_*$ are chain homotopic, and $\{f^s\}_s$ and $\{\bar{f}^s\}_s$ are homotopic in the weak sense that the composites $f^s \circ i$ and $\bar{f}^s \circ i\colon Y^{s+1} \to Z^s$ are homotopic for all $s \geq 0$.*

Proof. Freeness of each Q_s is clear from the wedge sum decomposition of L^s. Exactness of $\epsilon\colon P_* \to H^*(Y) \to 0$ is clear from the vanishing of i^*. The existence of a chain map g_* lifting f^* is then standard homological algebra. We need to construct the maps f^s and g^s in a diagram

$$\begin{tikzcd}
\cdots \arrow[r, "i"] & Y^2 \arrow[rr, "i"] \arrow[dl, "j"'] \arrow[dd, "f^2"] & & Y^1 \arrow[rr, "i"] \arrow[dl, "j"] \arrow[dd, "f^1"] & & Y \arrow[dl, "j"] \arrow[dd, "f"] \\
\cdots & & K^1 \arrow[ul, dashed, "\partial"] \arrow[dd, "g^1"] & & K^0 \arrow[ul, dashed, "\partial"] \arrow[dd, "g^0"] & \\
\cdots \arrow[r] & Z^2 \arrow[rr, "i"] \arrow[dl, "j"'] & & Z^1 \arrow[rr, "i"] \arrow[dl, "j"] & & Z \arrow[dl, "j"] \\
\cdots & & L^1 \arrow[ul, dashed, "\partial"] & & L^0 \arrow[ul, dashed, "\partial"] &
\end{tikzcd}$$

of spectra, inducing a commutative diagram

$$
\begin{tikzcd}
 & H^*(\Sigma^2 Y^2) \arrow[dr, hook, "\partial^*"] & & H^*(\Sigma Y^1) \arrow[dr, hook, "\partial^*"] & & H^*(Y) \\
\dots \arrow[ur, two heads, "j^*"] \arrow[rr] & & H^*(\Sigma K^1) \arrow[ur, two heads, "j^*"] \arrow[rr] & & H^*(K^0) \arrow[ur, two heads, "j^*"] & \\
 & H^*(\Sigma^2 Z^2) \arrow[uu, "(\Sigma^2 f^2)^*"] \arrow[dr, "\partial^*"] & & H^*(\Sigma Z^1) \arrow[uu, "(\Sigma f^1)^*"] \arrow[dr, "\partial^*"] & & H^*(Z) \arrow[uu, "f^*"'] \\
\dots \arrow[ur, "j^*"] \arrow[rr] & & H^*(\Sigma L^1) \arrow[uu, "(\Sigma g^1)^*"'] \arrow[ur, "j^*"] \arrow[rr] & & H^*(L^0) \arrow[uu, "(g^0)^*"'] \arrow[ur, "j^*"] &
\end{tikzcd}
$$

of $\mathscr{A}$-modules, with $g_s = (\Sigma^s g^s)^*$.

Inductively, suppose the maps $f = f^0, \dots, f^s$ and $g^0, \dots, g^{s-1}$ are given, for some $s \geq 0$, making the diagram to the right of f^s commute up to homotopy. Then $j^* \circ g_s = (\Sigma^s f^s)^* \circ j^*$, by the assumption that g_0 lifts f^* for $s = 0$, and by the assumption that $\partial^* j^* \circ g_s = g_{s-1} \circ \partial^* j^* = \partial^* (\Sigma^s f^s)^* \circ j^*$ and the injectivity of ∂^* for $s \geq 1$.

We have an isomorphism $[K^s, L^s] \cong \mathrm{Hom}_{\mathscr{A}}(H^*(L^s), H^*(K^s))$, so there is a unique homotopy class of maps $g^s \colon K^s \to L^s$ with $(\Sigma^s g^s)^* = g_s$. Note that $g^s \circ j \colon Y^s \to L^s$ is homotopic to $j \circ f^s \colon Y^s \to L^s$, because of the isomorphism $[Y^s, L^s] \cong \mathrm{Hom}_{\mathscr{A}}(H^*(L^s), H^*(Y^s))$ and the fact that $(g^s \circ j)^* = (j \circ f^s)^*$. (Both isomorphisms follow from hypothesis (a)).

Choosing a commuting homotopy and passing to mapping cones, or appealing to the triangulated structure on the stable category of spectra, we can find a map of homotopy fibers $f^{s+1} \colon Y^{s+1} \to Z^{s+1}$ making the diagram

$$
\begin{tikzcd}
Y^{s+1} \arrow[r, "i"] \arrow[d, "f^{s+1}"'] & Y^s \arrow[r, "j"] \arrow[d, "f^s"'] & K^s \arrow[r, "\partial"] \arrow[d, "g^s"'] & \Sigma Y^{s+1} \arrow[d, "\Sigma f^{s+1}"'] \\
Z^{s+1} \arrow[r, "i"] & Z^s \arrow[r, "j"] & L^s \arrow[r, "\partial"] & \Sigma Z^{s+1}
\end{tikzcd}
$$

commute up to homotopy. This completes the inductive step.

The uniqueness of g_* up to chain homotopy, meaning that any other lift $\bar{g}_*$ is chain homotopic to g_*, is standard homological algebra. We prove that $f^s \circ i$ is homotopic to $\bar{f}^s \circ i$ by induction on s. This is clear for $s = 0$, since $f_0 = \bar{f}_0 = f$. Suppose that $i \circ f^s \simeq f^{s-1} \circ i$ is homotopic to $i \circ \bar{f}^s \simeq \bar{f}^{s-1} \circ i \colon Y^s \to Z^{s-1}$, for some $s \geq 1$.

$$
\begin{tikzcd}
Y^{s+1} \arrow[r, "i"] & Y^s \arrow[r, "i"] \arrow[d, shift right, "f^s"'] \arrow[d, shift left, "\bar{f}^s"] & Y^{s-1} \arrow[d, shift right, "f^{s-1}"'] \arrow[d, shift left, "\bar{f}^{s-1}"] \\
 & Z^s \arrow[r, "i"] & Z^{s-1} \\
 & \Sigma^{-1} L^{s-1} \arrow[u, "\partial"] &
\end{tikzcd}
$$

Then $i \circ (\bar{f}^s - f^s)$ is null-homotopic, so that $\bar{f}^s - f^s$ factors through a map $h \colon Y^s \to \Sigma^{-1} L^{s-1}$. Then $\bar{f}^s \circ i - f^s \circ i = (\bar{f}^s - f^s) \circ i$ factors through $h \circ i \colon Y^{s+1} \to \Sigma^{-1} L^{s-1}$. This map induces $i^* \circ h^* = 0$ in cohomology, hence is null-homotopic because of the isomorphism $[Y^{s+1}, \Sigma^{-1} L^{s-1}] \cong \mathrm{Hom}_{\mathscr{A}}(H^*(\Sigma^{-1} L^{s-1}), H^*(Y^{s+1}))$. In other words, $f^s \circ i \simeq \bar{f}^s \circ i$. $\square$

Corollary 12.8. *Let $f \colon Y \to Z$ be a map of bounded below spectra with $H_*(Y)$ and $H_*(Z)$ of finite type. Then there is a well-defined map*

$$f_* \colon \{E_r(Y), d_r\}_r \longrightarrow \{E_r(Z), d_r\}_r$$

of Adams spectral sequences for $r \geq 2$, given at the E_2-level by the homomorphism

$$(f^*)^* \colon \mathrm{Ext}_{\mathscr{A}}^{s,t}(H^*(Y), \mathbb{F}_2) \longrightarrow \mathrm{Ext}_{\mathscr{A}}^{s,t}(H^*(Z), \mathbb{F}_2)$$

induced by the $\mathscr{A}$-module homomorphism $f^* \colon H^*(Z) \to H^*(Y)$, with expected abutment the homomorphism

$$f_* \colon \pi_*(Y) \to \pi_*(Z)\,.$$

(Similarly for the Adams spectral sequences converging to $[X,Y]_*$ and $[X,Z]_*$, for any spectrum X.)

Lemma 12.9. *Let $\{Y^s\}_s$ and $\{Z^s\}_s$ be Adams resolutions of a bounded below spectrum Y with $H_*(Y)$ of finite type. Then there is a homotopy equivalence $\operatorname{holim}_s Y^s \simeq \operatorname{holim}_s Z^s$.*

Proof. There are maps $\{f^s\colon Y^s \to Z^s\}_s$ and $\{\tilde{f}^s\colon Z^s \to Y^s\}_s$ of resolutions covering the identity map of $Y = Y^0 = Z^0$, and homotopies $\tilde{f}^s \circ f^s \circ i \simeq i\colon Y^{s+1} \to Y^s$ and $f^s \circ \tilde{f}^s \circ i \simeq i\colon Z^{s+1} \to Z^s$, for all $s \ge 0$. Hence $\operatorname{holim}_s f^s$ and $\operatorname{holim}_s \tilde{f}^s$ are homotopy inverses. □

Theorem 12.10. *Let $\{Y^s\}_s$ be an Adams resolution of Y, and let X be any spectrum. (The case $X = S$ is of particular interest.) A class $[f] \in [X,Y]_n$ has Adams filtration $\ge s$, i.e., is in the image F^s of $i^s\colon [X,Y^s]_n \to [X,Y]_n$, if and only if the representing map $f\colon \Sigma^n X \to Y$ can be factored as the composite of s maps*

$$\Sigma^n X = X_s \xrightarrow{z_s} X_{s-1} \xrightarrow{z_{s-1}} \dots \xrightarrow{z_2} X_1 \xrightarrow{z_1} X_0 = Y$$

where $0 = z_u^*\colon H^*(X_{u-1}) \to H^*(X_u)$ for each $1 \le u \le s$. In particular, $F^s \subset [X,Y]_*$ is independent of the choice of Adams resolution.

Proof. If $[f]$ has Adams filtration $\ge s$, let $g\colon \Sigma^n X \to Y^s$ be a lift, with $i^s \circ g \simeq f$. Let $X_u = Y^u$ and $z_u = i$ for $0 \le u \le s-1$, and let $z_s = ig$:

$$\Sigma^n X \xrightarrow{ig} Y^{s-1} \xrightarrow{i} \dots \xrightarrow{i} Y^1 \xrightarrow{i} Y$$

Conversely, given a factorization $f = z_1 \circ \dots \circ z_s$ as above, let $f^0\colon Y \to Y$ be the identity map. We can inductively find lifts $f^u\colon X_u \to Y^u$ making the diagram

$$\begin{tikzcd}
X_s \arrow[r, "z_s"] \arrow[d, "f^s"'] & X_{s-1} \arrow[r, "z_{s-1}"] \arrow[d, "f^{s-1}"'] & \dots \arrow[r, "z_2"] & X_1 \arrow[r, "z_1"] \arrow[d, "f^1"'] & Y \arrow[d] \\
Y^s \arrow[r, "i"] & Y^{s-1} \arrow[r, "i"] & \dots \arrow[r, "i"] & Y^1 \arrow[r, "i"] & Y
\end{tikzcd}$$

commute, since the obstruction to lifting $f^{u-1} \circ z_u\colon X_u \to Y^{u-1}$ over $i\colon Y^u \to Y^{u-1}$ is the homotopy class of the composite $j \circ f^{u-1} \circ z_u\colon X_u \to K^{u-1}$, which is zero because $z_u^* = 0$. Let $g = f^s\colon \Sigma^n X \to Y^s$. Then $i^s \circ g \simeq f$, and $[f]$ has Adams filtration $\ge s$. □

12.3. **Convergence.**

Definition 12.11. For each natural number m let the mod m Moore spectrum $S/m = S \cup_m e^1$ be defined by the homotopy cofiber sequence

$$S \xrightarrow{m} S \longrightarrow S/m \longrightarrow S^1$$

where the map m induces multiplication by m in integral (co-)homology. Note that $H_*(S/m;\mathbb{Z}) \cong \mathbb{Z}/m$ is concentrated in degree 0. For any spectrum Y let $Y/m = Y \wedge S/m$, so that there is a cofiber sequence

$$Y \xrightarrow{m} Y \longrightarrow Y/m \longrightarrow \Sigma Y\,.$$

Applying $F(-,Y)$ to the cofiber sequence

$$S^{-1} \longrightarrow S^{-1}/m \longrightarrow S \xrightarrow{m} S$$

leads to the cofiber sequence

$$Y \xrightarrow{m} Y \longrightarrow F(S^{-1}/m, Y) \longrightarrow \Sigma Y$$

and an equivalence $Y/m \simeq F(S^{-1}/m, Y)$.

Definition 12.12. For each prime p there is a horizontal tower of vertical cofiber sequences

$$\begin{tikzcd}
\dots \arrow[r, "p"] & S \arrow[r, "p"] \arrow[d, "p^e"'] & \dots \arrow[r, "p"] & S \arrow[r, "p"] \arrow[d, "p^2"] & S \arrow[d, "p"] \\
\dots \arrow[r, "="] & S \arrow[r, "="] \arrow[d] & \dots \arrow[r, "="] & S \arrow[r, "="] \arrow[d] & S \arrow[d] \\
\dots \arrow[r] & S/p^e \arrow[r] \arrow[d] & \dots \arrow[r] & S/p^2 \arrow[r] \arrow[d] & S/p \arrow[d] \\
\dots \arrow[r, "p"] & S^1 \arrow[r, "p"] & \dots \arrow[r, "p"] & S^1 \arrow[r, "p"] & S^1
\end{tikzcd}$$

We define the *p-completion* of Y as the homotopy limit $Y^\wedge_p = \operatorname{holim}_e Y/p^e$ of the tower

$$\cdots \to Y \wedge S/p^e \to \cdots \to Y \wedge S/p^2 \to Y \wedge S/p\,.$$

The maps $S \to S/p^e$ induce the p-completion map $Y \to Y^\wedge_p$.

Dually there is a horizontal sequence of vertical cofiber sequence

$$\begin{tikzcd}
S^{-1} \arrow[r, "p"] \arrow[d] & S^{-1} \arrow[r, "p"] \arrow[d] & \dots \arrow[r, "p"] & S^{-1} \arrow[r, "p"] \arrow[d] & \dots \\
S^{-1}/p \arrow[r] \arrow[d] & S^{-1}/p^2 \arrow[r] \arrow[d] & \dots \arrow[r] & S^{-1}/p^e \arrow[r] \arrow[d] & \dots \\
S \arrow[r, "="] \arrow[d, "p"'] & S \arrow[r, "="] \arrow[d, "p^2"'] & \dots \arrow[r, "="] & S \arrow[r, "="] \arrow[d, "p^e"'] & \dots \\
S \arrow[r, "p"] & S \arrow[r, "p"] & \dots \arrow[r, "p"] & S \arrow[r, "p"] & \dots
\end{tikzcd}$$

Let $S^{-1}/p^\infty = \operatorname{hocolim}_e S^{-1}/p^e$. Note that $H_*(S^{-1}/p^\infty;\mathbb{Z}) \cong \mathbb{Z}/p^\infty \cong \mathbb{Q}/\mathbb{Z}_{(p)} \cong \mathbb{Q}_p/\mathbb{Z}_p$. Applying $F(-,Y)$ we get the tower defining the p-completion, so

$$Y^\wedge_p \simeq F(S^{-1}/p^\infty, Y)\,.$$

The map $S^{-1}/p^\infty \to S$ induces the p-completion map $Y \to Y^\wedge_p$.

((See Bousfield.))

Lemma 12.13. *The p-completion map induces an equivalence $Y/p^e \to (Y^\wedge_p)/p^e$ for each e. Hence it induces an isomorphism $H_*(Y) \cong H_*(Y^\wedge_p)$ in mod p homology (and cohomology). The p-completion map $Y/p^e \to (Y/p^e)^\wedge_p$ for Y/p^e is also an equivalence.*

Proof. The map $S^{-1}/p^\infty \to S$ induces an equivalence $S^{-1}/p^e \wedge S^{-1}/p^\infty \to S^{-1}/p^e \wedge S = S^{-1}/p^e$, for each e, since $p^{-1}\pi_*(S/p^e) = 0$. Apply $F(-,Y)$ to get the first conclusion. Apply integral homology to the equivalence $Y/p \to (Y^\wedge_p)/p$ to get the second conclusion. Applying $F(-,Y)$ to the interchanged equivalence $S^{-1}/p^\infty \wedge S^{-1}/p^e \to S \wedge S^{-1}/p^e$ leads to the third conclusion. □

Lemma 12.14. *The p-completion of the p-completion map for Y, and the p-completion map for $Y^\wedge_p$, are equivalences $Y^\wedge_p \to (Y^\wedge_p)^\wedge_p$. In either sense, p-completion is idempotent up to equivalence.*

Proof. Use that the map $S^{-1}/p^\infty \to S$ induces equivalences $S^{-1}/p^\infty \wedge S^{-1}/p^\infty \to S^{-1}/p^\infty \wedge S$ and $S^{-1}/p^\infty \wedge S^{-1}/p^\infty \to S \wedge S^{-1}/p^\infty$, and apply $F(-,Y)$, or pass to homotopy limits over e from the previous lemma. □

Lemma 12.15. *Let $\pi_n(Y)^\wedge_p = \lim_e \pi_n(Y) \otimes \mathbb{Z}/p^e$ be the algebraic p-completion of $\pi_n(Y)$. There is a short exact sequence*

$$0 \to \pi_n(Y)^\wedge_p \to \lim_e \pi_n(Y/p^e) \to \operatorname{Hom}(\mathbb{Z}/p^\infty, \pi_{n-1}(Y)) \to 0$$

and an isomorphism $\operatorname{Rlim}_e \pi_{n+1}(Y/p^e) \cong \operatorname{Rlim}_e \operatorname{Hom}(\mathbb{Z}/p^e, \pi_n(Y))$. If $\pi_*(Y)$ is of finite type, i.e., if $\pi_n(Y)$ is finitely generated for each n, then $\pi_n(Y) \otimes \mathbb{Z}_p \cong \pi_n(Y)^\wedge_p \cong \pi_n(Y^\wedge_p)$ for all n.

Proof. We have a tower of short exact sequences

$$0 \to \pi_n(Y) \otimes \mathbb{Z}/p^e \longrightarrow \pi_n(Y/p^e) \longrightarrow \operatorname{Hom}(\mathbb{Z}/p^e, \pi_{n-1}(Y)) \to 0$$

for $e \geq 1$. Each homomorphism $\pi_n(Y)\otimes\mathbb{Z}/p^{e+1} \to \pi_n(Y)\otimes\mathbb{Z}/p^e$ is surjective, so $\operatorname{Rlim}_e \pi_n(Y)\otimes\mathbb{Z}/p^e = 0$. Hence the associated lim-Rlim exact sequence breaks up into a short exact sequence

$$0 \to \lim_e \pi_n(Y) \otimes \mathbb{Z}/p^e \longrightarrow \lim_e \pi_n(Y/p^e) \longrightarrow \lim_e \operatorname{Hom}(\mathbb{Z}/p^e, \pi_{n-1}(Y)) \to 0$$

and an isomorphism

$$\operatorname*{Rlim}_e \pi_n(Y/p^e) \cong \operatorname*{Rlim}_e \operatorname{Hom}(\mathbb{Z}/p^e, \pi_{n-1}(Y))\,.$$

Here $\lim_e \pi_n(Y) \otimes \mathbb{Z}/p^e = \pi_n(Y)^\wedge_p$ and $\lim_e \operatorname{Hom}(\mathbb{Z}/p^e, \pi_{n-1}(Y)) \cong \operatorname{Hom}(\mathbb{Z}/p^\infty, \pi_{n-1}(Y))$.

If $\pi_n(Y)$ is finitely generated, then clearly $\pi_n(Y)\otimes\mathbb{Z}_p \cong \pi_n(Y)^\wedge_p$. Furthermore, each $\operatorname{Hom}(\mathbb{Z}/p^e, \pi_n(Y))$ is finite, so $\operatorname{Rlim}_e \pi_{n+1}(Y/p^e) = \operatorname{Rlim}_e \operatorname{Hom}(\mathbb{Z}/p^e, \pi_n(Y)) = 0$. If also $\pi_{n-1}(Y)$ is finitely generated, then its p-torsion subgroup is annihilated by p^N for some fixed N. Hence $\operatorname{Hom}(\mathbb{Z}/p^e, \pi_{n-1}(Y)) \subset \pi_{n-1}(Y)$

equals that p-torsion subgroup for all $e \ge N$, and the homomorphisms in the limit system induce multiplication by p, hence are nilpotent. Thus $\lim_e \operatorname{Hom}(\mathbb{Z}/p^e, \pi_{n-1}(Y)) = 0$. Thus the lim-Rlim exact sequence

$$0 \to \operatorname{Rlim}_e \pi_{n+1}(Y/p^e) \longrightarrow \pi_n(Y_p^\wedge) \longrightarrow \lim_e \pi_n(Y/p^e) \to 0$$

for $Y_p^\wedge = \operatorname{holim}_e Y/p^e$ simplifies to an isomorphism $\pi_n(Y_p^\wedge) \cong \lim_e \pi_n(Y/p^e)$, and the short exact sequence above simplifies to another isomorphism $\pi_n(Y)_p^\wedge \cong \lim_e \pi_n(Y/p^e)$. □

Example 12.16. (1) $H \simeq H_p^\wedge$ and $(H\mathbb{Z})_p^\wedge \simeq (H\mathbb{Z}_{(p)})_p^\wedge \simeq H\mathbb{Z}_p$.

(2) For $Y = H\mathbb{Z}[1/p]$ or $H\mathbb{Q}$ we have $Y/p^e \simeq *$ for all e, so $(H\mathbb{Z}[1/p])_p^\wedge \simeq (H\mathbb{Q})_p^\wedge \simeq *$.

(3) For $Y = H(\mathbb{Z}[1/p]/\mathbb{Z}) = H\mathbb{Z}/p^\infty$ or $H(\mathbb{Q}/\mathbb{Z})$ we have $Y/p^e \simeq \Sigma H\mathbb{Z}/p^e$ for all e, so $H(\mathbb{Z}[1/p]/\mathbb{Z})_p^\wedge = H(\mathbb{Z}/p^\infty)_p^\wedge \simeq H(\mathbb{Q}/\mathbb{Z})_p^\wedge \simeq \Sigma H\mathbb{Z}_p$.

Lemma 12.17. *Let $0 \to \bigoplus_\alpha \mathbb{Z} \to \bigoplus_\beta \mathbb{Z} \to \mathbb{Z}_p \to 0$ be a short free resolution of $\mathbb{Z}_p$. There is a corresponding cofiber sequence $\bigvee_\alpha S \to \bigvee_\beta S \to S\mathbb{Z}_p$, where $H_*(S\mathbb{Z}_p;\mathbb{Z}) \cong \mathbb{Z}_p$ is concentrated in degree 0. Then $\pi_n(Y \wedge S\mathbb{Z}_p) \simeq \pi_n(Y) \otimes \mathbb{Z}_p$ for all n. In particular, $S_p^\wedge \simeq (S\mathbb{Z}_p)_p^\wedge \simeq S\mathbb{Z}_p$. If $\pi_*(Y)$ is of finite type then the natural map $Y \wedge S\mathbb{Z}_p \to Y_p^\wedge$ is an equivalence, and $H_*(Y) \to H_*(Y_p^\wedge)$ is an isomorphism.*

Proof. ((Straightforward. TBW.)) □

Let $H\mathbb{Z}$ be the integral Eilenberg–Mac Lane spectrum, with $\pi_0(H\mathbb{Z}) = \mathbb{Z}$ and $\pi_i(H\mathbb{Z}) = 0$ for $i \ne 0$. It is a ring spectrum, with multiplication $\phi\colon H\mathbb{Z} \wedge H\mathbb{Z} \to H\mathbb{Z}$ and unit $\eta\colon S \to H\mathbb{Z}$. (Not to be confused with the Hopf map $\eta\colon S^1 \to S$.) Let $\overline{H\mathbb{Z}} = H\mathbb{Z}/S$ be the cofiber.

Lemma 12.18. *$H^*(H\mathbb{Z}) \cong \mathscr{A}/\mathscr{A}\{Sq^1\}$ for $p = 2$, and $H^*(H\mathbb{Z}) \cong \mathscr{A}/\mathscr{A}\{\beta\}$ for p odd.*

Proof. Since the unit map $S \to H\mathbb{Z}$ induces an isomorphism on π_0 and a surjection on π_1, we find that $\overline{H\mathbb{Z}}$ is 1-connected. Hence $H^1(H\mathbb{Z}) \cong H^1(\overline{H\mathbb{Z}}) = 0$.

There is a short exact sequence of $\mathscr{A}$-modules

$$0 \longleftarrow \mathscr{A}/\mathscr{A}\{Sq^1\} \longleftarrow \mathscr{A} \longleftarrow \Sigma\mathscr{A}/\mathscr{A}\{Sq^1\} \longleftarrow 0$$

where the right hand arrow takes $\Sigma 1$ to Sq^1. It is clear that $\Sigma Sq^I \mapsto Sq^I \circ Sq^1$ maps to 0, for admissible I, if and only if $I = (i_1, \dots, i_\ell)$ with $i_\ell = 1$. These Sq^I generate precisely the left ideal $\mathscr{A}\{Sq^1\}$.

There is also a cofiber sequence $H\mathbb{Z} \xrightarrow{2} H\mathbb{Z} \longrightarrow H \longrightarrow \Sigma H\mathbb{Z}$, where $2^* = 0$, so that there is an associated short exact sequence

$$0 \longleftarrow H^*(H\mathbb{Z}) \longleftarrow H^*(H) \longleftarrow \Sigma H^*(H\mathbb{Z}) \longleftarrow 0\,.$$

in cohomology. Let $\mathscr{A} \to H^*(H)$ be the isomorphism taking Sq^I to its value on the generator $1 \in H^0(H)$. The composite $\Sigma\mathscr{A}/\mathscr{A}\{Sq^1\} \to \mathscr{A} \to H^*(H) \to H^*(H\mathbb{Z})$ is zero, since the source is generated by $\Sigma 1$ in degree 1, and $H^1(H\mathbb{Z}) = 0$. Hence there is a map from the first short exact sequence of $\mathscr{A}$-modules to the second one. By induction, we may assume that the left hand homomorphism $f\colon \mathscr{A}/\mathscr{A}\{Sq^1\} \to H^*(H\mathbb{Z})$ is an isomorphism in degrees $* < t$. Then the right hand homomorphism $\Sigma f\colon \Sigma\mathscr{A}/\mathscr{A}\{Sq^1\} \to \Sigma H^*(H\mathbb{Z})$ is an isomorphism in degrees $* \le t$. Since the middle map is an isomorphism, it follows that the left hand homomorphism is an isomorphism, also in degree t.

The proof for odd p is similar, comparing the short exact sequence

$$0 \longleftarrow \mathscr{A}/\mathscr{A}\{\beta\} \longleftarrow \mathscr{A} \longleftarrow \Sigma\mathscr{A}/\mathscr{A}\{\beta\} \longleftarrow 0$$

to the short exact sequence

$$0 \longleftarrow H^*(H\mathbb{Z}) \longleftarrow H^*(H) \longleftarrow H^*(\Sigma H\mathbb{Z}) \longleftarrow 0\,.$$

□

Recall Boardman's notion of conditional convergence, meaning that $\lim_s A^s = 0$ and $\operatorname{Rlim}_s A^s = 0$, and the result that strong convergence follows from conditional convergence and the vanishing of the derived E_∞-term RE_∞. For the spectral sequence associated to an Adams resolution $\{Y^s\}_s$, conditional convergence is equivalent to the contractibility of the homotopy limit $Y^\infty = \operatorname{holim}_s Y^s$, in view of Milnor's short exact sequence

$$0 \to \operatorname{Rlim}_s \pi_{n+1}(Y^s) \to \pi_n(\operatorname{holim}_s Y^s) \to \lim_s \pi_n(Y^s) \to 0\,.$$

As we have seen before, the condition $\operatorname{holim}_s Y^s \simeq *$ is independent of the choice of Adams resolution.

Lemma 12.19. *Let Y be bounded below with $H_*(Y)$ of finite type. Then there is an Adams resolution $\{Z^s\}_s$ of $Z = Y/p$ with $\operatorname{holim}_s Z^s \simeq *$.*

((Enough that Y/p is bounded below with $H_*(Y/p)$ of finite type?))

Proof. The "canonical $H\mathbb{Z}$-based resolution"

$$\begin{tikzcd}
\dots \arrow[r] & (\Sigma^{-1}\overline{H\mathbb{Z}})^{\wedge p} \arrow[r, "i"] \arrow[d, "j"] & \Sigma^{-1}\overline{H\mathbb{Z}} \arrow[r, "i"] \arrow[d, "j"] & S \arrow[d, "j"] \\
& H\mathbb{Z} \wedge (\Sigma^{-1}\overline{H\mathbb{Z}})^{\wedge p} & H\mathbb{Z} \wedge \Sigma^{-1}\overline{H\mathbb{Z}} & H\mathbb{Z}
\end{tikzcd}$$

is not an Adams resolution, since $H\mathbb{Z}$ is not a wedge sum of mod p Eilenberg–Mac Lane spectra, but the ring spectrum structure ensures that $j = \eta \wedge 1 \colon X \to H\mathbb{Z} \wedge X$ induces a split injection $1 \wedge j \colon H \wedge X \to H \wedge H\mathbb{Z} \wedge X$, so that $j^* \colon H^*(H\mathbb{Z} \wedge X) \to H^*(X)$ is surjective, for each spectrum X.

Smashing this diagram with $Z = Y/p$, we get a diagram

$$\begin{tikzcd}
\dots \arrow[r] & (\Sigma^{-1}\overline{H\mathbb{Z}})^{\wedge p} \wedge Y/p \arrow[r, "i"] \arrow[d, "j"] & \Sigma^{-1}\overline{H\mathbb{Z}} \wedge Y/p \arrow[r, "i"] \arrow[d, "j"] & Y/p \arrow[d, "j"] \\
& H \wedge (\Sigma^{-1}\overline{H\mathbb{Z}})^{\wedge p} \wedge Y & H \wedge \Sigma^{-1}\overline{H\mathbb{Z}} \wedge Y & H \wedge Y
\end{tikzcd}$$

where we have identified $H\mathbb{Z} \wedge X \wedge Y/p$ with $H \wedge X \wedge Y$, for suitable X. This is the desired Adams resolution, with $Z^s = (\Sigma^{-1}\overline{H\mathbb{Z}})^{\wedge s} \wedge Y/p$ and cofibers $L^s = H \wedge (\Sigma^{-1}\overline{H\mathbb{Z}})^{\wedge s} \wedge Y$. The maps j are split injective, so each j^* is surjective, as before. Since $(\overline{H\mathbb{Z}})^{\wedge s} \wedge Y$ is bounded below and $H_*((\overline{H\mathbb{Z}})^{\wedge s} \wedge Y) \cong H_*(\overline{H\mathbb{Z}})^{\otimes s} \otimes H_*(Y)$ is of finite type, it follows that each L^s is a wedge sum of suspended mod p Eilenberg–Mac Lane spectra, satisfying the finiteness condition required for an Adams resolution.

It remains to show that $\operatorname{holim}_s Z^s \simeq *$. This is true in the strong sense that in each topological degree n, $\pi_n(Z^s) = 0$ for all sufficiently large s. By assumption there is an integer N such that $\pi_n(Y) = 0$ for all $n < N$. We have seen that $\overline{H\mathbb{Z}}$ is 1-connected, so that $(\Sigma^{-1}\overline{H\mathbb{Z}})^{\wedge s}$ is $(s-1)$-connected. Then $Z^s = (\Sigma^{-1}\overline{H\mathbb{Z}})^{\wedge s} \wedge Y/p$ is $(N+s-1)$-connected. Hence $\pi_n(Z^s) = 0$ for all $n \le N+s-1$, or equivalently, for all $s > n - N$. $\square$

Theorem 12.20. *Let Y be bounded below with $H_*(Y)$ of finite type. Then the Adams spectral sequence*

$$E_2^{s,t} = \operatorname{Ext}_{\mathscr{A}}^{s,t}(H^*(Y), \mathbb{F}_p) \Longrightarrow \pi_{t-s}(Y_p^\wedge)$$

is strongly convergent. In particular, there is a strongly convergent Adams spectral sequence

$$E_2^{s,t} = \operatorname{Ext}_{\mathscr{A}}^{s,t}(\mathbb{F}_p, \mathbb{F}_p) \Longrightarrow \pi_{t-s}(S)_p^\wedge \,.$$

More generally, the Adams spectral sequence

$$E_2^{s,t} = \operatorname{Ext}_{\mathscr{A}}^{s,t}(H^*(Y), H^*(X)) \Longrightarrow [X, Y_p^\wedge]_{t-s}$$

is conditionally convergent. It is strongly convergent when $RE_\infty = 0$, which happens, for instance, if $H^*(X)$ is of finite type and bounded above, or if the spectral sequence collapses at a finite stage.

Proof. Let $\{Y^s\}_s$ be an Adams resolution of $Y^0 = Y$, with cofiber sequences

$$Y^{s+1} \xrightarrow{i} Y^s \xrightarrow{j} K^s \xrightarrow{\partial} \Sigma Y^{s+1} \,.$$

Smashing with S/p^e for each $e \ge 1$, we get a tower of Adams resolutions $\{Y^s/p^e\}_s$ of $Y^0/p^e = Y/p^e$, with cofiber sequences

$$Y^{s+1}/p^e \xrightarrow{i} Y^s/p^e \xrightarrow{j} K^s/p^e \xrightarrow{\partial} \Sigma Y^{s+1}/p^e \,.$$

(We check that these diagrams satisfy the conditions to be Adams resolutions: Each homomorphism $j^* \colon H^*(K^s/p^e) \to H^*(Y^s/p^e)$ can be rewritten as $j^* \otimes 1 \colon H^*(K^s) \otimes H^*(S/p^e) \to H^*(Y^s) \otimes H^*(S/p^e)$, hence remains surjective. Each cofiber K^s/p^e sits in a cofiber sequence

$$K^s \xrightarrow{p^e} K^s \longrightarrow K^s/p^e \longrightarrow \Sigma K^s$$

where p^e is null-homotopic, so that $K^s/p^e \simeq K^s \vee \Sigma K^s$ is still a suitably finite wedge sum of mod p Eilenberg–Mac Lane spectra.) Now pass to the homotopy limit over e of these Adams resolutions. The result is a diagram $\{(Y^s)_p^\wedge\}_s$ of spectra, with cofiber sequences

$$(Y^{s+1})_p^\wedge \xrightarrow{i} (Y^s)_p^\wedge \xrightarrow{j} (K^s)_p^\wedge \xrightarrow{\partial} \Sigma(Y^{s+1})_p^\wedge \,.$$

(Cofiber sequences are fiber sequences, up to a sign, hence are preserved by passage to homotopy limits, such as completions.) It is again an Adams resolution, since the completion map $K^s \to (K^s)^\wedge_p$ is an equivalence ($K^s \simeq \bigvee_u \Sigma^{n_u} H \simeq \prod_u \Sigma^{n_u} H$ and $H \to H^\wedge_p$ is easily seen to be an equivalence, see Lemma 12.13) and $j\colon (Y^s)^\wedge_p \to (K^s)^\wedge_p$ induces the "same" map as $j\colon Y^s \to K^s$ in mod p cohomology. We get the following vertical maps of Adams resolutions:

$$\begin{tikzcd}
\operatorname{holim}_s Y^s \arrow[rr] \arrow[dd] & & Y^2 \arrow[rr, "i"] \arrow[dl, "j"] \arrow[dd] & & Y^1 \arrow[rr, "i"] \arrow[dl, "j"] \arrow[dd] & & Y \arrow[dl, "j"] \arrow[dd] \\
& K^2 \arrow[dd, "\simeq"'] & & K^1 \arrow[dd, "\simeq"'] & & K^0 \arrow[dd, "\simeq"'] & \\
\operatorname{holim}_s (Y^s)^\wedge_p \arrow[rr] \arrow[dd] & & (Y^2)^\wedge_p \arrow[rr, "i"] \arrow[dl, "j"] \arrow[dd] & & (Y^1)^\wedge_p \arrow[rr, "i"] \arrow[dl, "j"] \arrow[dd] & & Y^\wedge_p \arrow[dl, "j"] \arrow[dd] \\
& (K^2)^\wedge_p \arrow[dd] & & (K^1)^\wedge_p \arrow[dd] & & (K^0)^\wedge_p \arrow[dd] & \\
\operatorname{holim}_s Y^s/p^e \arrow[rr] & & Y^2/p^e \arrow[rr, "i"] \arrow[dl, "j"] & & Y^1/p^e \arrow[rr, "i"] \arrow[dl, "j"] & & Y/p^e \arrow[dl, "j"] \\
& K^2/p^e & & K^1/p^e & & K^0/p^e &
\end{tikzcd}$$

(We omit the maps $\partial\colon K^s \to \Sigma Y^{s+1}$, etc.) By the previous lemma, there exists an Adams resolution $\{Z^s\}_s$ for Y/p with $\operatorname{holim}_s Z^s \simeq *$. Since this homotopy limit is independent of the choice of resolution, we must also have $\operatorname{holim}_s Y^s/p \simeq *$.

There are cofiber sequences $S/p \to S/p^{e+1} \to S^e \to \Sigma S/p$, inducing cofiber sequences $Y^s/p \to Y^s/p^{e+1} \to Y^s/p^e \to \Sigma Y^s/p$ for all s, hence also

$$\operatorname*{holim}_s Y^s/p \longrightarrow \operatorname*{holim}_s Y^s/p^{e+1} \longrightarrow \operatorname*{holim}_s Y^s/p^e \longrightarrow \Sigma \operatorname*{holim}_s Y^s/p\,.$$

We deduce that $\operatorname{holim}_s Y^s/p^e \simeq *$ for all $e \geq 1$, by induction on e. Thus

$$\operatorname*{holim}_s (Y^s)^\wedge_p = \operatorname*{holim}_s \operatorname*{holim}_e Y^s/p^e \simeq \operatorname*{holim}_e \operatorname*{holim}_s Y^s/p^e \simeq *$$

by the standard exchange of homotopy limits equivalence.

Applying homotopy, we get a map of unrolled exact couples from the one for Y to the one for $Y^\wedge_p$:

$$\begin{tikzcd}
\cdots \arrow[rr] & & \pi_*(Y^2) \arrow[rr, "i"] \arrow[dl, "j"] \arrow[dd] & & \pi_*(Y^1) \arrow[rr, "i"] \arrow[dl, "j"] \arrow[dd] & & \pi_*(Y) \arrow[dl, "j"] \arrow[dd] \\
& \pi_*(K^2) \arrow[ul, dashed, "\partial"] \arrow[dd, "\cong"'] & & \pi_*(K^1) \arrow[ul, dashed, "\partial"] \arrow[dd, "\cong"'] & & \pi_*(K^0) \arrow[ul, dashed, "\partial"] \arrow[dd, "\cong"'] & \\
\cdots \arrow[rr] & & \pi_*((Y^2)^\wedge_p) \arrow[rr, "i"] \arrow[dl, "j"] & & \pi_*((Y^1)^\wedge_p) \arrow[rr, "i"] \arrow[dl, "j"] & & \pi_*(Y^\wedge_p) \arrow[dl, "j"] \\
& \pi_*((K^2)^\wedge_p) \arrow[ul, dashed, "\partial"] & & \pi_*((K^1)^\wedge_p) \arrow[ul, dashed, "\partial"] & & \pi_*((K^0)^\wedge_p) \arrow[ul, dashed, "\partial"] &
\end{tikzcd}$$

This induces a map of spectral sequences, from the Adams spectral sequence for Y to the one associated to the lower exact couple. The equivalences $K^s \to (K^s)^\wedge_p$ induce isomorphisms

$$E_1^{s,t} = \pi_{t-s}(K^s) \xrightarrow{\cong} \pi_{t-s}((K^s)^\wedge_p)$$

of E_1-terms between these spectral sequences. By induction on r, it follows that it also induces an isomorphism of E_r-terms, for all $r \geq 1$. Hence we have two different exact couples generating the same spectral sequence. The upper one is the Adams spectral sequence for Y. The lower one is conditionally convergent to $\pi_*(Y^\wedge_p)$, since $\operatorname{holim}_s (Y^s)^\wedge_p \simeq *$. Hence the Adams spectral sequence for Y, with $E_2^{*,*} =$

$\mathrm{Ext}_{\mathscr{A}}^{*,*}(H^*(Y), \mathbb{F}_p)$, is conditionally convergent to $\pi_*(Y_p^\wedge)$, as asserted. Replacing $\pi_*(-)$ by $[X, -]_*$ we get the same conclusion for the Adams spectral sequence for maps $X \to Y$.

To get strong convergence to $\pi_*(Y_p^\wedge)$ or $[X, Y_p^\wedge]_*$, we need to verify Boardman's criterion $RE_\infty = 0$. In the first case, this follows since $E_2^{s,t}(Y)$ is of finite type, i.e., is finite(-dimensional) in each bidegree (s, t). In fact, this holds already at the E_1-term if we use the canonical Adams resolution for Y, with $\Sigma^s K^s = H \wedge (\bar{H})^{\wedge s} \wedge Y$, since then

$$E_1^{s,t} = \pi_{t-s}(K^s) \cong \pi_t(\Sigma^s K^s) \cong H_t((\bar{H})^{\wedge s} \wedge Y) \cong [H_*(\bar{H})^{\otimes s} \otimes H_*(Y)]_t \,.$$

In the case of a general spectrum X, we have

$$\begin{aligned} E_1^{s,t} &= [X, K^s]_{t-s} \cong [X, \Sigma^s K^s]_t \cong \mathrm{Hom}_{\mathscr{A}}^t(H^*(\Sigma^s K^s), H^*(X)) \\ &\cong \mathrm{Hom}_{\mathscr{A}}^t(\mathscr{A} \otimes I(\mathscr{A})^{\otimes s} \otimes H^*(Y), H^*(X)) \cong \mathrm{Hom}^t(I(\mathscr{A})^{\otimes s} \otimes H^*(Y), H^*(X)) \,. \end{aligned}$$

This group is finite if $H^*(X)$ is of finite type and bounded above, in the sense that there exists an integer N with $H^n(X) = 0$ for $n > N$. For instance, this is the case of X is a finite CW spectrum. □

Proposition 12.21. *Let Y be bounded below with $H_*(Y)$ of finite type. There is a cofiber sequence*

$$\operatorname*{holim}_s Y^s \longrightarrow Y \longrightarrow Y_p^\wedge$$

where $\{Y^s\}_s$ is any Adams resolution of Y.

Proof. We use the notation of the proof above. In view of the equivalences $K^s \simeq (K^s)_p^\wedge$, we get a chain of equivalences

$$\operatorname*{holim}_s \mathrm{hofib}(Y^s \to (Y^s)_p^\wedge) \simeq \mathrm{hofib}(Y^s \to (Y^s)_p^\wedge) \simeq \dots \simeq \mathrm{hofib}(Y \to Y_p^\wedge)$$

for all s. Passing to homotopy limits, we find that

$$\operatorname*{holim}_s Y^s \simeq \mathrm{hofib}(\operatorname*{holim}_s Y^s \to \operatorname*{holim}_s (Y^s)_p^\wedge) \simeq \operatorname*{holim}_s \mathrm{hofib}(Y^s \to (Y^s)_p^\wedge) \simeq \mathrm{hofib}(Y \to Y_p^\wedge) \,.$$

In other words, the p-completion $Y \to Y_p^\wedge$ precisely annihilates the obstruction $\mathrm{holim}_s Y^s$ to conditional convergence for the unrolled exact couple associated to the Adams resolution of Y. □

((Mention Bousfield's E-nilpotent completion $Y_E^\wedge = Y/\,\mathrm{holim}_s Y_E^s$ where $Y_E^s = (\Sigma^{-1}\bar{E})^{\wedge s} \wedge Y$?))

13. Multiplicative structure

13.1. **Composition and the Yoneda product.** Let X, Y and Z be spectra. We have a composition pairing

$$\circ \colon [Y, Z]_* \otimes [X, Y]_* \longrightarrow [X, Z]_*$$

that takes $g \colon \Sigma^v Y \to Z$ and $f \colon \Sigma^t X \to Y$ to the composite $g \circ \Sigma^v f \colon \Sigma^{t+v} X \to Z$. More explicitly, $g \colon Y \wedge S^v \to Z$ and $f \colon X \wedge S^t \to Y$, so $\Sigma^v f = f \wedge 1 \colon X \wedge S^t \wedge S^v \to Y \wedge S^v$ and $g \circ \Sigma^v f \colon X \wedge S^t \wedge S^v \to Z$. To simplify the notation we refer to f and g as maps $f \colon X \to Y$ and $g \colon Y \to Z$ of degree t and v, respectively, and write $gf = g \circ f \colon X \to Y$ for the composite of degree $t + v$.

Suppose that Y and Z are bounded below, and that $H_*(Y)$ and $H_*(Z)$ are of finite type. Let $\{Y^s\}_s$ and $\{Z^u\}_u$ be Adams resolutions of Y and Z, respectively, with cofibers $Y^s/Y^{s+1} = K^s$ and $Z^u/Z^{u+1} = L^u$. If f and g have Adams filtrations $\geq s$ and $\geq u$, meaning that they factor as $f = i^s \tilde{f}$ and $g = i^u \tilde{g}$ with $\tilde{f} \colon X \to Y^s$ and $\tilde{g} \colon Y \to Z^u$ of degree t and v, respectively, then we can lift $\tilde{g}$ to a map $\{g^s\}_s$ of Adams resolutions

$$\begin{tikzcd} X \arrow[d, "\tilde{f}"'] & & \\ Y^s \arrow[r, "i"] \arrow[d, "g^s"'] & \dots \arrow[r, "i"] & Y \arrow[d, "\tilde{g}"] \\ Z^{s+u} \arrow[r, "i"] & \dots \arrow[r, "i"] & Z^u \,. \end{tikzcd}$$

Hence $gf = i^u \tilde{g} i^s \tilde{f} = i^{s+u} g^s f$ factors through $i^{s+u} \colon Z^{s+u} \to Z$, and has Adams filtration $\geq (s + u)$. We thus get a restricted pairing

$$F^u[Y, Z]_* \otimes F^s[X, Y]_* \longrightarrow F^{s+u}[X, Z]_*$$

that induces a pairing

$$F^u/F^{u+1} \otimes F^s/F^{s+1} \longrightarrow F^{s+u}/F^{s+u+1}$$

of filtration subquotients. When the respective spectral sequences converge, we can rewrite this as a pairing

$$E_\infty^{u,*} \otimes E_\infty^{s,*} \longrightarrow E_\infty^{s+u,*}$$

of E_∞-terms. Conversely, this pairing of E_∞-terms will determine the restricted pairings $F^u \otimes F^s \to F^{s+u}$ modulo F^{s+u+1}, i.e., modulo higher Adams filtrations. In this way the pairing of E_∞-terms determines the composition pairing $[Y,Z]_* \otimes [X,Y]_* \to [X,Z]_*$ modulo the Adams filtration.

Example 13.1. ((Example of this phenomenon: $h_2^3 = h_1^2 h_3$ so $\nu^3 \equiv \eta^2\sigma$ modulo Adams filtration ≥ 4. In fact, $\nu^3 = \eta^2\sigma + \eta\epsilon$.))

Let $P_s = H^*(\Sigma^s K^s)$ and $Q_u = H^*(\Sigma^u L^u)$, so that there are free resolutions

$$\cdots \to P_s \xrightarrow{\partial_s} P_{s-1} \to \cdots \to P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\epsilon} H^*(Y) \to 0$$

and

$$\cdots \to Q_u \xrightarrow{\partial_u} Q_{u-1} \to \cdots \to Q_1 \xrightarrow{\partial_1} Q_0 \xrightarrow{\epsilon} H^*(Z) \to 0\,.$$

By definition,

$$\begin{aligned}
\mathrm{Ext}_{\mathscr{A}}^{u,v}(H^*(Z), H^*(Y)) &= H^u(\mathrm{Hom}_{\mathscr{A}}^v(Q_*, H^*(Y))) \\
\mathrm{Ext}_{\mathscr{A}}^{s,t}(H^*(Y), H^*(X)) &= H^s(\mathrm{Hom}_{\mathscr{A}}^t(P_*, H^*(X))) \\
\mathrm{Ext}_{\mathscr{A}}^{u+s,v+t}(H^*(Z), H^*(X)) &= H^{u+s}(\mathrm{Hom}_{\mathscr{A}}^{v+t}(Q_*, H^*(X)))\,.
\end{aligned}$$

The (opposite) Yoneda product is a pairing

$$\mathrm{Ext}_{\mathscr{A}}^{*,*}(H^*(Z), H^*(Y)) \otimes \mathrm{Ext}_{\mathscr{A}}^{*,*}(H^*(Y), H^*(X)) \longrightarrow \mathrm{Ext}_{\mathscr{A}}^{*,*}(H^*(Z), H^*(X))\,,$$

and we shall see that the Adams spectral sequence relates the Yoneda product in $E_2 = \mathrm{Ext}_{\mathscr{A}}(-,-)$ to the composition product in homotopy. (This is the opposite of the usual Yoneda pairing, meaning that the two factors in the source have been interchanged. This comes about due to the contravariance of cohomology. Working at odd primes the interchange introduces a sign.)

Let $f\colon P_s \to \Sigma^t H^*(X)$ and $g\colon Q_u \to \Sigma^v H^*(Y)$ be $\mathscr{A}$-module homomorphisms. To simplify the notation, we will refer to these as homomorphisms $f\colon P_s \to H^*(X)$ and $g\colon Q_u \to H^*(Y)$ of degree t and v, respectively. We also suppose that f and g are cocycles, meaning that $0 = f\partial_{s+1}\colon P_{s+1} \to H^*(X)$ and $0 = g\partial_{u+1}\colon Q_{u+1} \to H^*(Y)$. The cohomology classes $[f]$ and $[g]$ are then elements in $\mathrm{Ext}_{\mathscr{A}}^{s,t}(H^*(Y), H^*(X))$ and $\mathrm{Ext}_{\mathscr{A}}^{u,v}(H^*(Z), H^*(Y))$, respectively. Then g lifts to a chain map $g_* = \{g_n\colon Q_{u+n} \to P_n\}_n$, where each g_n has degree v, making the diagram

$$\begin{tikzcd}
 & H^*(X) & & & & \\
\dots \arrow[r] & P_s \arrow[u, "f"] \arrow[r, "\partial_s"] & \dots \arrow[r] & P_1 \arrow[r, "\partial_1"] & P_0 \arrow[r, "\epsilon"] & H^*(Y) \\
\dots \arrow[r] & Q_{u+s} \arrow[u, "g_s"] \arrow[r, "\partial_{u+s}"] & \dots \arrow[r] & Q_{u+1} \arrow[u, "g_1"] \arrow[r, "\partial_{u+1}"] & Q_u \arrow[u, "g_0"] \arrow[ur, "g"'] &
\end{tikzcd}$$

commute. The composite $fg_s\colon Q_{u+s} \to H^*(X)$ is then an $\mathscr{A}$-module homomorphism of degree $(v+t)$, and satisfies $fg_s\partial_{u+s+1} = 0$. It is therefore a cocycle in $\mathrm{Hom}_{\mathscr{A}}^{v+t}(H^*(Z), H^*(X))$, and its cohomology class $[fg_s]$ in $\mathrm{Ext}_{\mathscr{A}}^{u+s,v+t}(H^*(Z), H^*(X))$ is by definition the Yoneda product of $[g]$ and $[f]$. It is not hard to check that a different choice of chain map lifting g only changes the cocycle fg_s by a coboundary, i.e., a homomorphism that factors through $\partial_{u+s}\colon Q_{u+s} \to Q_{u+s-1}$, so that its cohomology class is unchanged. Likewise, changing f or g by a coboundary only changes fg_s by a coboundary, so that the Yoneda product is well defined. [[TODO: Rewrite this as a clear definition.]]

Example 13.2. Let $X = Y = Z = S$ and let $P_* = Q_*$ be the minimal resolution of $\mathbb{F}_2$ computed earlier. We can compute the Yoneda product

$$\mathrm{Ext}_{\mathscr{A}}^{u,v}(\mathbb{F}_2, \mathbb{F}_2) \otimes \mathrm{Ext}_{\mathscr{A}}^{s,t}(\mathbb{F}_2, \mathbb{F}_2) \longrightarrow \mathrm{Ext}_{\mathscr{A}}^{u+s,v+t}(\mathbb{F}_2, \mathbb{F}_2)$$

that makes $\mathrm{Ext}_{\mathscr{A}}^{*,*}(\mathbb{F}_2, \mathbb{F}_2)$ into a bigraded algebra, by choosing cocycle representatives $f\colon P_s \to \mathbb{F}_2$ and $g\colon P_u \to \mathbb{F}_2$, lifting g to a chain map $g_*\colon P_{u+*} \to P_*$, and computing the composite fg_s.

Let $f = \gamma_{1,0} = h_0 \colon P_1 \to \mathbb{F}_2$ be dual to $g_{1,0} \in P_1$ and let $g = \gamma_{1,2} = h_2 \colon P_1 \to \mathbb{F}_2$ be dual to $g_{1,2} \in P_1$. A lift $g_0 \colon P_1 \to P_0$ of g is given by $g_{1,2} \mapsto g_{0,0}$ and $g_{1,i} \mapsto 0$ for $i \neq 2$.

$$\begin{tikzcd}
\mathbb{F}_2 & & \\
P_1 \arrow[u, "f=h_0"] \arrow[r, "\partial_1"] & P_0 \arrow[r, "\epsilon"] & \mathbb{F}_2 \\
P_2 \arrow[u, "g_1"] \arrow[r, "\partial_2"] & P_1 \arrow[u, "g_0"] \arrow[ur, "g=h_2"'] &
\end{tikzcd}$$

The composite $g_0\partial_2 \colon P_2 \to P_0$ is then given by $g_{2,0} \mapsto 0$, $g_{2,1} \mapsto 0$, $g_{2,2} \mapsto Sq^1 g_{0,0}$, $g_{2,3} \mapsto Sq^4 g_{0,0}$ etc. A lift $g_1 \colon P_2 \to P_1$ is given by $g_{2,0} \mapsto 0$, $g_{2,1} \mapsto 0$, $g_{2,2} \mapsto g_{1,0}$, $g_{2,3} \mapsto g_{1,2}$ etc. Hence $fg_1 \colon P_2 \to \mathbb{F}_2$ is given by $g_{2,2} \mapsto 1$ and $g_{2,i} \mapsto 0$ for $i \neq 2$ (for degree reasons), so that $[fg_1] = \gamma_{2,2}$. Thus $h_0h_2 = \gamma_{2,2}$ in bidegree $(s,t) = (2,4)$ of $\mathrm{Ext}_{\mathscr{A}}^{*,*}(\mathbb{F}_2, \mathbb{F}_2)$. In hindsight, this it the only possible nonzero value of the product, and it is realized because of the summand $Sq^1 g_{1,2}$ in $\partial_2(g_{2,2})$ and the summand $Sq^4 g_{0,0}$ in $\partial_1(g_{1,2})$, with Sq^1 detecting h_0 and Sq^4 detecting h_2.

Proposition 13.3. *Let $P_* \to \mathbb{F}_2$ and $Q_* \to H^*(Z)$ be minimal resolutions, with $P_0 = \mathscr{A}\{[]\}$, $P_1 = \mathscr{A}\{[Sq^{2^i}]\} \mid i \geq 0\}$, $Q_u = \mathscr{A}\{g_{u,j}\}_j$ and $Q_{u+1} = \mathscr{A}\{g_{u+1,k}\}_k$. Here $\partial_1([Sq^{2^i}]) = Sq^{2^i}[]$, and we can write*

$$\partial_{u+1}(g_{u+1,k}) = \sum_{i,j} \theta^k_{i,j} Sq^{2^i} g_{u,j}$$

for suitable coefficients $\theta^k_{i,j} \in \mathscr{A}$. Let $h_i \in \mathrm{Ext}^{1,2^i}_{\mathscr{A}}(\mathbb{F}_2, \mathbb{F}_2)$, $\gamma_{u,j} \in \mathrm{Ext}^{u,*}_{\mathscr{A}}(H^*(Z), \mathbb{F}_2)$ and $\gamma_{u+1,k} \in \mathrm{Ext}^{u+1,*}_{\mathscr{A}}(H^*(Z), \mathbb{F}_2)$ be dual to $[Sq^{2^i}]$, $g_{u,j}$ and $g_{u+1,k}$, respectively. Then

$$h_i \cdot \gamma_{u,j} = \sum_k \epsilon(\theta^k_{i,j}) \gamma_{u+1,k} \,,$$

where $\epsilon \colon \mathscr{A} \to \mathbb{F}_2$ is the augmentation. Hence $h_i \cdot \gamma_{u,j}$ contains the term $\gamma_{u+1,k}$ if and only if $\partial_{u+1}(g_{u+1,k})$ contains the term $Sq^{2^i} g_{u,j}$.

Proof. The coefficient of $g_{u,j}$ in $\partial_{u+1}(g_{u+1,k})$ can be written as a sum $\sum_i \theta^k_{i,j} Sq^{2^i}$, since the Sq^{2^i} generate $I(\mathscr{A})$ as a left $\mathscr{A}$-module, and the coefficient lies in this augmentation ideal, by the assumption that the resolution is minimal. Consider the following diagram, for fixed choices of $i \geq 0$ and j.

$$\begin{tikzcd}
\mathbb{F}_2 & & \\
P_1 \arrow[u, "h_i"] \arrow[r, "\partial_1"] & P_0 \arrow[r, "\epsilon"] & \mathbb{F}_2 \\
Q_{u+1} \arrow[u, "g_1"] \arrow[r, "\partial_{u+1}"'] & Q_u \arrow[u, "g_0"] \arrow[ur, "\gamma_{u,j}"'] &
\end{tikzcd}$$

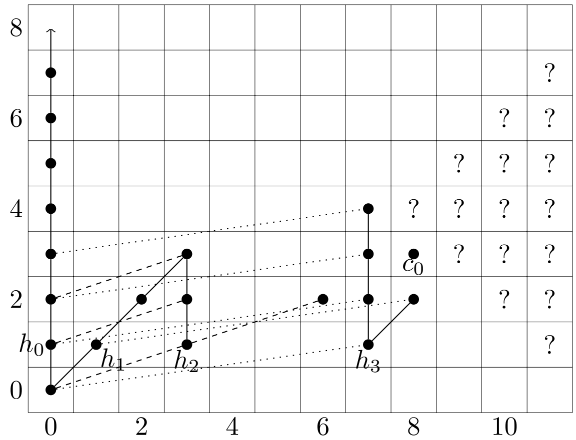
Here $f = h_i$ maps $[Sq^{2^i}]$ to 1 and the remaining generators of P_1 to 0. Likewise $g = \gamma_{u,j}$ maps $g_{u,j}$ to 1 and the remaining generators of Q_u to 0. We lift g to a chain map $g_* \colon Q_{u+*} \to P_*$, by first letting g_0 map $g_{u,j}$ to $1[]$ and sending the other generators of Q_u to 0. Then

$$g_0\partial_{u+1}(g_{u+1,k}) = g_0(\sum_{i,j} \theta^k_{i,j} Sq^{2^i} g_{u,j}) = \sum_i \theta^k_{i,j} Sq^{2^i} [] \,,$$

so we can set $g_1(g_{u+1,k}) = \sum_i \theta^k_{i,j} [Sq^{2^i}]$. Hence the Yoneda product $f \circ g_1 \colon Q_{u+1} \to \mathbb{F}_2$ maps $g_{u+1,k}$ to $\epsilon(\theta^k_{i,j})$, and therefore contains $\gamma_{u+1,k}$ with that coefficient. □

Example 13.4. From the minimal resolution in Theorem 10.11, we can read off the following nontrivial products: $h_0\gamma_{0,0} = \gamma_{1,0}$, $h_1\gamma_{0,0} = \gamma_{1,1}$, $h_2\gamma_{0,0} = \gamma_{1,2}$, $h_3\gamma_{0,0} = \gamma_{1,3}$, $h_0\gamma_{1,0} = \gamma_{2,0}$, $h_1\gamma_{1,1} = \gamma_{2,1}$, $h_2\gamma_{1,0} = \gamma_{2,2}$, $h_0\gamma_{1,2} = \gamma_{2,2}$, $h_2\gamma_{1,2} = \gamma_{2,3}$, $h_3\gamma_{1,0} = \gamma_{2,4}$, $h_0\gamma_{1,3} = \gamma_{2,4}$, $h_3\gamma_{1,1} = \gamma_{2,5}$, $h_1\gamma_{1,3} = \gamma_{2,5}$,

$h_0\gamma_{2,0} = \gamma_{3,0}$, $h_2\gamma_{2,0} = \gamma_{3,1}$, $h_1\gamma_{2,1} = \gamma_{3,1}$, $h_0\gamma_{2,2} = \gamma_{3,1}$, $h_3\gamma_{2,0} = \gamma_{3,2}$, $h_0\gamma_{2,4} = \gamma_{3,2}$, $h_0\gamma_{3,0} = \gamma_{4,0}$, $h_3\gamma_{3,0} = \gamma_{4,1}$, $\dots$, $h_0\gamma_{10,0} = \gamma_{11,0}$. This gives the following multiplicative structure.



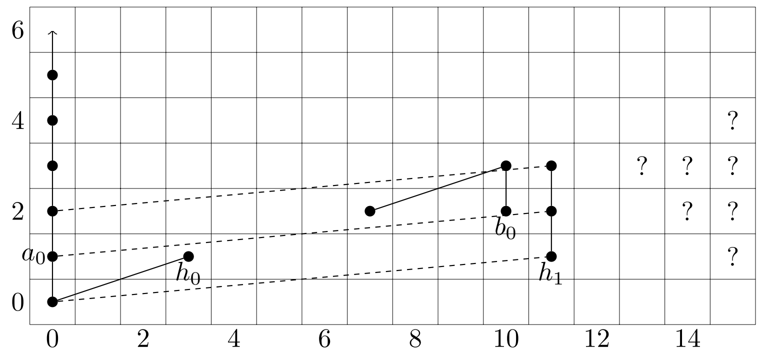
Proposition 13.5. *For p odd, let $Q_* \to H^*(Z)$ be a minimal resolution, with $Q_u = \mathscr{A}\{g_{u,j}\}_j$ and $Q_{u+1} = \mathscr{A}\{g_{u+1,k}\}_k$. Let $\gamma_{u,j} \in \operatorname{Ext}^{u,*}_{\mathscr{A}}(H^*(Z), \mathbb{F}_p)$ and $\gamma_{u+1,k} \in \operatorname{Ext}^{u+1,*}_{\mathscr{A}}(H^*(Z), \mathbb{F}_p)$ be dual to $g_{u,j}$ and $g_{u+1,k}$, respectively. Then the coefficient (in $\mathbb{F}_p$) of $\gamma_{u+1,k}$ in the Yoneda product $a_0 \cdot \gamma_{u,j}$ equals the coefficient of $\beta g_{u,j}$ in $\partial_{u+1}(g_{u+1,k})$, and the coefficient of $\gamma_{u+1,k}$ in $h_i \cdot \gamma_{u,j}$ equals the coefficients of $P^{p^i} g_{u,j}$ in $\partial_{u+1}(g_{u+1,k})$.*

Proof. The proof is similar to the case $p = 2$. We write $\partial_{u+1}(g_{u+1,k})$ as

$$\sum_j (\theta^k_j \beta + \sum_i \theta^k_{i,j} P^{p^i}) g_{u,j}$$

with θ^k_j and $\theta^k_{i,j}$ in $\mathscr{A}$. This is possible, since the resolution is assumed to be minimal. Then $a_0 \cdot \gamma_{u,j} = \epsilon(\theta^k_j)\gamma_{u+1,k}$ and $h_i \cdot \gamma_{u,j} = \epsilon(\theta^k_{i,j})\gamma_{u+1,k}$. □

Example 13.6. From the minimal resolution in Theorem 10.14, we can read off the following nontrivial products: $a_0\gamma_{0,0} = \gamma_{1,0}$, $h_0\gamma_{0,0} = \gamma_{1,1}$, $h_1\gamma_{0,0} = \gamma_{1,2}$, $a_0\gamma_{1,0} = \gamma_{2,0}$, $h_1\gamma_{1,0} = \gamma_{2,3}$, $a_0\gamma_{1,2} = -\gamma_{2,3}$, $a_0\gamma_{2,0} = \gamma_{3,0}$, $h_0\gamma_{2,1} = \gamma_{3,1}$, $a_0\gamma_{2,2} = -\gamma_{3,1}$, $h_1\gamma_{2,0} = \gamma_{3,2}$, $a_0\gamma_{2,3} = -\gamma_{3,2}$, $a_0\gamma_{3,0} = \gamma_{4,0}$, $\dots$, $a_0\gamma_{14,0} = \gamma_{15,0}$. This gives the following multiplicative structure.



Definition 13.7. Consider any two complexes P_* and Q_* of $\mathscr{A}$-modules. Let

$$\mathrm{HOM}^{u,v}_{\mathscr{A}}(Q_*, P_*) = \prod_s \mathrm{Hom}^v_{\mathscr{A}}(Q_{u+s}, P_s)$$

be the abelian group of sequences $\{g_s : Q_{u+s} \to P_s\}_s$ of $\mathscr{A}$-module homomorphisms, each of degree v. Thus $\mathrm{HOM}^u_{\mathscr{A}}(Q_*, P_*)$ is a graded abelian group. Let

$$\delta_u : \mathrm{HOM}^u_{\mathscr{A}}(Q_*, P_*) \to \mathrm{HOM}^{u+1}_{\mathscr{A}}(Q_*, P_*)$$

map $\{g_s\}_s$ to $\{\partial_{s+1} g_{s+1} + g_s \partial_{u+s+1}\}_s$. ((We are working mod 2, so there is no sign.)) Then $\delta_{u+1}\delta_u = 0$, so $\mathrm{HOM}^*_{\mathscr{A}}(Q_*, P_*)$ is a cocomplex of graded abelian groups.

Lemma 13.8. *The kernel*

$$\ker(\delta_0) \subset \mathrm{HOM}^0_{\mathscr{A}}(Q_*, P_*)$$

consists of the chain maps $g_*\colon Q_* \to P_*$, meaning the sequences $\{g_s\colon Q_s \to P_s\}_s$ of $\mathscr{A}$-module homomorphisms such that $\partial_{s+1}g_{s+1} = g_s\partial_{s+1}$ for all s. The image

$$\mathrm{im}(\delta_{-1}) \subset \ker(\delta_0)$$

consists of the chain maps that are chain homotopic to 0, i.e., those of the form $\{\partial_{s+1}h_{s+1} + h_s\partial_s\}_s$ for some collection of $\mathscr{A}$-module homomorphisms $h_{s+1}\colon Q_s \to P_{s+1}$ for all s. Hence the 0-th cohomology

$$H^0(\mathrm{HOM}^*_{\mathscr{A}}(Q_*, P_*)) \cong \{g_*\colon Q_* \to P_*\}/(\simeq) = [Q_*, P_*]$$

is the (graded abelian) group of chain homotopy classes of chain maps $Q_* \to P_*$. More generally, $H^u(\mathrm{HOM}^*_{\mathscr{A}}(Q_*, P_*))$ is the group $[Q_{u+*}, P_*]$ of chain homotopy classes of chain maps $Q_{u+*} \to P_*$.

In the special case when $P_* = H^*(Y)$ is concentrated in filtration $s = 0$, so that $P_0 = H^*(Y)$ and $P_s = 0$ for $s \neq 0$, then $\mathrm{HOM}^{u,v}_{\mathscr{A}}(Q_*, H^*(Y)) \cong \mathrm{Hom}^v_{\mathscr{A}}(Q_u, H^*(Y))$ and $\delta_u = (\partial_{u+1})^*$, so that $H^u(\mathrm{HOM}_{\mathscr{A}}(Q_*, H^*(Y))) \cong H^u(\mathrm{Hom}_{\mathscr{A}}(Q_*, H^*(Y)))$. When Q_* is a free resolution of $H^*(Z)$, this is $\mathrm{Ext}^u_{\mathscr{A}}(H^*(Z), H^*(Y))$.

Proposition 13.9. *Let $\epsilon\colon P_* \to H^*(Y)$ and $\epsilon\colon Q_* \to H^*(Z)$ be free $\mathscr{A}$-module resolutions. Then*

$$\epsilon_*\colon \mathrm{HOM}^*_{\mathscr{A}}(Q_*, P_*) \xrightarrow{\simeq} \mathrm{HOM}^*_{\mathscr{A}}(Q_*, H^*(Y)) \cong \mathrm{Hom}_{\mathscr{A}}(Q_*, H^*(Y))$$

is a quasi-isomorphism, in the sense that it induces an isomorphism

$$\epsilon_*\colon H^u(\mathrm{HOM}^*_{\mathscr{A}}(Q_*, P_*)) \xrightarrow{\cong} \mathrm{Ext}^u_{\mathscr{A}}(H^*(Z), H^*(Y))$$

in cohomology, in each filtration u.

This is standard homological algebra. The first assertion only requires that Q_* is free and $P_* \to H^*(Y)$ is exact, but the final identification with Ext requires that $Q_* \to H^*(Z)$ is exact.

The composition pairing and the quasi-isomorphism

$$\begin{array}{ccc}
\mathrm{HOM}^*_{\mathscr{A}}(Q_*, P_*) \otimes \mathrm{Hom}_{\mathscr{A}}(P_*, H^*(X)) & \longrightarrow & \mathrm{Hom}_{\mathscr{A}}(Q_*, H^*(X)) \\
\simeq \big\downarrow & & \\
\mathrm{Hom}^*_{\mathscr{A}}(Q_*, H^*(Y)) \otimes \mathrm{Hom}_{\mathscr{A}}(P_*, H^*(X)) & &
\end{array}$$

thus induce a pairing and an isomorphism

$$\begin{array}{ccc}
H^u(\mathrm{Hom}^*_{\mathscr{A}}(Q_*, P_*)) \otimes \mathrm{Ext}^s_{\mathscr{A}}(H^*(Y), H^*(X)) & \longrightarrow & \mathrm{Ext}^{u+s}_{\mathscr{A}}(H^*(Z), H^*(X)) \\
\cong \big\downarrow & \nearrow & \\
\mathrm{Ext}^u_{\mathscr{A}}(H^*(Z), H^*(Y)) \otimes \mathrm{Ext}^s_{\mathscr{A}}(H^*(Y), H^*(X)) & &
\end{array}$$

in cohomology, and the Yoneda product is given by the dashed arrow. From this description it is easy to see that the Yoneda product is associative and unital. [[No evident commutativity in this generality.]]

13.2. **Pairings of spectral sequences.**

Definition 13.10. Let $\{'E_r\}_r$, $\{''E_r\}_r$ and $\{E_r\}_r$ be three spectral sequence. A pairing of these spectral sequences is a sequence of homomorphisms

$$\phi_r\colon {}'E_r^{*,*} \otimes {}''E_r^{*,*} \longrightarrow E_r^{*,*}$$

((for $r \geq 1$)) such that the Leibniz rule

$$d_r(\phi_r(x \otimes y)) = \phi_r(d_r(x) \otimes y) + (-1)^n \phi_r(x \otimes d_r(y))$$

holds, where $n = |x|$ is the total degree of x, and

$$\phi_{r+1}([x] \otimes [y]) = [\phi_r(x \otimes y)]$$

where $[x] \in {}'E_{r+1}^{*,*}$ is the homology class of a d_r-cycle $x \in {}'E_r^{*,*}$, and similarly for $[y]$ and the right hand side. In other words, the diagrams

$$\begin{array}{ccc}
{}'E_r^{*,*} \otimes {}''E_r^{*,*} & \xrightarrow{\phi_r} & E_r^{*,*} \\
{\scriptstyle d_r\otimes 1 \pm 1 \otimes d_r}\big\downarrow & & \big\downarrow{\scriptstyle d_r} \\
{}'E_r^{*,*} \otimes {}''E_r^{*,*} & \xrightarrow{\phi_r} & E_r^{*,*}
\end{array}$$

and

$$\begin{tikzcd}
H^{*,*}('E_r) \otimes H^{*,*}(''E_r) \arrow[r] \arrow[d, "\cong"'] & H^{*,*}('E_r \otimes ''E_r) \arrow[r, "(\phi_r)_*"] & H^{*,*}(E_r) \arrow[d, "\cong"] \\
'E_{r+1}^{*,*} \otimes ''E_{r+1}^{*,*} \arrow[rr, "\phi_{r+1}"] & & E_{r+1}^{*,*}
\end{tikzcd}$$

commute.

A spectral sequence pairing $\{\phi_r\}_r$ induces a pairing

$$\phi_\infty \colon 'E_\infty^{*,*} \otimes ''E_\infty^{*,*} \longrightarrow E_\infty^{*,*}$$

of E_∞-terms. ((Clear if each spectral sequence vanishes in negative filtrations, so that in each bidegree (s,t) the E_r-terms eventually form a descending sequence, with intersection equal to the E_∞-term.))

When the Künneth homomorphism $H^{*,*}('E_r) \otimes H^{*,*}(''E_r) \to H^{*,*}('E_r \otimes ''E_r)$ is an isomorphism, for each r, one can readily define a tensor product spectral sequence $\{'E_r \otimes ''E_r\}_r$, and the pairing of spectral sequences is the same as a morphism $\{'E_r \otimes ''E_r\}_r \to \{E_r\}_r$ of spectral sequences.

Definition 13.11. Suppose that the spectral sequences above converge to the graded abelian groups G', G'' and G, respectively, in the sense that there are filtrations $\{'F^s\}_s$, $\{''F^s\}_s$ and $\{F^s\}_s$ of these groups, and isomorphisms $'F^s/'F^{s+1} \cong 'E_\infty^s$, $''F^s/''F^{s+1} \cong ''E_\infty^s$ and $F^s/F^{s+1} \cong E_\infty^s$, for all s.

We say that a pairing $\{\phi_r\}_r$ of spectral sequences, as above, *converges* to a pairing $\phi \colon G' \otimes G'' \to G$ if the latter pairing restricts to homomorphisms $\phi \colon 'F^u \otimes ''F^s \to F^{u+s}$ for all u and s, and if the induced homomorphisms $\phi \colon 'F^u/'F^{u+1} \otimes ''F^s/''F^{s+1} \to F^{u+s}/F^{u+s+1}$ agree with the limit $\phi_\infty \colon 'E_\infty^u \otimes ''E_\infty^s \to E_\infty^{u+s}$ of the pairings ϕ_r.

In other words, the diagram

$$\begin{tikzcd}
'E_\infty^u \otimes ''E_\infty^s \arrow[d, "\phi_\infty"'] & 'F^u/'F^{u+1} \otimes ''F^s/''F^{s+1} \arrow[l, "\cong"'] \arrow[d, "\phi"'] & 'F^u \otimes ''F^s \arrow[l, two heads] \arrow[r] \arrow[d, "\phi"'] & G' \otimes G'' \arrow[d, "\phi"'] \\
E_\infty^{u+s} & F^{u+s}/F^{u+s+1} \arrow[l, "\cong"'] & F^{u+s} \arrow[l, two heads] \arrow[r, tail] & G
\end{tikzcd}$$

commutes. ((Consequences?))

Definition 13.12. An algebra spectral sequence is a spectral sequence $\{E_r\}_r$ with a spectral sequence pairing $\{\phi_r \colon E_r \otimes E_r \to E_r\}_r$ that is associative and unital. It is commutative if the pairing satisfies $\phi_r(y \otimes x) = (-1)^{mn}\phi_r(x \otimes y)$ for all x, y and r, where $n = |x|$ and $m = |y|$ are the total degrees. ((Elaborate?))

Adams (1958) defined a join pairing in his spectral sequence for S, which is stably equivalent to a smash product pairing in that spectral sequence. We shall return to those pairings later, but first look at the case of composition pairings, since these are most closely related to the Yoneda product. ((We may also need to look at this for Moss' later theorem on Toda brackets and Massey products.))

Theorem 13.13 (Moss (1968))**.** *Let X, Y and Z be spectra, with Y and Z bounded below and $H_*(Y)$ and $H_*(Z)$ of finite type. There is a pairing of spectral sequences*

$$E_r^{*,*}(Y,Z) \otimes E_r^{*,*}(X,Y) \longrightarrow E_r^{*,*}(X,Z)$$

which agrees for $r = 2$ with the (opposite) Yoneda pairing

$$\mathrm{Ext}_{\mathscr{A}}^{*,*}(H^*(Z), H^*(Y)) \otimes \mathrm{Ext}_{\mathscr{A}}^{*,*}(H^*(Y), H^*(X)) \longrightarrow \mathrm{Ext}_{\mathscr{A}}^{*,*}(H^*(Z), H^*(X))$$

and which converges to the composition pairing

$$[Y, Z_2^\wedge]_* \otimes [X, Y_2^\wedge]_* \longrightarrow [X, Z_2^\wedge]_* \,.$$

The pairing is associative and unital.

[[We omit this proof, and will instead deduce the theorem (for X and Y finite CW spectra) from a similar theorem about the smash product pairing.]]

13.3. **Modules over cocommutative Hopf algebras.** The Künneth isomorphism $H^*(Y \wedge Z) \cong H^*(Y) \otimes H^*(Z)$ and the universal coefficient theorem $H^*(F(X,S)) \cong \mathrm{Hom}^*(H^*(X), \mathbb{F}_p)$ (for finite CW spectra X) can be refined from being statements about graded $\mathbb{F}_p$-vector spaces to statements about left $\mathscr{A}$-modules. This requires making sense of the tensor product $M \otimes N = M \otimes_{\mathbb{F}_p} N$ and the homomorphism group $\mathrm{Hom}(M,N) = \mathrm{Hom}_{\mathbb{F}_p}(M,N)$ as left $\mathscr{A}$-modules, for given left $\mathscr{A}$-modules M and N.

By the Cartan formula

$$Sq^k(y \wedge z) = \sum_{i+j=k} Sq^i(y) \wedge Sq^j(z)$$

in $H^*(Y \wedge Z)$, for $y \in H^*(Y)$ and $z \in H^*(Z)$, it is clear that Sq^k must act on $y \otimes z$ in $H^*(Y) \otimes H^*(Z)$ as the sum over $i+j=k$ of the action by $Sq^i \otimes Sq^j$. Milnor (1958) proved that for $p=2$ the rule

$$Sq^k \longmapsto \sum_{i+j=k} Sq^i \otimes Sq^j$$

extends in a unique manner to an algebra homomorphism

$$\psi\colon \mathscr{A} \longrightarrow \mathscr{A} \otimes \mathscr{A} .$$

Here $\mathscr{A} \otimes \mathscr{A}$ is given the algebra structure given by the composition

$$\mathscr{A} \otimes \mathscr{A} \otimes \mathscr{A} \otimes \mathscr{A} \xrightarrow{1\otimes\gamma\otimes 1} \mathscr{A} \otimes \mathscr{A} \otimes \mathscr{A} \otimes \mathscr{A} \xrightarrow{\phi\otimes\phi} \mathscr{A} \otimes \mathscr{A} ,$$

where $\gamma\colon \mathscr{A}\otimes\mathscr{A} \to \mathscr{A}\otimes\mathscr{A}$ is the (graded) twist isomorphism, so that $(\theta_1\otimes\theta_2)\cdot(\theta_3\otimes\theta_4) = (-1)^{|\theta_2||\theta_3|}\theta_1\theta_3\otimes\theta_2\theta_4$. In general, $\gamma\colon M \otimes N \to N \otimes M$ is given by

$$\gamma(m \otimes n) = (-1)^{|m||n|} n \otimes m .$$

For $p=2$ the sign can be ignored. For p odd the rules

$$\beta \longmapsto \beta \otimes 1 + 1 \otimes \beta$$

and

$$P^k \longmapsto \sum_{i+j=k} P^i \otimes P^j$$

likewise extend uniquely to an algebra homomorphism $\psi\colon \mathscr{A} \longrightarrow \mathscr{A} \otimes \mathscr{A}$. [[Give Milnor's proof?]]

It follows that the Künneth isomorphism is an isomorphism of $\mathscr{A}$-modules, if we define the tensor product $M \otimes N$ of two A-modules M and N as follows.

Definition 13.14. Let M and N be left $\mathscr{A}$-modules, with module action maps $\lambda\colon \mathscr{A} \otimes M \to M$ and $\lambda\colon \mathscr{A} \otimes N \to N$. We give $M \otimes N$ the left $\mathscr{A}$-module structure given by the composition

$$\mathscr{A} \otimes M \otimes N \xrightarrow{\psi\otimes 1\otimes 1} \mathscr{A} \otimes \mathscr{A} \otimes M \otimes N \xrightarrow{1\otimes\gamma\otimes 1} \mathscr{A} \otimes M \otimes \mathscr{A} \otimes N \xrightarrow{\lambda\otimes\lambda} M \otimes N .$$

If we write $\psi(\theta) = \sum_i \theta'_i \otimes \theta''_i$ for $\theta \in \mathscr{A}$, which we usually abbreviate to $\sum \theta' \otimes \theta''$, then

$$\theta \cdot (m \otimes n) = \sum (-1)^{|\theta''||m|} \theta' \cdot m \otimes \theta'' \cdot n$$

for $m \in M$ and $n \in N$. The sign enters from the interchange of θ'' and m, and can be ignored for $p=2$.

The *coproduct* ψ is counital and coassociative, in the sense that the diagrams

$$\begin{tikzcd} & \mathscr{A} \arrow[dl, "\cong"'] \arrow[d, "\psi"] \arrow[dr, "\cong"] & \\ \mathbb{F}_p \otimes \mathscr{A} & \mathscr{A} \otimes \mathscr{A} \arrow[l, "\epsilon\otimes 1"] \arrow[r, "1\otimes\epsilon"'] & \mathscr{A} \otimes \mathbb{F}_p \end{tikzcd}$$

and

$$\begin{tikzcd} \mathscr{A} \arrow[r, "\psi"] \arrow[d, "\psi"'] & \mathscr{A} \otimes \mathscr{A} \arrow[d, "\psi\otimes 1"] \\ \mathscr{A} \otimes \mathscr{A} \arrow[r, "1\otimes\psi"'] & \mathscr{A} \otimes \mathscr{A} \otimes \mathscr{A} \end{tikzcd}$$

commute. Hence $\sum \epsilon(\theta')\theta'' = \theta = \sum \theta'\epsilon(\theta'')$ and $\sum\sum (\theta')' \otimes (\theta')'' \otimes \theta'' = \sum\sum \theta' \otimes (\theta'')' \otimes (\theta'')''$. Furthermore, it is cocommutative, in the sense that the diagram

$$\begin{tikzcd}
 & \mathscr{A} \arrow[dl, "\psi"'] \arrow[dr, "\psi"] & \\
\mathscr{A} \otimes \mathscr{A} \arrow[rr, "\gamma"'] & & \mathscr{A} \otimes \mathscr{A}
\end{tikzcd}$$

commutes, so that $\sum \theta' \otimes \theta'' = \sum (-1)^{|\theta'||\theta''|}\theta'' \otimes \theta'$. All of these properties are easily verified for the algebra generators (Sq^k for $p = 2$, β and P^k for p odd) of $\mathscr{A}$.

The counital and coassociative augmentation $\epsilon\colon \mathscr{A} \to \mathbb{F}_p$ and coproduct $\psi\colon \mathscr{A} \to \mathscr{A} \otimes \mathscr{A}$ give $\mathscr{A}$ the structure of a *coalgebra*. By cocommutativity of ψ, it is in fact a *cocommutative coalgebra*. Both the augmentation and the coproduct are algebra morphisms. This means that $\mathscr{A}$ is a *bialgebra*, or more precisely, a *cocommutative bialgebra*.

The cocommutativity of $\mathscr{A}$ ensures that the twist isomorphism $\gamma\colon M \otimes N \to N \otimes M$ is $\mathscr{A}$-linear, since $\gamma\psi = \psi$ implies that the left hand square in the following diagram commutes. The remainder of the diagram also commutes.

$$\begin{tikzcd}
\mathscr{A} \otimes M \otimes N \arrow[r, "\psi\otimes 1\otimes 1"] \arrow[d, "1\otimes\gamma"'] & \mathscr{A} \otimes \mathscr{A} \otimes M \otimes N \arrow[r, "1\otimes\gamma\otimes 1"] \arrow[d, "\gamma\otimes\gamma"'] & \mathscr{A} \otimes M \otimes \mathscr{A} \otimes N \arrow[r, "\lambda\otimes\lambda"] \arrow[d, "\gamma"'] & M \otimes N \arrow[d, "\gamma"'] \\
\mathscr{A} \otimes N \otimes M \arrow[r, "\psi\otimes 1\otimes 1"] & \mathscr{A} \otimes \mathscr{A} \otimes N \otimes M \arrow[r, "1\otimes\gamma\otimes 1"] & \mathscr{A} \otimes N \otimes \mathscr{A} \otimes M \arrow[r, "\lambda\otimes\lambda"] & N \otimes M
\end{tikzcd}$$

[[If we arrange that $\otimes$ is strictly unital and associative, as we implicitly arrange when we treat the unitality and associativity isomorphisms as identities, then $\mathscr{A}$-Mod is a permutative category.]]

Furthermore, $\mathscr{A}$ admits a *conjugation* $\chi\colon \mathscr{A} \to \mathscr{A}$, a linear homomorphism satisfying the relations

$$\phi(1 \otimes \chi)\psi = \eta\epsilon = \phi(\chi \otimes 1)\psi\,.$$

Equivalently χ makes the diagram

$$\begin{tikzcd}
 & \mathscr{A} \otimes \mathscr{A} \arrow[rr, "1\otimes\chi"] & & \mathscr{A} \otimes \mathscr{A} \arrow[dr, "\phi"] & \\
\mathscr{A} \arrow[ur, "\psi"] \arrow[rr, "\epsilon"] \arrow[dr, "\psi"'] & & \mathbb{F}_p \arrow[rr, "\eta"] & & \mathscr{A} \\
 & \mathscr{A} \otimes \mathscr{A} \arrow[rr, "\chi\otimes 1"'] & & \mathscr{A} \otimes \mathscr{A} \arrow[ur, "\phi"'] &
\end{tikzcd}$$

commute. It follows that χ is an anti-homomorphism, i.e., satisfies $\chi(\theta_1\theta_2) = \chi(\theta_2)\chi(\theta_1)$ for all $\theta_1, \theta_2 \in \mathscr{A}$, and it is an *involution*, i.e., χ^2 equals the identity. [[Give a proof? Milnor–Moore?]]

For $p = 2$, $\chi(1) = 1$ and $\sum_{i+j=k} Sq^i\chi(Sq^j) = 0$ for all $k \geq 1$, so that

$$\chi(Sq^k) = Sq^k + \sum_{i=1}^{k-1} Sq^i\chi(Sq^{k-i})\,.$$

For example, $\chi(Sq^1) = Sq^1$, $\chi(Sq^2) = Sq^2$, $\chi(Sq^3) = Sq^2Sq^1$ and $\chi(Sq^2Sq^1) = Sq^3$. For p odd we get $\chi(\beta) = -\beta$ and

$$\chi(P^k) = -P^k - \sum_{i=1}^{k-1} P^i\chi(P^{k-i})\,.$$

A bialgebra with a conjugation is called a *Hopf algebra*. The Steenrod algebra $\mathscr{A}$ is thus an example of a *cocommutative Hopf algebra*.

Let M be a left $\mathscr{A}$-module. The functor $L \mapsto L \otimes M$ is left adjoint to the functor $N \mapsto \mathrm{Hom}^*(M, N)$, in the sense that there is a natural bijection

$$\mathrm{Hom}^*(L \otimes M, N) \cong \mathrm{Hom}^*(L, \mathrm{Hom}^*(M, N))$$

taking $f\colon L \otimes M \to N$ to $g\colon L \to \mathrm{Hom}^*(M, N)$ given by $g(\ell)(m) = f(\ell \otimes m)$. The identity map of $L \otimes M$ on the left corresponds to the adjunction unit $in\colon L \to \mathrm{Hom}^*(M, L \otimes M)$ on the right, with $in(\ell)(m) = \ell \otimes m$. The identity map of $\mathrm{Hom}^*(M, N)$ on the right corresponds to the adjunction coinit

$ev\colon \operatorname{Hom}^*(M, N) \otimes N \to M$ on the left, with $ev(f \otimes n) = f(n)$. These adjunctions do not involve the symmetric structure, and do not require the introduction of signs.

Definition 13.15. Let M and N be left $\mathscr{A}$-modules, with action maps $\lambda\colon \mathscr{A} \otimes M \to M$ and $\lambda\colon \mathscr{A} \otimes N \to N$. We give $\operatorname{Hom}(M, N)$ the $\mathscr{A}$-module structure given by the homomorphism $\mathscr{A} \otimes \operatorname{Hom}(M, N) \longrightarrow \operatorname{Hom}(M, N)$ with left adjoint $\mathscr{A} \otimes \operatorname{Hom}(M, N) \otimes M \longrightarrow N$ given by the composite

$$\begin{aligned} \mathscr{A} \otimes \operatorname{Hom}(M,N) \otimes M &\xrightarrow{\psi\otimes1\otimes1} \mathscr{A} \otimes \mathscr{A} \otimes \operatorname{Hom}(M,N) \otimes M \xrightarrow{1\otimes\chi\otimes1\otimes1} \mathscr{A} \otimes \mathscr{A} \otimes \operatorname{Hom}(M,N) \otimes M \\ &\xrightarrow{1\otimes\gamma\otimes1} \mathscr{A} \otimes \operatorname{Hom}(M,N) \otimes \mathscr{A} \otimes M \xrightarrow{1\otimes1\otimes\lambda} \mathscr{A} \otimes \operatorname{Hom}(M,N) \otimes M \xrightarrow{1\otimes ev} \mathscr{A} \otimes N \xrightarrow{\lambda} N \,. \end{aligned}$$

For $\theta \in \mathscr{A}$, $f\colon M \to N$ in $\operatorname{Hom}(M, N)$ and $m \in M$, this composite is

$$\sum (-1)^{|\theta''||f|} \theta' \cdot f(\chi(\theta'') \cdot m) \,.$$

Proposition 13.16. *The category $\mathscr{A}$-Mod of left $\mathscr{A}$-modules, with respect to the tensor product $- \otimes -$, the unit object $\mathbb{F}_p$, the twist isomorphism γ and the mapping object $\operatorname{Hom}(-,-)$, is closed symmetric monoidal.*

Example 13.17. For another example of this situation, consider a discrete group G and a field k. The group algebra $k[G]$ has unit and product given by the neutral element e and the multiplication in G. It admits a cocommutative coproduct $\psi\colon k[G] \to k[G] \otimes k[G]$, given by $\psi(g) = g \otimes g$ for each $g \in G$. The augmentation $\epsilon\colon k[G] \to k$ satisfies $\epsilon(e) = 1$ and $\epsilon(g) = 0$ for all group elements $g \neq e$. The conjugation $\chi\colon k[G] \to k[G]$ is the anti-homomorphism given by $\chi(g) = g^{-1}$. These maps make $k[G]$ a cocommutative Hopf algebra. The tensor product of two $k[G]$-modules M and N is again a $k[G]$-module $M \otimes N = M \otimes_k N$, with the diagonal action $g \cdot (m \otimes n) = gm \otimes gn$. The twist isomorphism $\gamma\colon M \otimes N \to N \otimes M$ is $k[G]$-linear. The homomorphism module $\operatorname{Hom}(M, N) = \operatorname{Hom}_k(M, N)$ has the conjugate $k[G]$-action, given by $(g \cdot f)(m) = gf(g^{-1}m))$. Each $k[G]$-linear homomorphism $M \otimes N \to P$ corresponds bijectively to a $k[G]$-linear homomorphism $M \to \operatorname{Hom}(N, P)$.

The following result should be compared with Lemma 9.20.

Proposition 13.18. *Let M be any $\mathscr{A}$-module, with underlying graded $\mathbb{F}_p$-module $|M|$. There is an untwisting isomorphism of $\mathscr{A}$-modules,*

$$\mathscr{A} \otimes |M| \xrightarrow{\cong} \mathscr{A} \otimes M$$

from the induced $\mathscr{A}$-module on the left hand side (with $\mathscr{A}$ acting only on the first tensor factor), to the tensor product of $\mathscr{A}$-modules on the right hand side (with the diagonal $\mathscr{A}$-action). In particular, the diagonal tensor product $\mathscr{A} \otimes M$ is a free $\mathscr{A}$-module.

Proof. The isomorphism from left to right is the composite

$$\mathscr{A} \otimes |M| \xrightarrow{\psi\otimes1} \mathscr{A} \otimes \mathscr{A} \otimes |M| \xrightarrow{1\otimes\lambda} \mathscr{A} \otimes M \,.$$

It sends $\theta \otimes m$ to $\sum \theta' \otimes \theta'' m$, where $\psi(\theta) = \sum \theta' \otimes \theta''$. It is $\mathscr{A}$-linear, because the induced $\mathscr{A}$-module action on the left corresponds to the diagonal $\mathscr{A}$-module action on the tensor product of $\mathscr{A}$ and $\mathscr{A} \otimes |M|$ in the middle, and the left action map $\lambda\colon \mathscr{A} \otimes |M| \to M$ is $\mathscr{A}$-linear.

The inverse isomorphism, from right to left, is the composite

$$\mathscr{A} \otimes M \xrightarrow{\psi\otimes1} \mathscr{A} \otimes \mathscr{A} \otimes M \xrightarrow{1\otimes\chi\otimes1} \mathscr{A} \otimes \mathscr{A} \otimes M \xrightarrow{1\otimes\lambda} \mathscr{A} \otimes |M| \,.$$

It sends $\theta \otimes m$ to $\sum \theta' \otimes \chi(\theta'') m$.

One composite is visible along the upper and right hand edges of the following commutative diagram.

$$\begin{tikzcd}
\mathscr{A} \otimes |M| \arrow[r, "\psi\otimes1"] \arrow[d, "\psi\otimes1"'] & \mathscr{A} \otimes \mathscr{A} \otimes |M| \arrow[r, "1\otimes\lambda"] \arrow[d, "\psi\otimes1\otimes1"] & \mathscr{A} \otimes M \arrow[d, "\psi\otimes1"] \\
\mathscr{A} \otimes \mathscr{A} \otimes |M| \arrow[r, "1\otimes\psi\otimes1"] \arrow[dd, "1\otimes\epsilon\otimes1"'] & \mathscr{A} \otimes \mathscr{A} \otimes \mathscr{A} \otimes |M| \arrow[d, "1\otimes\chi\otimes1\otimes1"] & \mathscr{A} \otimes \mathscr{A} \otimes M \arrow[d, "1\otimes\chi\otimes1"] \\
 & \mathscr{A} \otimes \mathscr{A} \otimes \mathscr{A} \otimes |M| \arrow[r, "1\otimes1\otimes\lambda"] \arrow[d, "1\otimes\phi\otimes1"] & \mathscr{A} \otimes \mathscr{A} \otimes M \arrow[d, "1\otimes\lambda"] \\
\mathscr{A} \otimes \mathbb{F}_p \otimes |M| \arrow[r, "1\otimes\eta\otimes1"] & \mathscr{A} \otimes \mathscr{A} \otimes |M| \arrow[r, "1\otimes\lambda"] & \mathscr{A} \otimes |M|
\end{tikzcd}$$

The upper left hand rectangle commutes because ψ is coassociative, the lower left hand rectangle commutes because χ is a conjugation, the upper right hand rectangles commutes by naturality of the tensor product, and the lower right hand rectangle commutes by associativity for λ. The left hand and lower edges are the (mutually inverse) canonical isomorphisms, by counitality of ψ and unitality of λ.

[[The other composite is similar.]] □

13.4. **Smash product and tensor product.** Let T, V, Y and Z be spectra. We have a smash product pairing

$$\wedge\colon [T,Y]_* \otimes [V,Z]_* \longrightarrow [T\wedge V, Y\wedge Z]_*$$

taking $f\colon T\to Y$ and $g\colon V\to Z$ to $f\wedge g\colon T\wedge V\to Y\wedge Z$, and similarly for graded maps. In particular, for $T=V=S$ we have a pairing

$$\wedge\colon \pi_*(Y)\otimes\pi_*(Z)\longrightarrow \pi_*(Y\wedge Z)\,.$$

If Y is a ring spectrum, with unit $\eta\colon S\to Y$ and multiplication $\phi\colon Y\wedge Y\to Y$, we have a unit homomorphism

$$\eta_*\colon \pi_*(S)\longrightarrow \pi_*(Y)$$

and a product

$$\pi_*(Y)\otimes\pi_*(Y)\xrightarrow{\wedge}\pi_*(Y\wedge Y)\xrightarrow{\phi_*}\pi_*(Y)$$

that make $\pi_*(Y)$ an algebra over $\pi_*(S)$. If Y is homotopy commutative, then $\pi_*(Y)$ is a (graded) commutative $\pi_*(S)$-algebra.

When $Y=S$, the smash product $\wedge\colon \pi_*(S)\otimes\pi_*(S)\to\pi_*(S)$ agrees up to sign with the composition product $\circ\colon \pi_*(S)\otimes\pi_*(S)\to\pi_*(S)$. The smash product of $f\colon S^t\to S$ and $g\colon S^v\to S$ is $f\wedge g\colon S^{t+v}\cong S^t\wedge S^v\to S\wedge S=S$, while the composition product is $f\circ\Sigma^t g\colon S^{v+t}=\Sigma^t S^v\to\Sigma^t S=S^t\to S$. These agree up to the twist equivalence $\gamma\colon S^t\wedge S^v\cong S^v\wedge S^t$, which is a a map of degree $(-1)^{tv}$.

Now suppose that Y and Z are bounded below with $H_*(Y)$ and $H_*(Z)$ of finite type, and let $\{Y^s\}_s$ and $\{Z^u\}_u$ be Adams resolutions. If $f\colon T\to Y$ and $g\colon V\to Z$ have Adams filtrations $\ge s$ and $\ge u$, respectively, then they factor as the composites of s maps

$$T=T_s\to\cdots\to T_0=Y$$

and u maps

$$V=V_u\to\cdots\to V_0=Z\,,$$

all inducing zero on cohomology. By the Künneth theorem, the smash product $f\wedge g$ then factors as the composite of $(s+u)$ cohomologically trivial maps

$$T\wedge V=T_s\wedge V_u\to\cdots\to T_0\wedge V_u\to\cdots\to T_0\wedge V_0=Y\wedge Z\,.$$

Hence we get a restricted pairing

$$F^s[T,Y]_*\otimes F^u[V,Z]_*\longrightarrow F^{s+u}[T\wedge V,Y\wedge Z]_*$$

that descends to a pairing

$$F^s/F^{s+1}\otimes F^u/F^{u+1}\longrightarrow F^{s+u}/F^{s+u+1}$$

of filtration quotients. When the respective spectral sequences converge, we can write this as a pairing

$$E_\infty^{s,*}\otimes E_\infty^{u,*}\longrightarrow E_\infty^{s+u,*}$$

of E_∞-terms. We will relate this to an algebraically defined pairing

$$\mathrm{Ext}_{\mathscr{A}}^{s,*}(H^*(Y),H^*(T))\otimes\mathrm{Ext}_{\mathscr{A}}^{u,*}(H^*(Z),H^*(V))\longrightarrow\mathrm{Ext}_{\mathscr{A}}^{s+u,*}(H^*(Y\wedge Z),H^*(T\wedge V))$$

of the Adams spectral sequence E_2-terms.

Let M, N, T and V be $\mathscr{A}$-modules.

Lemma 13.19. *Let $\epsilon\colon P_*\to M$ and $\epsilon\colon Q_*\to N$ be two resolutions. Then $\epsilon\otimes\epsilon\colon P_*\otimes Q_*\to M\otimes N$ is a resolution. If P_* or Q_* is free, then so is $P_*\otimes Q_*$.*

Proof. If $\epsilon_*\colon H_*(P_*)\to M$ and $\epsilon_*\colon H_*(Q_*)\to N$ are isomorphisms, then $(\epsilon\otimes\epsilon)_*\colon H_*(P_*\otimes Q_*)\to M\otimes N$ must also be an isomorphism, due to the Künneth isomorphism $H_*(P_*)\otimes H_*(Q_*)\cong H_*(P_*\otimes Q_*)$.

If P_* is free in each degree, then $P_s\otimes Q_u$ is a sum of copies of $\mathscr{A}\otimes Q_u$, for each s and u, hence is free by Proposition 13.18. Hence $P_*\otimes Q_*$ is free in each degree. The same argument applies if Q_* is free in each degree. □

Definition 13.20. The *tensor product pairing*

$$\otimes\colon \operatorname{Ext}^{s,t}_{\mathscr{A}}(M,T)\otimes \operatorname{Ext}^{u,v}_{\mathscr{A}}(N,V)\longrightarrow \operatorname{Ext}^{s+u,t+v}_{\mathscr{A}}(M\otimes N, T\otimes V)$$

is given by choosing free $\mathscr{A}$-module resolutions $P_*\to M$ and $Q_*\to N$. The tensor product $P_*\otimes Q_*\to M\otimes N$ is then a free $\mathscr{A}$-module resolution of $M\otimes N$, and $T\otimes V$ is a left $\mathscr{A}$-module, in both cases using the coproduct $\psi\colon \mathscr{A}\to\mathscr{A}\otimes\mathscr{A}$ to restrict the external $\mathscr{A}\otimes\mathscr{A}$-module structure to an internal $\mathscr{A}$-module structure. The tensor product of $\mathscr{A}$-module homomorphisms induces a pairing

$$\operatorname{Hom}^*_{\mathscr{A}}(P_*,T)\otimes \operatorname{Hom}^*_{\mathscr{A}}(Q_*,V)\longrightarrow \operatorname{Hom}^*_{\mathscr{A}}(P_*\otimes Q_*, T\otimes V)$$

of complexes, and the tensor product pairing is the induced pairing in homology.

More explicitly, the pairing takes cocycles $f\colon P_s\to\Sigma^t T$ and $g\colon Q_u\to\Sigma^v V$, with $f\partial_{s+1}=0$ and $g\partial_{u+1}=0$, to the tensor product

$$f\otimes g\colon P_s\otimes Q_u\longrightarrow \Sigma^t T\otimes\Sigma^v V\cong\Sigma^{t+v}T\otimes V\,.$$

This is extended by zero on the remaining summands of $(P_*\otimes Q_*)_{s+u}$. Equivalently, f and g can be viewed as chain maps $P_*\to\Sigma^t T[s]$ and $Q_*\to\Sigma^v V[u]$, respectively, where $\Sigma^t T[s]$ is the chain complex with $\Sigma^t T$ concentrated in degree s, and similarly for $\Sigma^v V[u]$. Then $(f\otimes g)\partial_{s+u+1} = f\partial_{s+1}\otimes g + (-1)^{|f|} f\otimes g\partial_{u+1} = 0$, so the tensor product is a cocycle.

In particular, for $s=0$ and $u=0$, the tensor product pairing on Ext agrees with the Hom-pairing

$$\otimes\colon \operatorname{Hom}^t_{\mathscr{A}}(M,T)\otimes\operatorname{Hom}^v_{\mathscr{A}}(M,V)\longrightarrow\operatorname{Hom}^{t+v}_{\mathscr{A}}(M\otimes N,T\otimes V)$$

that maps $f\colon M\to\Sigma^t T$ and $g\colon N\to\Sigma^v V$ to $f\otimes g\colon M\otimes N\to\Sigma^t T\otimes\Sigma^v V\cong\Sigma^{t+v}T\otimes V$.

Alternatively, if we have given another free $\mathscr{A}$-module resolution $R_*\to M\otimes N$, then we can cover the identity of $M\otimes N$ by a chain map $\Delta\colon R_*\to P_*\otimes Q_*$, unique up to chain homotopy. Then the composite

$$R_*\xrightarrow{\Delta} P_*\otimes Q_*\xrightarrow{f\otimes g}\Sigma^t T\otimes\Sigma^v V\cong\Sigma^{t+v}T\otimes V$$

is a cocycle that represents the tensor product $[f]\otimes[g]$ in $\operatorname{Ext}^{s+u,t+v}_{\mathscr{A}}(M\otimes N,T\otimes V)$.

13.5. **The smash product pairing of Adams spectral sequences.** Let Y and Z be spectra. We have a smash product pairing

$$\wedge\colon \pi_*(Y)\otimes\pi_*(Z)\longrightarrow\pi_*(Y\wedge Z)$$

that takes $f\colon S^t\to Y$ and $g\colon S^v\to Z$ to the smash product $f\wedge g\colon S^{t+v}\cong S^t\wedge S^v\to Y\wedge Z$.

Suppose that Y and Z are bounded below, and that $H_*(Y)$ and $H_*(Z)$ are of finite type. Let $\{Y^s\}_s$ and $\{Z^u\}_u$ be Adams resolutions of Y and Z, respectively, with cofibers $Y^s/Y^{s+1}=K^s$ and $Z^u/Z^{u+1}=L^u$. Let $P_s=H^*(\Sigma^s K^s)$ and $Q_u=H^*(\Sigma^u L^u)$ be the $\mathscr{A}$-modules that appear in the usual free resolutions $\epsilon\colon P_*\to H^*(Y)$ and $\epsilon\colon Q_*\to H^*(Z)$.

Let $W=Y\wedge Z$ be the smash product. Then W is bounded below and $H_*(W)\cong H_*(Y)\otimes H_*(Z)$ is of finite type. We shall construct an Adams resolution $\{W^n\}_n$ of W by geometrically mixing the Adams resolutions for Y and Z.

Traditionally, this is done by first replacing Y, Z and their Adams resolutions by homotopy equivalent spectra, so that each Y^s and Z^u is a CW spectrum, and each map $i\colon Y^{s+1}\to Y^s$ and $i\colon Z^{u+1}\to Z^u$ is the inclusion of a CW subspectrum. Then $Y^s\wedge Z^u$ is a CW subspectrum of $Y\wedge Z$, and one can form the union of these subspectra for all $s+u=n$. Hence one defines

$$W^n=\bigcup_{s+u=n} Y^s\wedge Z^u\,.$$

Then W^{n+1} is a CW subspectrum of W^n, and

$$W^n/W^{n+1}\cong\bigvee_{s+u=n}K^s\wedge L^u\,.$$

Lemma 13.21. *The diagram*

$$\begin{tikzcd}
\dots \arrow[r] & W^2 \arrow[r, "i"] \arrow[d, "j"] & W^1 \arrow[r, "i"] \arrow[d, "j"] & W \arrow[d, "j"] \\
& W^2/W^3 \arrow[ul, dashed, "\partial"] & W^1/W^2 \arrow[ul, dashed, "\partial"] & W/W^1 \arrow[ul, dashed, "\partial"]
\end{tikzcd}$$

is an Adams resolution of $W=Y\wedge Z$. The associated free resolution $R_*\to H^*(W)$ is the tensor product of the free resolutions $P_*\to H^*(Y)$ and $Q_*\to H^*(Z)$.

Proof. Since each K^s is a wedge sum of suspended copies of H, of finite type, and each L^u is of finite type, we know that W^n/W^{n+1} is a wedge sum of suspended copies of H, of finite type. Let

$$R_n = H^*(\Sigma^n(W^n/W^{n+1})) \cong \bigoplus_{s+u=n} P_s \otimes Q_u \,.$$

This is a free $\mathscr{A}$-module of finite type, by its geometric origin as the cohomology of W^n/W^{n+1}. The composite $W^{n-1}/W^n \to \Sigma W^n \to \Sigma(W^n/W^{n+1})$ splits as the direct sum of the maps $j\partial\wedge 1\colon K^{s-1}\wedge L^u \to \Sigma K^s \wedge L^u \cong \Sigma(K^s \wedge L^u)$ and $1 \wedge j\partial\colon K^s \wedge L^{u-1} \to K^s \wedge \Sigma L^u \cong \Sigma(K^s \wedge L^u)$. Hence the boundary map $\partial_n\colon R_n \to R_{n-1}$ is given by the usual formula

$$\partial_n(x \otimes y) = \partial_n(x) \otimes y + x \otimes \partial_n(y)$$

(we work at $p = 2$, hence there is no sign), so that $R_* = P_* \otimes Q_*$ is the tensor product of the two resolutions. By the Künneth theorem, the homology of R_* is the tensor product of the homologies of P_* and Q_*, so $\epsilon\colon R_* \to H^*(Y) \otimes H^*(Z) \cong H^*(Y \wedge Z)$ is a free resolution.

In particular, $j\colon W^0 = Y \wedge Z \to K^0 \wedge L^0$ induces a surjection j^* in cohomology. It follows that $\partial\colon W/W^1 \to \Sigma W^1$ induces an injection ∂^* in cohomology, with image in $R_0 = H^*(W/W^1)$ equal to the kernel of $j^* = \epsilon$. This equals the image of $\partial_1 = \partial^* j^* : R_1 \to R_0$, by exactness at R_0 of the free resolution, which implies that j^*, induced by $j\colon W^1 \to W^1/W^2$, is surjective. Suppose inductively that $j\colon W^{n-1} \to W^{n-1}/W^n$ induces a surjection j^* in cohomology, for some $n \geq 2$. Then $\partial\colon W^{n-1}/W^n \to \Sigma W^n$ induces an injection ∂^* in cohomology. The image of ∂^* equals the kernel of j^*, hence lies in the kernel of $\partial_{n-1} = \partial^* j^*\colon R_{n-1} \to R_{n-2}$. This equals the image of $\partial_n = \partial^* j^*\colon R_n \to R_{n-1}$, by exactness at R_{n-1}, which implies that j^*, induced by $j\colon W^n \to W^n/W^{n+1}$, is surjective. $\square$

Granting a little more technology, the substitution by CW spectra can be replaced by the passage to a homotopy colimit. For a fixed $n \geq 0$, one considers the diagram of all spectra $Y^s \wedge Z^u$ for $s + u \geq n$, and forms the homotopy colimit

$$W^n = \operatorname*{hocolim}_{s+u\geq n} Y^s \wedge Z^u \,.$$

There is a natural diagram

$$\cdots \to W^2 \xrightarrow{i} W^1 \xrightarrow{i} W^0 \simeq Y \wedge Z$$

and an identification

$$W^n/W^{n+1} \cong \bigvee_{s+u=n} \operatorname{hocofib}(Y^{s+1} \to Y^s) \wedge \operatorname{hocofib}(Z^{u+1} \to Z^u)$$

where $\operatorname{hocofib}(Y^{s+1} \to Y^s) \simeq K^s$ denotes the mapping cone of the given map, etc. The proof of the lemma goes through in the same way with these conventions.

The following theorem is similar to that proved in §4 of Adams (1958).

Theorem 13.22. *There is a natural pairing*

$$E_r^{s,t}(Y) \otimes E_r^{u,v}(Z) \longrightarrow E_r^{s+u,t+v}(Y \wedge Z)$$

of Adams spectral sequences, given at the E_2-term by the tensor product pairing

$$\operatorname{Ext}_{\mathscr{A}}^{s,t}(H^*(Y), \mathbb{F}_p) \otimes \operatorname{Ext}_{\mathscr{A}}^{u,v}(H^*(Z), \mathbb{F}_p) \longrightarrow \operatorname{Ext}_{\mathscr{A}}^{s+u,t+v}(H^*(Y \wedge Z), \mathbb{F}_p)$$

and converging to the smash product pairing

$$\pi_{t-s}(Y_p^\wedge) \otimes \pi_{v-u}(Z_p^\wedge) \longrightarrow \pi_{t-s+v-u}((Y \wedge Z)_p^\wedge) \,.$$

More generally, there is a natural pairing

$$E_r^{s,t}(T, Y) \otimes E_r^{u,v}(V, Z) \longrightarrow E_r^{s+u,t+v}(T \wedge V, Y \wedge Z)$$

of spectral sequences, given at the E_2-term by the tensor product pairing

$$\operatorname{Ext}_{\mathscr{A}}^{s,t}(H^*(Y), H^*(T)) \otimes \operatorname{Ext}_{\mathscr{A}}^{u,v}(H^*(Z), H^*(V)) \longrightarrow \operatorname{Ext}_{\mathscr{A}}^{s+u,t+v}(H^*(Y \wedge Z), H^*(T \wedge V))$$

and converging to the smash product pairing

$$[T, Y_p^\wedge]_{t-s} \otimes [V, Z_p^\wedge]_{v-u} \longrightarrow [T \wedge V, (Y \wedge Z)_p^\wedge]_{t-s+v-u} \,.$$

((Discuss the role of completion in the pairing?))

Proof. Recall that $E_r^s = Z_r^s / B_r^s$, where

$$Z_r^s = \partial^{-1} \operatorname{im}(i_*^{r-1} \colon \pi_*(Y^{s+r}) \to \pi_*(Y^{s+1}))$$

and

$$B_r^s = j \ker(i_*^{r-1} \colon \pi_*(Y^s) \to \pi_*(Y^{s+r-1}))$$

are subgroups of $E_s^1 = \pi_*(K^s)$. For the purpose of this proof, it is convenient to rewrite these groups as

$$Z_r^s = \operatorname{im}(\pi_*(Y^s/Y^{s+r}) \to \pi_*(K^s))$$

and

$$B_r^s = \operatorname{im}(\pi_*(\Sigma^{-1}(Y^{s-r+1}/Y^s)) \to \pi_*(K^s))\,.$$

These formulas can be obtained by chases in the diagrams

$$\begin{tikzcd}
Y^{s+r} \arrow[r, "i^r"] \arrow[d] & Y^s \arrow[r] \arrow[d, "j"] & Y^s/Y^{s+r} \arrow[d] \\
* \arrow[r] \arrow[d] & K^s \arrow[r, "="] \arrow[d, "\partial"] & K^s \arrow[d] \\
\Sigma Y^{s+r} \arrow[r, "\Sigma i^{r-1}"] & \Sigma Y^{s+1} \arrow[r] & \Sigma(Y^{s+1}/Y^{s+r})
\end{tikzcd}$$

and

$$\begin{tikzcd}
* \arrow[r] \arrow[d] & \Sigma^{-1}(Y^{s-r+1}/Y^s) \arrow[r, "="] \arrow[d] & \Sigma^{-1}(Y^{s-r+1}/Y^s) \arrow[d] \\
Y^{s+1} \arrow[r, "i"] \arrow[d, "="'] & Y^s \arrow[r, "j"] \arrow[d, "i^{r-1}"] & K^s \arrow[d] \\
Y^{s+1} \arrow[r, "i^r"] & Y^{s-r+1} \arrow[r] & Y^{s-r+1}/Y^{s+1}
\end{tikzcd}$$

of horizontal and vertical cofiber sequences.

The differential $d_r^s \colon E_r^s \to E_r^{s+r}$ is determined by the homomorphism $\delta \colon \pi_*(Y^s/Y^{s+r}) \to Z_r^{s+r}$ induced by $Y^s/Y^{s+r} \to \Sigma K^{s+r}$ and the surjection $\pi \colon \pi_*(Y^s/Y^{s+r}) \to Z_r^s$ induced by $Y^s/Y^{s+r} \to K^s$:

$$\begin{tikzcd}
E_1^s & Z_r^s \arrow[l, tail] \arrow[d, two heads] & \pi_*(Y^s/Y^{s+r}) \arrow[l, two heads, "\pi"'] \arrow[r, "\delta"] & Z_r^{s+t} \arrow[r, tail] \arrow[d, two heads] & E_1^{s+r} \\
& E_r^s \arrow[rr, "d_r^s"] & & E_r^{s+r} &
\end{tikzcd}$$

To compare this with the exact couple definition of d_r^s, consider the commutative diagram

$$\begin{tikzcd}
K^s \arrow[d, "k"'] & Y^s/Y^{s+r} \arrow[l] \arrow[r] \arrow[d] & \Sigma K^{s+r} \arrow[d, equal] \\
\Sigma Y^{s+1} & \Sigma Y^{s+r} \arrow[l, "i^{r-1}"'] \arrow[r, "j"] & \Sigma K^{s+r}
\end{tikzcd}$$

where the left hand square is homotopy (co-)cartesian. (It follows that $B_{r+1}^{s+r}/B_r^{s+r} \subset E_r^{s+r}$ equals the image of d_r^s.)

So far we have discussed the Adams spectral sequence for a single spectrum Y. We now relate the Adams spectral sequences for Y, Z and $W = Y \wedge Z$, where W has the Adams resolution obtained from given Adams resolutions of Y and Z.

There is a preferred inclusion $Y^s \wedge Z^u \to W^n$ for all $s, u \geq 0$ and $n = s + u$. It restricts to inclusions $Y^{s+r} \wedge Z^u \to W^{n+r}$ and $Y^s \wedge Z^{u+r} \to W^{n+r}$, that agree on $Y^{s+r} \wedge Z^{u+r}$. Hence we have a main commutative diagram

$$\begin{tikzcd}
U \arrow[r] \arrow[d] & Y^{s+r} \wedge Z^u \cup Y^s \wedge Z^{u+r} \arrow[r] \arrow[d] \arrow[rr, bend left, "a_r"] & Y^{s+1} \wedge Z^u \cup Y^s \wedge Z^{u+1} \arrow[r, "a_1"] \arrow[d] & Y^s \wedge Z^u \arrow[r] \arrow[d] & Y \wedge Z \arrow[d, "\simeq"] \\
W^{n+r+1} \arrow[r, "i"] & W^{n+r} \arrow[r, "i^{r-1}"] \arrow[rr, bend right, "i^r"'] & W^{n+1} \arrow[r, "i"] & W^n \arrow[r] & W
\end{tikzcd}$$

where $Y^{s+r} \wedge Z^u \cup Y^s \wedge Z^{u+r}$ denotes the pushout of $Y^{s+r} \wedge Z^u$ and $Y^s \wedge Z^{u+r}$ along $Y^{s+r} \wedge Z^{u+r}$, and U is brief notation for a similar union of $Y^{s+r+1} \wedge Z^u$, $Y^{s+r} \wedge Z^{u+1}$, $Y^{s+1} \wedge Z^u$ and $Y^s \wedge Z^{u+1}$.

Passing to horizontal cofibers for the middle part of the diagram, we get a commutative diagram

$$\begin{tikzcd}
Y^s \wedge Z^u \arrow[r] \arrow[d] & Y^s/Y^{s+r} \wedge Z^u/Z^{u+r} \arrow[r] \arrow[d] & K^s \wedge L^u \arrow[d] \\
W^n \arrow[r] & W^n/W^{n+r} \arrow[r] & W^n/W^{n+1}
\end{tikzcd} \tag{5}$$

where the maps in the upper row are smash products of the standard maps $Y^s \to Y^s/Y^{s+r}$, $Y^s/Y^{s+r} \to K^s$, etc. The vertical map $K^s \wedge L^u \to W^n/W^{n+1}$ agrees with the inclusion of a summand in $W^n/W^{n+1} \cong \bigvee_{s+u=n} K^s \wedge L^u$. Hence it induces a pairing

$$\phi_1 \colon E_1^s(Y) \otimes E_1^u(Z) \longrightarrow E_1^n(W)$$

that corresponds to the previously discussed pairing

$$\operatorname{Hom}_{\mathscr{A}}(P_*, \mathbb{F}_p) \otimes \operatorname{Hom}_{\mathscr{A}}(Q_*, \mathbb{F}_p) \longrightarrow \operatorname{Hom}_{\mathscr{A}}(P_* \otimes Q_*, \mathbb{F}_p)$$

under the d-invariant isomorphisms $\pi_{t-s}(K^s) \cong \operatorname{Hom}^t_{\mathscr{A}}(P_s, \mathbb{F}_p)$, etc.

Passing to horizontal cofibers further to the left in the main diagram, we get a commutative diagram

$$\begin{tikzcd}
Y^s/Y^{s+r} \wedge Z^u/Z^{u+r} \arrow[r] \arrow[d] & \Sigma(Y^{s+r} \wedge Z^u \cup Y^s \wedge Z^{u+r}) \arrow[r] \arrow[d] & \Sigma(K^{s+r} \wedge L^u \vee K^s \wedge L^{u+r}) \arrow[d] \\
W^n/W^{n+r} \arrow[r] & \Sigma W^{n+r} \arrow[r] & \Sigma(W^{n+r}/W^{n+r+1})
\end{tikzcd} \tag{6}$$

where the composite map in the upper row is the wedge sum of the smash product of the standard maps $Y^s/Y^{s+r} \to \Sigma K^{s+r}$ and $Z^u/Z^{u+r} \to L^u$, and the smash product of the standard maps $Y^s/Y^{s+r} \to K^s$ and $Z^u/Z^{u+r} \to \Sigma L^{u+r}$. The right hand vertical map is the suspension of the wedge sum of the pairings $K^{s+r} \wedge L^u \to W^{n+r}/W^{n+r+1}$ and $K^s \wedge L^{u+r} \to W^{n+r}/W^{n+r+1}$.

We now claim that (a) $\phi_1 = \tilde{\phi}_1$ restricts to a pairing

$$\tilde{\phi}_r \colon Z_r^s(Y) \otimes Z_r^u(Z) \longrightarrow Z_r^n(W)\,,$$

(b) $\tilde{\phi}_r$ descends to a pairing

$$\phi_r \colon E_r^s(Y) \otimes E_r^u(Z) \longrightarrow E_r^n(W)$$

and (c) ϕ_r satisfies the Leibniz rule

$$d_r(\phi_r(y \otimes z)) = \phi_r(d_r(y) \otimes z) + (-1)^{|y|} \phi_r(y \otimes d_r(z))\,.$$

Here $r \geq 1$ and $n = s + u$.

Assuming these claims, which are similar to the conditions of Lemma 2.2 of Moss (1968), we can easily finish the proof of the theorem. The pairings $(\phi_r)_*$ and ϕ_{r+1} agree, under the identification $H^s(E_r^*, d_r) \cong E_{r+1}^s$, since they are both induced by a passage to quotients from $\tilde{\phi}_{r+1}$. Hence the sequence $\{\phi_r\}_r$ qualifies as a pairing of spectral sequences. In particular, $\phi_2 = (\phi_1)_*$ is the tensor product pairing of Ext-groups. This spectral sequence pairing converges to the smash product pairing in homotopy, since the pairing of E_∞-terms is induced by the pairing

$$\pi_*(Y^s) \otimes \pi_*(Z^u) \longrightarrow \pi_*(Y^s \wedge Z^u) \longrightarrow \pi_*(W^n)$$

via the surjections $\pi_*(Y^s) \to E_\infty^s$, etc., and the pairing of filtration quotients is induced by the same pairing via the surjections $\pi_*(Y^s) \to F^s \to F^s/F^{s+1}$, etc. These surjections have the same kernel, so the induced pairings of quotients are compatible under the identifications $F^s/F^{s+1} \cong E_\infty^s$.

It remains to prove the three parts of the claim.

(a) Applying $\pi_*(-)$ to the right hand square in diagram (5), we get the outer rectangle of the following map of pairings:

$$\begin{tikzcd}
\pi_*(Y^s/Y^{s+r}) \otimes \pi_*(Z^u/Z^{u+r}) \arrow[r, two heads] \arrow[d] & Z^s_r(Y) \otimes Z^u_r(Z) \arrow[r, tail] \arrow[d, dashed, "\tilde{\phi}_r"'] & E^s_1(Y) \otimes E^u_1(Z) \arrow[d, "\phi_1"] \\
\pi_*(W^n/W^{n+r}) \arrow[r, two heads, "\pi"] & Z^n_r(W) \arrow[r, tail] & E^n_1(W)
\end{tikzcd}$$

In view of the description of $Z^n_r(W)$ as the image of $\pi_*(W^n/W^{n+r}) \to \pi_*(W^n/W^{n+1}) = E^n_1(W)$, and similarly for Y and Z, it follows that there is a unique pairing $\tilde{\phi}_r$ that makes the whole diagram commute.

(b) To check that $\tilde{\phi}_r$ descends to a pairing $\phi_r \colon E^s_r(Y) \otimes E^u_r(Z) \to E^n_r(W)$, we use the diagram

$$\begin{tikzcd}
E^s_r(Y) \otimes E^u_r(Z) \arrow[d, dashed, "\phi_r"'] & Z^s_r(Y) \otimes Z^u_r(Z) \arrow[l, two heads] \arrow[r, tail] \arrow[d, "\tilde{\phi}_r"'] & Z^s_{r-1}(Y) \otimes Z^u_{r-1}(Z) \arrow[r, two heads] \arrow[d, "\tilde{\phi}_{r-1}"'] & E^s_{r-1}(Y) \otimes E^u_{r-1}(Z) \arrow[d, "\phi_{r-1}"'] \\
E^n_r(W) & Z^n_r(W) \arrow[l, two heads] \arrow[r, tail] & Z^n_{r-1}(W) \arrow[r, two heads] & E^n_{r-1}(W) \,.
\end{tikzcd}$$

There is only something to prove for $r \geq 2$. We assume, by induction on r, that the Leibniz rule in (c) holds for d_{r-1} and ϕ_{r-1}.

Given $y \in B^s_r(Y) \subset Z^s_r(Y)$ and $z \in Z^u_r(Z)$ we must show that $\tilde{\phi}_r(y \otimes z) \in B^n_r(W) \subset Z^n_r(W)$. The image of y in $E^s_{r-1}(Y)$ has the form $[y] = d_{r-1}(x)$ for some $x \in E^{s-r+1}_{r-1}(Y)$, and the image of z in $E^u_{r-1}(Z)$ satisfies $d_{r-1}([z]) = 0$. Then $d_{r-1}(\phi_{r-1}(x \otimes [z])) = \phi_{r-1}(d_{r-1}(x) \otimes [z]) \pm \phi_{r-1}(x \otimes d_{r-1}([z])) = \phi_{r-1}([y] \otimes [z]) \pm 0 = [\tilde{\phi}_r(y \otimes z)]$. Hence $\tilde{\phi}_r(y \otimes z)$ is congruent modulo $B^n_{r-1}(W)$ to a class in $B^n_r(W)$, as we asserted. The same argument shows that $\tilde{\phi}_r$ maps $Z^s_r(Y) \otimes B^u_r(Z)$ into $B^n_r(W)$. Hence $\tilde{\phi}_r$ descends to ϕ_r, and this uniquely determines ϕ_r.

(c) [[TODO: Account for signs.]] Applying $\pi_*(-)$ to the outer rectangle in diagram (6), we get the outer rectangle of the following map of pairings:

$$\begin{tikzcd}
\pi_*(Y^s/Y^{s+r}) \otimes \pi_*(Z^u/Z^{u+r}) \arrow[r, "\left[\begin{smallmatrix} \delta\otimes\pi \\ \pi\otimes\delta \end{smallmatrix}\right]"] \arrow[d] & \begin{matrix} Z^{s+r}_r(Y) \otimes Z^u_r(Z) \\ \oplus \\ Z^s_r(Y) \otimes Z^{u+r}_r(Z) \end{matrix} \arrow[r, tail] \arrow[d, "{[\tilde{\phi}_r \ \tilde{\phi}_r]}"'] & \begin{matrix} E^{s+r}_1(Y) \otimes E^u_1(Z) \\ \oplus \\ E^s_1(Y) \otimes E^{u+r}_1(Z) \end{matrix} \arrow[d, "{[\phi_1 \ \phi_1]}"] \\
\pi_*(W^n/W^{n+r}) \arrow[r, "\delta"] & Z^{n+r}_r(W) \arrow[r, tail] & E^{n+r}_1(W)
\end{tikzcd}$$

Since the pairings $\tilde{\phi}_r$ have been defined to make the right hand square commute, the whole diagram commutes.

Combining parts of four of these diagrams, we have the commutative sprawl:

$$\begin{tikzcd}
E^s_r(Y) \otimes E^u_r(Z) \arrow[rrr, bend left, "\phi_r"] \arrow[dd, "\left[\begin{smallmatrix} d^s_r\otimes 1 \\ 1\otimes d^u_r \end{smallmatrix}\right]"'] & Z^s_r(Y) \otimes Z^u_r(Z) \arrow[l, two heads] \arrow[r, "\tilde{\phi}_r"] & Z^n_r(W) \arrow[r, two heads] & E^n_r(W) \arrow[dd, "d^n_r"] \\
& \pi_*(Y^s/Y^{s+r}) \otimes \pi_*(Z^u/Z^{u+r}) \arrow[r] \arrow[u, two heads, "\pi\otimes\pi"] \arrow[d, "\left[\begin{smallmatrix} \delta\otimes\pi \\ \pi\otimes\delta \end{smallmatrix}\right]"'] & \pi_*(W^n/W^{n+r}) \arrow[u, two heads, "\pi"'] \arrow[d, "\delta"'] & \\
\begin{matrix} E^{s+r}_r(Y) \otimes E^u_r(Z) \\ \oplus \\ E^s_r(Y) \otimes E^{u+r}_r(Z) \end{matrix} \arrow[rrr, bend right, "{[\phi_r \ \phi_r]}"'] & \begin{matrix} Z^{s+r}_r(Y) \otimes Z^u_r(Z) \\ \oplus \\ Z^s_r(Y) \otimes Z^{u+r}_r(Z) \end{matrix} \arrow[l, two heads] \arrow[r, "{[\tilde{\phi}_r \ \tilde{\phi}_r]}"] & Z^{n+r}_r(W) \arrow[r, two heads] & E^{n+r}_r(W)
\end{tikzcd}$$

Going around the outer boundary of the diagram we see that $d^n_r(\phi_r(y \otimes z)) = \phi_r(d^s_r(y) \otimes z) + \phi_r(y \otimes d^u_r(z))$, proving the Leibniz rule. $\square$

Remark 13.23. If $y \in \pi_*(K^s)$ and $z \in \pi_*(L^u)$ lift to $\tilde{y} \in \pi_*(Y^s/Y^{s+r})$ and $\tilde{z} \in \pi_*(Z^u/Z^{u+r})$, respectively, with images $\delta y \in \pi_*(\Sigma K^{s+r})$ and $\delta z \in \pi_*(\Sigma L^{u+r})$, then $y \wedge z \in \pi_*(K^s \wedge L^u)$ lifts to $\tilde{y} \wedge \tilde{z} \in \pi_*(Y^s/Y^{s+r} \wedge Z^u/Z^{u+r})$.

$$\begin{tikzcd}
\Sigma K^{s+r} & Y^s/Y^{s+r} \arrow[l] \arrow[r] & K^s
\end{tikzcd}$$

$$\begin{tikzcd}
\Sigma K^{s+r} \wedge L^u & & K^s \wedge L^u & L^u \\
& Y^s/Y^{s+r} \wedge Z^u/Z^{u+r} \arrow[ul] \arrow[ur] \arrow[dr] & & Z^u/Z^{u+r} \arrow[u] \arrow[d] \\
& & \Sigma K^s \wedge L^{u+r} & \Sigma L^{u+r}
\end{tikzcd}$$

The maps $Y^s \wedge Z^u \to W^{s+u} = W^n$ induce a commutative diagram

$$\begin{tikzcd}
\Sigma K^{s+r} \wedge L^u \vee \Sigma K^s \wedge L^{u+r} \arrow[d] & Y^s/Y^{s+r} \wedge Z^u/Z^{u+r} \arrow[l] \arrow[r] \arrow[d] & K^s \wedge L^u \arrow[d] \\
\Sigma(W^{n+r}/W^{n+r+1}) & W^n/W^{n+r} \arrow[l] \arrow[r] & W^n/W^{n+1}
\end{tikzcd}$$

and $\tilde{y} \wedge \tilde{z}$ maps to a lift $\tilde{y} \cdot \tilde{z}$ in $\pi_*(W^n/W^{n+r})$ of the image $y \cdot z$ of $y \wedge z$ in W^n/W^{n+1}. Hence $\delta(y \cdot z)$ is the image $\delta y \cdot z + y \cdot \delta z$ of $\delta y \wedge z + y \wedge \delta z$ in $\pi_*(\Sigma K^{s+r} \wedge L^u \vee \Sigma K^s \wedge L^{u+r})$. The key point is that, even if $Y^s/Y^{s+r} \wedge Z^u/Z^{u+r}$ is attached to all of $Y^{s+r} \wedge Z^u \cup Y^s \wedge Z^{u+r}$ in $Y^s \wedge Z^u$, the composite map to $W^{n+r} \to W^{n+r}/W^{n+r+1}$ factors through the quotient $K^{s+r} \wedge L^u \vee K^s \wedge L^{u+r}$, making the left hand square above commute. The bookkeeping shows that δy represents $d_r([y])$, and so on, so that $\delta(y \cdot z) = \delta y \cdot z + y \cdot \delta z$ implies the Leibniz rule for d_r.

Corollary 13.24. *Suppose that Y is a ring spectrum, with multiplication $\phi\colon Y \wedge Y \to Y$ and unit $\eta\colon S \to Y$. Then there is a natural pairing*

$$E_r^{*,*}(Y) \otimes E_r^{*,*}(Y) \longrightarrow E_r^{*,*}(Y)\,,$$

given at the E_2-term by the composite

$$\mathrm{Ext}_{\mathscr{A}}^{*,*}(H^*(Y), \mathbb{F}_p) \otimes \mathrm{Ext}_{\mathscr{A}}^{*,*}(H^*(Y), \mathbb{F}_p) \longrightarrow \mathrm{Ext}_{\mathscr{A}}^{*,*}(H^*(Y \wedge Y), \mathbb{F}_p) \xrightarrow{\phi_*} \mathrm{Ext}_{\mathscr{A}}^{*,*}(H^*(Y), \mathbb{F}_p)\,,$$

and a unit map

$$E_r^{*,*}(S) \xrightarrow{\eta_*} E_r^{*,*}(Y)\,,$$

given at the E_2-term by

$$\mathrm{Ext}_{\mathscr{A}}^{*,*}(\mathbb{F}_p, \mathbb{F}_p) \xrightarrow{\eta_*} \mathrm{Ext}_{\mathscr{A}}^{*,*}(H^*(Y), \mathbb{F}_p)\,,$$

that make the Adams spectral sequence $E_r^{*,*}(Y)$ an algebra spectral sequence over $E_r^{*,*}(S)$. If Y is homotopy commutative, then it is a commutative algebra spectral sequence.

13.6. **The bar resolution.** Let k be a commutative ring, and consider any k-algebra A. (Our principal example will be the case $k = \mathbb{F}_p$ and $A = \mathscr{A}$, the mod p Steenrod algebra.) We write $\otimes$ for $\otimes_k$ and Hom for Hom_k. Let M and N be left and right A-modules, respectively. The two-sided bar construction $\beta_\bullet(N, A, M)$ is the simplicial k-module, given in degree $q \geq 0$ by

$$\beta_q(N, A, M) = N \otimes A^{\otimes q} \otimes M\,.$$

Following Eilenberg–Mac Lane (see Mac Lane (1963, Sect. X.2)) we use the notation $n[a_1|\dots|a_q]m$ for the tensor $n \otimes a_1 \otimes \cdots \otimes a_q \otimes m$ in $\beta_q(N, A, M)$, and the use of vertical bars in this notation gives the construction its name. The face operators $d_i\colon \beta_q(N, A, M) \to \beta_{q-1}(N, A, M)$ for $0 \leq i \leq q$ are given by

$$d_i(n[a_1|\dots|a_q]m) = \begin{cases} na_1[a_2|\dots|a_q]m & \text{for } i = 0, \\ n[a_1|\dots|a_{i-1}|a_i a_{i+1}|a_{i+2}|\dots|a_q]m & \text{for } 0 < i < q, \text{ and} \\ n[a_1|\dots|a_{q-1}]a_q m & \text{for } i = q. \end{cases}$$

The degeneracy operators $s_j\colon \beta_q(N, A, M) \to \beta_{q+1}(N, A, M)$ for $0 \leq j \leq q$ are given by

$$s_j(n[a_1|\dots|a_q]m) = n[a_1|\dots|a_j|1|a_{j+1}|\dots|a_q]m\,,$$

where $1 = \eta(1) \in A$ denotes the algebra unit. There is an augmentation

$$\epsilon \colon \beta_\bullet(N, A, M) \longrightarrow N \otimes_A M$$

to the balanced tensor product $N \otimes_A M$, considered as a constant simplicial object, given in degree $q = 0$ by $\epsilon(n[]m) = n \otimes_A m$. In the special case $N = A$, with the right A-module structure given by the k-algebra multiplication, there is an extra degeneracy operator $s_{-1} \colon \beta_q(A, A, M) \to \beta_{q+1}(A, A, M)$ given by

$$s_{-1}(a_0[a_1|\dots|a_q]m) = 1[a_0|a_1|\dots|a_q]m$$

and the augmentation specializes to $\epsilon \colon \beta_\bullet(A, A, M) \to M$ given by $\epsilon(a[]m) = am$. [[Can discuss how s_{-1} specifies a simplicial contraction of $\beta_\bullet(A, A, M)$ to M.]] In this case the left A-module structure on $N = A$ induces an A-module structure on each $\beta_q(A, A, M)$, making $\beta_\bullet(A, A, M)$ a simplicial A-module. (The extra degeneracy s_{-1} is not A-linear. We use the induced, not the diagonal, A-module structure on each $\beta_q(A, A, M) = A \otimes A^{\otimes q} \otimes M$, even when the latter exists.)

The associated normalized chain complex is the normalized bar construction. It is the chain complex of k-modules given in degree $q \geq 0$ by

$$B_q(N, A, M) = N \otimes J(A)^{\otimes q} \otimes M$$

where $J(A) = \operatorname{cok}(\eta \colon k \to A)$ is the unit coideal. [[Often denoted $\bar{A}$.]] The boundary operator $\partial_q \colon B_q(N, A, M) \to B_{q-1}(N, A, M)$ is the alternating sum

$$\partial_q = \sum_{i=0}^{q} (-1)^i d_i$$

of the face operators, which descends over the surjection $\beta_q(N, A, M) \to B_q(N, A, M)$. Hence

$$\partial_q(n[a_1|\dots|a_q]m) = na_1[a_2|\dots|a_q]m + \sum_{0<i<q} (-1)^i n[a_1|\dots|a_i a_{i+1}|\dots|a_q]m + (-1)^q n[a_1|\dots|a_{q-1}]a_q m$$

for $n \in N$, $a_i \in J(A)$ and $m \in M$. [[A sign may be introduced, depending on the degrees of the terms a_i.]] There is still an augmentation

$$\epsilon \colon B_0(N, A, M) \longrightarrow N \otimes_A M$$

given by the canonical surjection $N \otimes M \to N \otimes_A M$, and we get an augmented chain complex

$$\dots \to B_2(N, A, M) \xrightarrow{\partial_2} B_1(N, A, M) \xrightarrow{\partial_1} B_0(N, A, M) \xrightarrow{\epsilon} N \otimes_A M \to 0$$

of k-modules. In the special case when $N = A$, the augmentation can be rewritten as $\epsilon \colon B_0(N, A, M) \to M$, sending $a[]m$ to am. In this case the extra degeneracy gives rise to a chain contraction S of $B_*(A, A, M)$ to M. This is a chain homotopy

$$S_q \colon B_q(A, A, M) \longrightarrow B_{q+1}(A, A, M)$$

given by

$$S_q(a_0[a_1|\dots|a_q]m) = 1[a_0|a_1|\dots|a_q]m$$

for all $q \geq 0$. It satisfies

$$\partial S + S\partial = 1 - \eta\epsilon\,.$$

Here 1 denotes the identity, and $\eta \colon M \to B_0(A, A, M)$ sends m to $1[]m$, so that $\eta\epsilon(a[]m) = 1[]am$. [[Prove the chain homotopy relation. Clarify that $\partial = 0$ on $B_0(N, A, M)$. Conversely, the boundaries ∂ are inductively determined by this relation and A-linearity.]] Hence ϵ and η are chain homotopy inverse equivalences between $B_*(A, A, M)$ and M, where M is viewed as a trivial chain complex concentrated in degree 0.

The left A-module structure on $N = A$ makes $B_*(A, A, M)$ a chain complex of left A-modules. In other words,

$$\epsilon \colon B_*(A, A, M) \longrightarrow M$$

is an A-module resolution of M, called the *bar resolution*. (Note that we use the induced A-module structure on each $B_q(A, A, M) = A \otimes J(A)^{\otimes q} \otimes M$, not the diagonal structure, in case the latter exists.) If $J(A)$ and M are flat, resp. projective, as k-modules, then so is $J(A)^{\otimes q} \otimes M$. This will imply that $B_q(A, A, M)$ is flat, resp. projective, as a left A-module. Hence, under these conditions, the bar resolution is a flat, resp. projective, resolution. If k is a field, as it will be in the case when A is the mod p Steenrod algebra, then these conditions are automatically satisfied.

In particular, we can in principle use the bar resolution to calculate $\mathrm{Tor}^A_*(N, M)$ and $\mathrm{Ext}^*_A(M, L)$. If $J(A)$ and M are flat k-modules, then

$$\mathrm{Tor}^A_s(N, M) = H_s(N \otimes_A B_*(A, A, M), 1 \otimes \partial) \cong H_s(B_*(N, A, M), \partial)$$

for each $s \geq 0$, as graded k-modules. This uses the evident isomorphism $N \otimes_A B_q(A, A, M) \cong B_q(N, A, M)$ for each $q \geq 0$. If $J(A)$ and M are projective k-modules, and L is any left A-module, then

$$\mathrm{Ext}^s_A(M, L) = H^s(\mathrm{Hom}_A(B_*(A, A, M), L), \mathrm{Hom}(\partial, 1))$$

for each $s \geq 0$, again as graded k-modules. [[Can rewrite $\mathrm{Hom}_A(B_q(A, A, M), L) = \mathrm{Hom}_A(A \otimes J(A)^{\otimes q} \otimes M, L) \cong \mathrm{Hom}_k(J(A)^{\otimes q} \otimes M, L)$ in terms of $L \otimes (J(A)^*)^{\otimes q} \otimes M^*$, leading to the cobar complex $C^q(L, A^*, M^*)$ for the dual coalgebra A^*. Return to this later.]]

13.7. **Comparison of pairings.** The bar resolution grows too fast in size to be useful for efficient machine calculation, but its explicit form makes it useful for theoretical analysis. [[Calculate Yoneda composition and tensor product in terms of bar resolutions.]]

Let $f : X \to Y$ and $g : Y \to Z$ be maps of spectra, and let $\tilde{g} : S \to F(Y, Z)$ be right adjoint to g. The smash product $\tilde{g} \wedge f : X \cong S \wedge X \to F(Y, Z) \wedge Y$ followed by the evaluation map $ev : F(Y, Z) \wedge Y \to Z$ defines a map

$$ev \circ (\tilde{g} \wedge f) : X \longrightarrow Z$$

that equals the composite $gf = g \circ f : X \to Z$. Hence the composition pairing

$$\circ : [Y, Z]_* \otimes [X, Y]_* \longrightarrow [X, Z]_*$$

can be rewritten in terms of the smash product pairing as the composite

$$\pi_* F(Y, Z) \otimes [X, Y]_* \xrightarrow{\wedge} [S \wedge X, F(Y, Z) \wedge Y]_* \xrightarrow{ev_*} [X, Z]_* \,.$$

In particular, for $Y = S$, the composition pairing

$$\circ : [S, Z]_* \otimes [X, S]_* \longrightarrow [X, Z]_*$$

equals the smash product pairing. Specializing to $X = S$, the composition and smash products give the same module action

$$\pi_*(Z) \otimes \pi_*(S) \longrightarrow \pi_*(Z) \,.$$

Specializing further to $Z = S$ the two ring structures

$$\pi_*(S) \otimes \pi_*(S) \longrightarrow \pi_*(S)$$

on $\pi_*(S)$ agree. The smash product pairing is graded commutative, since $\mu : S \wedge S \cong S$ and $\mu\gamma$ are homotopic (or equal, in some models). It follows that also the composition product is graded commutative, which is not so evident from its definition.

Conversely, given maps $f : T \to Y$ and $g : V \to Z$ of spectra, the smash product $f \wedge g : T \wedge V \to Y \wedge Z$ can be factored in two ways, as

$$(f \wedge 1) \circ (1 \wedge g) = f \wedge g = (1 \wedge g) \circ (f \wedge 1) \,.$$

Hence the smash product pairing

$$\wedge : [T, Y]_* \otimes [V, Z]_* \longrightarrow [T \wedge V, Y \wedge Z]_*$$

can be rewritten in terms of the composition pairing as the composite

$$\pi_* F(T, Y) \otimes \pi_* F(V, Z) \xrightarrow{\sigma_* \otimes \tau_*} \pi_* F(T \wedge Z, Y \wedge Z) \otimes \pi_* F(T \wedge V, T \wedge Z) \xrightarrow{\circ} \pi_* F(T \wedge V, Y \wedge Z) \,.$$

[[Explain the stabilization maps $\sigma : F(T, Y) \to F(T \wedge Z, Y \wedge Z)$ and $\tau : F(V, Z) \to F(T \wedge V, T \wedge Z)$, perhaps in terms of the adjoints $ev \wedge 1 : F(T, Y) \wedge T \wedge Z \to Y \wedge Z$ and $\gamma(ev \wedge 1)(1 \wedge \gamma) : F(V, Z) \wedge T \wedge V \to T \wedge Z$.]] In particular, for $T = Z = S$, the smash product pairing

$$\wedge : [S, Y]_* \otimes [V, S]_* \longrightarrow [V, Y]_*$$

equals the composition pairing.

Let L, M and N be left A-modules, for a k-algebra A that is projective as a k-module. Let $\epsilon : B_*(A, A, M) \to M$ and $\epsilon : B_*(A, A, N) \to N$ be the normalized bar resolutions. The Yoneda composition

$$\circ : \mathrm{Ext}^{s,t}_A(M, L) \otimes \mathrm{Ext}^{u,v}_A(N, M) \longrightarrow \mathrm{Ext}^{s+u,t+v}_A(N, L)$$

takes $[f] \otimes [g]$ to $[\Sigma^v f \circ g_s]$, where $[f]$ and $[g]$ are the cohomology classes of cocycles $f : B_s(A, A, M) \to \Sigma^t L$ and $g : B_u(A, A, N) \to \Sigma^v M$, or equivalently, of chain maps $f : B_*(A, A, M) \to \Sigma^t L[s]$ and $g : B_*(A, A, N) \to \Sigma^v M[u]$, where $\Sigma^t L[s]$ denotes the chain complex with $\Sigma^t L$ in cohomological degree s and 0 in all other degrees, and similarly for $\Sigma^v M[u]$. Furthermore, $g_* \colon B_*(A, A, N) \to \Sigma^v B_*(A, A, M)[u]$ is a chain map lifting g, so that $\epsilon g_* = g$. It consists of A-module maps $g_q \colon B_{q+u}(A, A, N) \to \Sigma^v B_q(A, A, M)$ for all $q \geq 0$, commuting [[up to a sign]] with the boundary maps, and $\epsilon g_0 = g$. [[Explain the cohomological shift by u, denoted $[u]$, of a chain complex, including sign convention?]]

$$\begin{tikzcd}
B_*(A,A,N) \arrow[r, "g_*"] \arrow[d, "\simeq"', "\epsilon"] \arrow[dr, "g"] & B_*(A,A,M) \arrow[d, "\simeq"', "\epsilon"] \arrow[dr, "f"] & \\
N & M & L
\end{tikzcd}$$

The opposite Yoneda composition

$$\circ^{op} \colon \mathrm{Ext}_A^{u,v}(N, M) \otimes \mathrm{Ext}_A^{s,t}(M, L) \longrightarrow \mathrm{Ext}_A^{s+u,t+v}(N, L)$$

is given by the twist map $\gamma \colon [g] \otimes [f] \mapsto (-1)^{(t-s)(v-u)}[f] \otimes [g]$ followed by the Yoneda composition, hence maps $[g] \otimes [f]$ to $(-1)^{(t-s)(v-u)}[\Sigma^v f \circ g_s]$ in the notation above. [[Is this the correct sign?]]

It is possible to write down an explicit chain map g_* lifting g. Compare Adams (1960, p. 33).

Lemma 13.25. *Given a cocycle $g \colon B_*(A, A, N) \to \Sigma^v M[u]$, a chain map*

$$g_* \colon B_*(A, A, N) \to \Sigma^v B_*(A, A, M)[u]$$

that lifts g is given [[up to sign]] by the formula

$$g_q(a_0[a_1|\dots|a_{q+u}]n) = a_0[a_1|\dots|a_q]g(1[a_{q+1}|\dots|a_{q+u}]n)$$

for each $q \geq 0$. Hence the Yoneda product $[f] \circ [g]$ is represented by the cocycle $\Sigma^v f \circ g_s \colon B_{s+u}(A, A, N) \to \Sigma^{t+v} L$ given by

$$a_0[a_1|\dots|a_{s+u}]n \longmapsto f(a_0[a_1|\dots|a_s]g(1[a_{s+1}|\dots|a_{s+u}]n)) .$$

Proof. Let $g_0 \colon B_u(A, A, N) \to B_0(A, A, M)$ of internal degree v be given by

$$g_0(a_0[a_1|\dots|a_u]n) = a_0[]g(1[a_1|\dots|a_u]n) .$$

Then g_0 is A-linear, and $\epsilon g_0 = g$. Next, define $g_1 \colon B_{u+1}(A, A, N) \to B_1(A, A, M)$ to be the A-linear homomorphism of internal degree v that agrees with $S_0 g_0 \partial_{u+1}$ when restricted along $\eta \otimes 1 \colon k \otimes A^{\otimes(u+1)} \otimes N \to B_{u+1}(A, A, N)$. It is given by

$$g_1(a_0[a_1|\dots|a_{u+1}]n) = a_0[a_1]g(1[a_2|\dots|a_{u+1}]n) ,$$

since the remaining summands from ∂_{u+1} are mapped to terms of the form $a_0[1]m = 0$ in $B_1(A, A, M)$. Then $\partial_1 g_1 = \partial_1 S_0 g_0 \partial_{u+1}$ on $k \otimes A^{\otimes(u+1)} \otimes N$, which by the relation $\partial_1 S_0 + \eta\epsilon = 1$ differs from $g_0 \partial_{u+1}$ by $\eta\epsilon g_0 \partial_{u+1} = \eta g \partial_{u+1} = 0$, where we use that g_0 lifts g and g is a cocycle. Hence $\partial_1 g_1 = g_0 \partial_{u+1}$ on $k \otimes A^{\otimes(u+1)} \otimes N$. Since both sides are A-linear, it follows that $\partial_1 g_1 = g_0 \partial_{u+1}$ on all of $B_{u+1}(A, A, N)$.

For $q \geq 2$, suppose inductively we have defined g_* as a chain map, of internal degree v, up to and including $g_{q-1} \colon B_{u+q-1}(A, A, N) \to B_{q-1}(A, A, M)$. In particular, we are assuming that $\partial_{q-1} g_{q-1} = g_{q-2} \partial_{u+q-1}$. Define $g_q \colon B_{q+u}(A, A, N) \to B_q(A, A, M)$ to be the A-linear homomorphism that agrees with $S_{q-1} g_{q-1} \partial_{q+u}$ when restricted over $\eta \otimes 1 \colon k \otimes A^{\otimes(q+u)} \otimes N \to B_{q+u}(A, A, N)$. Here $S_{q-1} \colon B_{q-1}(A, A, N) \to B_q(A, A, N)$ is part of the chain contraction of $B_*(A, A, N)$, and is not A-linear. By induction it follows that

$$g_q(a_0[a_1|\dots|a_{q+u}]n) = a_0[a_1|\dots|a_q]g(1[a_{q+1}|\dots|a_{q+u}]n) ,$$

since

$$\begin{aligned}
g_q(1[a_1|\dots|a_{q+u}]n) &= S_{q-1} g_{q-1} \partial_{q+u}(1[a_1|\dots|a_{q+u}]n) \\
&= S_{q-1} g_{q-1}(a_1[a_2|\dots|a_{q+u}]n - 1[a_1 a_2|\dots|a_{q+u}]n + \dots) \\
&= S_{q-1}(a_1[a_2|\dots|a_q]g(1[a_{q+1}|\dots|a_{q+u}]n) - 1[a_1 a_2|\dots|a_q]g(1[a_{q+1}|\dots|a_{q+u}]n) + \dots) \\
&= 1[a_1|a_2|\dots|a_q]g(1[a_{q+1}|\dots|a_{q+u}]n) - 1[1|a_1 a_2|\dots|a_q]g(1[a_{q+1}|\dots|a_{q+u}]n) + \dots
\end{aligned}$$

where the second and the remaining terms are zero in the normalized bar resolution. To check that g_* is a chain map, we must prove that $\partial_q g_q = g_{q-1} \partial_{q+u} \colon B_{q+u}(A, A, N) \to B_{q-1}(A, A, M)$. [[This should involve a sign $(-1)^u$.]] Both sides are A-linear, so it suffices to prove this after restriction to $k \otimes A^{\otimes(q+u)} \otimes N$. Here $\partial_q g_q = \partial_q S_{q-1} g_{q-1} \partial_{q+u}$ differs from $g_{q-1} \partial_{q+u}$ by $S_{q-2} \partial q - 1 g_{q-1} \partial_{q+u}$, in view of the relation $\partial_q S_{q-1} + S_{q-2} \partial_{q-1} = 1$. This difference equals $S_{q-2} g_{q+u-1} \partial_{q+u-1} \partial_{q+u} = 0$, by the inductive hypothesis and the fact that $B_*(A, A, N)$ is a chain complex. □

[[When A is a Hopf algebra, and $M = k$, so that

$$f(a_0[a_1|\dots|a_s]g(1[a_{s+1}|\dots|a_{s+u}]n)) = f(a_0[a_1|\dots|a_s]1) \cdot g(1[a_{s+1}|\dots|a_{s+u}]n)\,,$$

the Yoneda product is induced by the diagonal approximation $B_*(A, A, N) \to B_*(A, A, k) \otimes B_*(A, A, N)$ mapping $a_0[a_1|\dots|a_q]n$ to the sum over $s + u = q$ of $a_0[a_1|\dots|a_s]1 \otimes 1[a_{s+1}|\dots|a_{s+q}]n$. Dualize to a concatenation pairing of cobar complexes. What about the case when $M \neq k$?]]

We can also write down explicit diagonal approximations to calculate the tensor product pairings. Compare Adams (1960, p. 35).

Let M, N, T and V be left A-modules, still for a k-algebra A that is projective as a k-module. Let $\epsilon\colon B_*(A, A, M) \to M$ and $\epsilon\colon B_*(A, A, N) \to N$ be the normalized bar resolutions. These are projective A-module resolutions. The tensor product $\epsilon \otimes \epsilon\colon B_*(A, A, M) \otimes B_*(A, A, N) \to M \otimes N$ is then a projective $A \otimes A$-module resolution, hence is chain homotopy equivalent to the normalized bar resolution $\epsilon\colon B_*(A \otimes A, A \otimes A, M \otimes N) \to M \otimes N$.

An explicit chain equivalence

$$AW\colon B_*(A \otimes A, A \otimes A, M \otimes N) \longrightarrow B_*(A, A, M) \otimes B_*(A, A, N)$$

is given by the Alexander–Whitney map

$$AW(a_0 \otimes b_0[a_1 \otimes b_1|\dots|a_q \otimes b_q]m \otimes n) = \sum_{i=0}^{q} a_0[a_1|\dots|a_i]a_{i+1}\dots a_q m \otimes b_0 b_1 \dots b_i[b_{i+1}|\dots|b_q]n\,.$$

See Mac Lane (1964, Cor. X.7.2). Now suppose that A is a Hopf algebra, with coproduct $\psi\colon A \to A \otimes A$. Viewing $M \otimes N$ as an A-module by restricting the $A \otimes A$-module stucture along the algebra homomorphism ψ, we get a chain equivalence

$$\Psi = B(\psi, \psi, 1)\colon B(A, A, M \otimes N) \longrightarrow B(A \otimes A, A \otimes A, M \otimes N)$$

of A-module resolutions of $M \otimes N$. Note that both of these are projective A-module resolutions, by our assuptions on A and the untwisting isomorphism from Proposition 13.18. The composite $\Delta = AW \circ \Psi$ is a chain equivalence

$$\Delta\colon B_*(A, A, M \otimes N) \longrightarrow B_*(A, A, M) \otimes B_*(A, A, N)$$

of A-modules, with the diagonal action on the right hand side, given by

$$\Delta(a_0[a_1|\dots|a_q]m \otimes n) = \sum_{i=0}^{q} a_0'[a_1'|\dots|a_i']a_{i+1}'\dots a_q' m \otimes a_0'' a_1'' \dots a_i''[a_{i+1}''|\dots|a_q'']n\,,$$

where $\psi(a_i) = \sum a_i' \otimes a_i''$ for all $0 \le i \le q$.

[[TODO: State the result above as a lemma.]]

As a special case, if $M = k$, and we arrange that $a_i' \in I(A)$ for all summands $a_i' \otimes a_i''$ of $\psi(a_i)$, except for a term $1 \otimes a_i$, then

$$\Delta(a_0[a_1|\dots|a_q]n) = \sum_{i=0}^{q} a_0'[a_1'|\dots|a_i']1 \otimes a_0'' a_1'' \dots a_i''[a_{i+1}|\dots|a_q]n\,.$$

This recovers Adams' formula. [[Explain how to construct this directly?]]

The tensor product pairing

$$\otimes\colon \operatorname{Ext}_A^{s,t}(M, T) \otimes \operatorname{Ext}_A^{u,v}(N, V) \longrightarrow \operatorname{Ext}_A^{s+u,t+v}(M \otimes N, T \otimes V)$$

takes $[f] \otimes [g]$ to $[(f \otimes g)\Delta]$, where $f\colon B_s(A, A, M) \to \Sigma^t T$ and $g\colon B_u(A, A, N) \to \Sigma^v V$ are cocycles, so that $f\partial_{s+1} = 0$ and $g\partial_{u+1} = 0$. When viewed as chain maps $f\colon B_*(A, A, M) \to \Sigma^t T[s]$ and $g\colon B_*(A, A, N) \to \Sigma^u V[u]$, mapping to 0 in degrees other than s and u, respectively, their tensor product is a chain map

$$f \otimes g\colon B_*(A, A, M) \otimes B_*(A, A, N) \longrightarrow \Sigma^t T[s] \otimes \Sigma^u V[u] \cong \Sigma^{t+v} T \otimes V[s+u]\,.$$

The composite $(f \otimes g)\Delta$ is then the chain map determined by the cocycle

$$B_{s+u}(A, A, M \otimes N) \longrightarrow \Sigma^{t+v} T \otimes V$$

given by

$$a_0[a_1|\dots|a_{s+u}]m \otimes n \longmapsto \sum f(a_0'[a_1'|\dots|a_s']a_{s+1}'\dots a_{s+u}' m) \otimes g(a_0'' a_1'' \dots a_s''[a_{s+1}''|\dots|a_{s+u}'']n)\,.$$

Proposition 13.26. *Let A be a Hopf algebra, projective as a k-module, and let L and N be A-modules. The Yoneda composition pairing*

$$\circ\colon \operatorname{Ext}_A^{s,t}(k,L)\otimes \operatorname{Ext}_A^{u,v}(N,k)\longrightarrow \operatorname{Ext}_A^{s+u,t+v}(N,L)$$

agrees with the tensor product pairing

$$\otimes\colon \operatorname{Ext}_A^{s,t}(k,L)\otimes \operatorname{Ext}_A^{u,v}(N,k)\longrightarrow \operatorname{Ext}_A^{s+u,t+v}(k\otimes N,L\otimes k)\,.$$

Proof. Let $f\colon B_s(A,A,k)\to\Sigma^t L$ and $g\colon B_u(A,A,N)\to\Sigma^v k$ be A-linear cocycles. The Yoneda composite $[f]\circ[g]$ is represented by the cocycle $\Sigma^v f\circ g_s\colon B_{s+u}(A,A,N)\to\Sigma^{t+v}L$ given by

$$a_0[a_1|\dots|a_{s+u}]n\longmapsto f(a_0[a_1|\dots|a_s]g(1[a_{s+1}|\dots|a_{s+u}]n))\,.$$

The tensor product $[f]\otimes[g]$ is represented by the cocycle

$$a_0[a_1|\dots|a_{s+u}]n\longmapsto \sum f(a_0'[a_1'|\dots|a_s']\epsilon(a_{s+1}'\dots a_{s+u}'))\cdot g(a_0''a_1''\dots a_s''[a_{s+1}''|\dots|a_{s+u}'']n)$$

where $\psi(a_i)=\sum a_i'\otimes a_i''$. The assumption $M=k$ implies that $g(a_0''a_1''\dots a_s''[\dots]n)=0$ if some $a_i''\in I(A)$, and $\epsilon(a_{s+1}'\dots a_{s+u}')=0$ if some $a_i'\in I(A)$, so only the summands $a_i\otimes 1$ of $\psi(a_i)$ contribute for $0\le i\le s$, and only the summands $1\otimes a_i$ contribute for $s+1\le i\le s+u$. Hence the sum simplifies to the single term

$$a_0[a_1|\dots|a_{s+u}]n\longmapsto f(a_0[a_1|\dots|a_s]1)\cdot g(1[a_{s+1}|\dots|a_{s+u}]n)\,.$$

Since f is k-linear, this equals the cocycle $\Sigma^v f\circ g_s$. □

Theorem 13.27. *There is a natural pairing*

$$E_r^{s,t}(S,Z)\otimes E_r^{u,v}(X,S)\longrightarrow E_r^{s+u,t+v}(X,Z)$$

of Adams spectral sequences, given at the E_2-term by the opposite Yoneda product

$$\operatorname{Ext}_{\mathscr{A}}^{s,t}(H^*(Z),\mathbb{F}_p)\otimes \operatorname{Ext}_{\mathscr{A}}^{u,v}(\mathbb{F}_p,H^*(X))\longrightarrow \operatorname{Ext}_{\mathscr{A}}^{s+u,t+v}(H^*(Z),H^*(X))$$

and converging to the composition pairing

$$\pi_{t-s}(Z_p^\wedge)\otimes[X,S_p^\wedge]_{v-u}\longrightarrow[X,Z_p^\wedge]_{t-s+v-u}\,.$$

Proof. This is the same as the smash and tensor product pairing of the Adams spectral sequences, since the tensor product of Ext-groups agrees with the opposite Yoneda product, and the smash product of homotopy classes agrees with the composition product. □

13.8. **The composition pairing, revisited.** Here is a geometric proof of Moss' theorem on the composition pairing, close to the one for the smash product pairing.

Proof. Let $\{Y^s\}_s$ and $\{Z^u\}_u$ be Adams resolutions of Y and Z, with cofibers $Y^s/Y^{s+1}=K^s$ and $Z^u/Z^{u+1}=L^u$, respectively. Let $P_s=H_*(\Sigma^s K^s)$ and $Q_u=H_*(\Sigma^u L^u)$, as usual.

Consider the homotopy limit of mapping spectra

$$M^u=\operatorname*{holim}_{n\le u+s}F(Y^s,Z^n)\,.$$

Restriction from $n\le u+s+1$ to $n\le u+s$ gives a map $i\colon M^{u+1}\to M^u$. Its homotopy fiber is the product over s of the iterated homotopy fiber in the square

$$\begin{tikzcd} F(Y^s,Z^{u+s+1}) \arrow[r] \arrow[d] & F(Y^s,Z^{u+s}) \arrow[d] \\ F(Y^{s+1},Z^{u+s+1}) \arrow[r] & F(Y^{s+1},Z^{u+s})\,, \end{tikzcd}$$

which is equivalent to $F(K^s,L^{u+s})$. Hence we get a tower

$$\begin{tikzcd} \dots \arrow[r] & M^{u+1} \arrow[r] & M^u \arrow[r] \arrow[d] & \dots \arrow[r] & M^1 \arrow[r] & M^0 \arrow[d] \\ & & \prod_s F(K^s,L^{u+s}) \arrow[ul, dashed] & & & \prod_s F(K^s,L^s)\,. \arrow[ul, dashed] \end{tikzcd}$$

Restriction to $(s,n)=(0,u)$ defines a map to the tower

$$\begin{tikzcd}
\dots \arrow[r] & F(Y,Z^{u+1}) \arrow[r] & F(Y,Z^u) \arrow[r] \arrow[d] & \dots \arrow[r] & F(Y,Z^1) \arrow[r] & F(Y,Z) \arrow[d] \\
 & & F(Y,L^u) \arrow[ul, dashed] & & & F(Y,L^0)\,. \arrow[ul, dashed]
\end{tikzcd}$$

Applying homotopy we get a map of unrolled exact couples, from

$$\begin{tikzcd}
\dots \arrow[r] & \pi_*(M^{u+1}) \arrow[r] & \pi_*(M^u) \arrow[r] \arrow[d] & \dots \arrow[r] & \pi_*(M^1) \arrow[r] & \pi_*(M^0) \arrow[d] \\
 & & \prod_s [K^s, L^{u+s}]_* \arrow[ul, dashed] & & & \prod_s [K^s, L^s]_* \arrow[ul, dashed]
\end{tikzcd}$$

to the one generating the Adams spectral sequence $\{E_r^{*,*}(Y,Z)\}_r$. Let $\{'E_r^{u,*}\}_r$ be the spectral sequence generated by the unrolled exact couple just displayed. The map $'E_1^{u,*} \to E_1^{u,*}(Y,Z)$ of E_1-terms can be identified, using the d-invariant isomorphisms

$$\prod_s [K^s, L^{u+s}]_* \cong \prod_s \mathrm{Hom}^*_{\mathscr{A}}(Q_{u+s}, P_s) = \mathrm{HOM}^{u,*}_{\mathscr{A}}(Q_*, P_*)$$
$$[Y, L^u]_* \cong \mathrm{Hom}^*_{\mathscr{A}}(Q_u, H^*(Y))\,,$$

with the quasi-isomorphism

$$\epsilon_* \colon \mathrm{HOM}^{u,*}_{\mathscr{A}}(Q_*, P_*) \longrightarrow \mathrm{Hom}^*_{\mathscr{A}}(Q_u, H^*(Y))$$

induced by $\epsilon \colon P_* \to H^*(Y)$. Hence the map of E_2-terms is an isomorphism, identifying $'E_2^{u,*}$ with the Adams E_2-term

$$E_2^{u,*}(Y,Z) = \mathrm{Ext}^{u,*}_{\mathscr{A}}(H^*(Z), H^*(X))\,.$$

We shall define a pairing of spectral sequences

$$\phi_r \colon {'E_r^{u,*}} \otimes E_r^{s,*}(X,Y) \longrightarrow E_r^{u+s,*}(X,Z)$$

for $r \geq 1$, which agrees with the composition pairing

$$\mathrm{HOM}^{u,*}_{\mathscr{A}}(Q_*, P_*) \otimes \mathrm{Hom}_{\mathscr{A}}(P_s, H^*(X)) \to \mathrm{Hom}_{\mathscr{A}}(Q_{u+s}, H^*(X))$$

for $r=1$. For $r \geq 2$ the source is isomorphic to

$$E_r^{u,*}(Y,Z) \otimes E_r^{s,*}(X,Y)$$

via $\epsilon_* \otimes 1$, which yields Moss' pairing and the compatibility with the Yoneda product for $r=2$.

The pairing $\phi_1 \colon {'E_1^{u,*}} \otimes E_1^{s,*}(X,Y) \to E_1^{u+s,*}(X,Z)$ is the composition pairing

$$\prod_s [K^s, L^{u+s}]_* \otimes [X, K^s]_* \longrightarrow [X, L^{u+s}]_*$$

that takes $(g^s)_s \otimes f$ to $g^s f$. We show that it restricts to a pairing $\tilde\phi_r \colon {'Z_r^{u,*}} \otimes Z_r^{s,*}(X,Y) \to Z_r^{u+s,*}(X,Z)$ of r-th cycles, that descends to a pairing $\phi_r \colon {'E_r^{u,*}} \otimes E_r^{s,*}(X,Y) \to E_r^{u+s,*}(X,Z)$ satisfying the Leibniz rule, for each $r \geq 1$.

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We shall use the identifications

$$'Z_r^{u,*} = \mathrm{im}(\pi_*(M^u/M^{u+r}) \to \pi_*(M^u/M^{u+1}))$$
$$Z_r^{s,*}(X,Y) = \mathrm{im}([X, Y^s/Y^{s+r}]_* \to [X, K^s]_*)$$
$$Z_r^{s,*}(X,Z) = \mathrm{im}([X, Z^{u+s}/Z^{u+s+r}]_* \to [X, L^{u+s}]_*)$$

where $M^u/M^{u+1} = \prod_s F(K^s, L^{u+s})$.

Consider the commutative square

$$\begin{tikzcd}
F(Y^s, Z^{u+s+r}) \arrow[r] \arrow[d] & F(Y^s, Z^{u+s}) \arrow[d] \\
F(Y^{s+r}, Z^{u+s+r}) \arrow[r] & F(Y^{s+r}, Z^{u+s})\,.
\end{tikzcd}$$

There are restriction maps from M^{u+r} to the upper left hand corner, and from M^u to the homotopy pullback of the rest of the square. Hence there is a map of homotopy fibers from $\Sigma^{-1}(M^u/M^{u+1})$ to $F(Y^s/Y^{s+r}, \Sigma^{-1}(Z^{u+s}/Z^{u+s+r}))$, giving a map

$$M^u/M^{u+r} \longrightarrow F(Y^s/Y^{s+r}, Z^{u+s}/Z^{u+s+r})$$

and an adjoint pairing

$$M^u/M^{u+r} \wedge Y^s/Y^{s+r} \longrightarrow Z^{u+s}/Z^{u+s+r}$$

compatible with the pairing $M^u/M^{u+1} \wedge K^s \to L^{u+s}$ for $r = 1$. This leads to the commutative diagram

$$\begin{tikzcd}
\pi_*(M^u/M^{u+r}) \otimes [X, Y^s/Y^{s+r}]_* \arrow[r] \arrow[d] & {[X, Z^{u+s}/Z^{u+s+r}]_*} \arrow[d] \\
\prod_s [K^s, L^{u+s}]_* \otimes [X, K^s]_* \arrow[r, "\phi_1"] & {[X, L^{u+s}]_*} \, .
\end{tikzcd}$$

The induced pairing of vertical images is ϕ_r.

((EDIT TO HERE)) □

14. Calculations

14.1. The minimal resolution, revisited. Recall the minimal resolution $\epsilon\colon P_* \to \mathbb{F}_2$.

Lemma 14.1. *The product $h_i \cdot \gamma_{s,n}$ contains the summand $\gamma_{s+1,m}$ if and only if $\partial_{s+1}(g_{s+1,m}) = \sum_j a_j g_{s,j}$ contains the summand $Sq^{2^i} g_{s,n}$.*

Proof. Let $\gamma_{s,n}\colon P_s \to \mathbb{F}_2$ be dual to the generator $g_{s,n} \in P_s$, and let $h_i = \gamma_{1,i}\colon P_1 \to \mathbb{F}_2$ be dual to $g_{1,i} = [Sq^{2^i}]$.

$$\begin{tikzcd}
P_{s+1} \arrow[r, "\gamma_1"] \arrow[d, "\partial_{s+1}"'] & P_1 \arrow[d, "\partial_1"'] \arrow[dr, "h_i"] & \\
P_s \arrow[r, "\gamma_0"] \arrow[dr, "\gamma_{s,n}"'] & P_0 \arrow[d, "\epsilon"] & \mathbb{F}_2 \\
 & \mathbb{F}_2 &
\end{tikzcd}$$

We lift $\gamma_{s,n}$ to $\gamma_0\colon P_s \to P_0$ mapping $g_{s,n} \mapsto g_{0,0}$ and $g_{s,j} \mapsto 0$ for $j \neq n$. Then $\gamma_0 \circ \partial_{s+1}$ sends $g_{s+1,m}$ to $a_n g_{0,0}$. To lift γ_0 to $\gamma_1\colon P_{s+1} \to P_1$ we write $a_n = \sum_k b_k Sq^{2^k}$, with each $b_k \in \mathscr{A}$. Then we may take $\gamma_1(g_{s+1,m}) = \sum_k b_k g_{1,k}$, since $\partial_1(g_{1,k}) = Sq^{2^k} g_{0,0}$. The coefficient of $g_{s+1,m}$ in the Yoneda product $h_i \cdot \gamma_{s,n}$ is then given by the value of $h_i \circ \gamma_1$ on $g_{s+1,m}$, which equals $h_i(\sum_k b_k g_{1,k}) = \epsilon(b_i)$. Hence $\gamma_{s+1,m}$ occurs as a summand in $h_i \cdot \gamma_{s,n}$ if and only if Sq^{2^i} occurs as a summand in $a_n = \sum_k b_k Sq^{2^k}$. This is equivalent to the condition that Sq^{2^i} occurs as a summand when a_n is written as a sum of admissible monomials. □

Proposition 14.2. *The Yoneda products in $\mathrm{Ext}_{\mathscr{A}}^{*,*}(\mathbb{F}_2, \mathbb{F}_2)$ in internal degrees $t \leq 11$ are given by:*

$\cdot$	$\gamma_{0,0}$	$\gamma_{1,0}$	$\gamma_{1,1}$	$\gamma_{1,2}$	$\gamma_{1,3}$	$\gamma_{2,0}$	$\gamma_{2,1}$	$\gamma_{2,2}$	$\gamma_{2,3}$	$\gamma_{2,4}$	$\gamma_{2,5}$	$\gamma_{3,0}$	$\gamma_{3,1}$	$\gamma_{3,2}$	$\gamma_{s,0}$
h_0	$\gamma_{1,0}$	$\gamma_{2,0}$	0	$\gamma_{2,2}$	$\gamma_{2,4}$	$\gamma_{3,0}$	0	$\gamma_{3,1}$	0	$\gamma_{3,2}$	0	$\gamma_{4,0}$	0	$\gamma_{4,1}$	$\gamma_{s+1,0}$
h_1	$\gamma_{1,1}$	0	$\gamma_{2,1}$	0	$\gamma_{2,5}$	0	$\gamma_{3,1}$	0	0	0	?	0	0	?	0
h_2	$\gamma_{1,2}$	$\gamma_{2,2}$	0	$\gamma_{2,3}$	?	$\gamma_{3,1}$	0	0	?	?	?	0	0	?	0
h_3	$\gamma_{1,3}$	$\gamma_{2,4}$	$\gamma_{2,5}$	?	?	$\gamma_{3,2}$	?	?	?	?	?	$\gamma_{4,1}$	?	?	?

for $5 \leq s \leq 10$.

Proof. This can be read off from the minimal resolution $\epsilon\colon P_* \to \mathbb{F}_2$, using the lemma above. □

Remark 14.3. The remaining summands, like $Sq^3 g_{1,0}$ in $\partial_2(g_{2,1})$ and $Sq^2 Sq^1 g_{1,1}$ in $\partial_2(g_{2,2})$, contribute to higher compositions like Massey products, like $h_1^2 \in \langle h_0, h_1, h_0 \rangle$ and $h_0 h_2 \in \langle h_1, h_0, h_1 \rangle$, which imply $\eta^2 \in \langle 2, \eta, 2 \rangle$ and $2\nu \in \langle \eta, 2, \eta \rangle$, respectively.

Definition 14.4. Let $c_0 \in \mathrm{Ext}_{\mathscr{A}}^{3,11}(\mathbb{F}_2, \mathbb{F}_2)$ be the class of the cocycle $\gamma_{3,3}\colon P_3 \to \mathbb{F}_2$ of degree 11, dual to $g_{3,3}$.

Corollary 14.5. *The algebra unit is* $1 = \gamma_{0,0}$*. The classes* $h_0 = \gamma_{1,0}$*,* $h_1 = \gamma_{1,1}$*,* $h_2 = \gamma_{1,2}$*,* $h_3 = \gamma_{1,3}$ *and* $c_0 = \gamma_{3,3}$ *are indecomposable. The remaining additive generators in internal degree* $t \le 11$ *are decomposable. These algebra generators commute with one another, so the Yoneda product is commutative (in this range). The decomposable generators have the following presentations:*

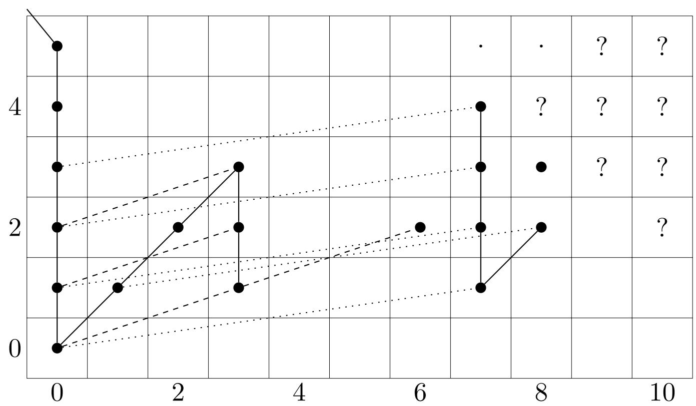
$$\begin{aligned}
\gamma_{2,0} &= h_0^2 & \qquad \gamma_{3,0} &= h_0^3 \\
\gamma_{2,1} &= h_1^2 & \qquad \gamma_{3,1} &= h_1^3 = h_0^2 h_2 \\
\gamma_{2,2} &= h_0 h_2 & \qquad \gamma_{3,2} &= h_0^2 h_3 \\
\gamma_{2,3} &= h_2^2 & \qquad \gamma_{4,1} &= h_0^3 h_3 \\
\gamma_{2,4} &= h_0 h_3 & \qquad \gamma_{s,0} &= h_0^s \\
\gamma_{2,5} &= h_1 h_3 & &
\end{aligned}$$

for $s \ge 5$*. The relations* $h_0 h_1 = 0$*,* $h_1 h_2 = 0$*,* $h_1^3 = h_0^2 h_2$ *and* $h_0 h_2^2 = 0$ *are satisfied, and generate all other relations for* $s \le 3$ *and* $t \le 11$*.*

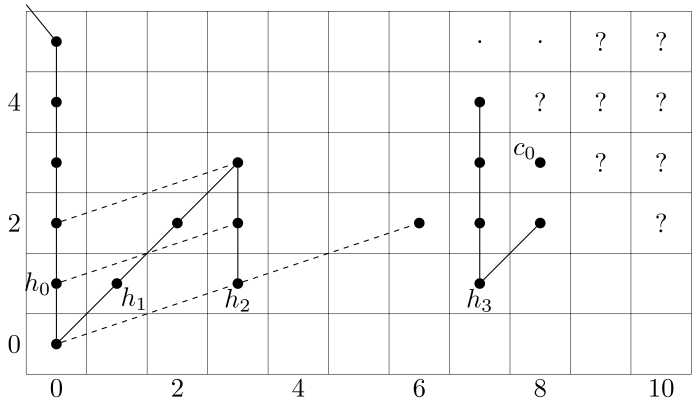
We redraw the Adams E_2-term with these standard names for the generators, in the usual chart with the topological degree $t - s$ on the horizontal axis and the filtration degree s on the vertical axis. (The class labeled h_1^3 could equally well have been called $h_0^2 h_2$.)

	h_0^5							$\cdot$	$\cdot$	?	?
4	h_0^4							$h_0^3 h_3$	?	?	?
	h_0^3			h_1^3				$h_0^2 h_3$	c_0	?	?
2	h_0^2		h_1^2	$h_0 h_2$			h_2^2	$h_0 h_3$	$h_1 h_3$		?
	h_0	h_1		h_2				h_3			
0	1										
	0		2		4		6		8		10

Another way to draw the chart is to use a $\bullet$ for each additive generator, a vertical line connecting x to $h_0 x$, a line of slope 1 connecting x to $h_1 x$, a (dashed) line of slope 1/3 connecting x to $h_2 x$, and a (dotted) line of slope 1/7 connecting x to $h_3 x$.



Here is the same chart without the h_3-multiplications, which tend to clutter the picture, but with labels for the indecomposables.



The reader might contemplate the relations $h_i h_{i+1} = 0$, $h_{i+1}^3 = h_i^2 h_{i+2}$ and $h_i h_{i+2}^2 = 0$, in view of this diagram.

Let us take for granted Adams' vanishing result, in the form that the groups $E_2^{s,t} = 0$ for $1 \le t-s \le 7$ and $s \ge 5$. Then:

Lemma 14.6. *$E_2^{s,t} = E_\infty^{s,t}$ for $t \le 11$.*

Proof. Since the h_i for $0 \le i \le 3$ represent homotopy classes, they are infinite cycles, meaning that $d_r(h_i) = 0$ for all $r \ge 2$. By the Leibniz rule, it follows that $d_r(x) = 0$ for each x in the subalgebra generated by these classes. The only remaining additive generator is c_0, but $d_r(c_0)$ lands in Adams' vanishing range, for all $r \ge 2$. □

Theorem 14.7. *[(a)]*

1. *$\pi_0(S)_2^\wedge \cong \mathbb{Z}_2$ is generated by the identity map $\iota\colon S \to S$, represented by $1 \in E_\infty^{0,0}$. The class of $2^s\iota$ is represented by $h_0^s \in E_\infty^{s,s}$, for all $s \ge 0$.*
2. *$\pi_1(S)_2^\wedge \cong \mathbb{Z}/2$ is generated by the complex Hopf map $\eta\colon S^1 \to S$, represented by $h_1 \in E_\infty^{1,2}$.*
3. *$\pi_2(S)_2^\wedge \cong \mathbb{Z}/2$ is generated by η^2, represented by $h_1^2 \in E_\infty^{2,4}$.*
4. *$\pi_3(S)_2^\wedge \cong \mathbb{Z}/8$ is generated by the quaternionic Hopf map $\nu\colon S^3 \to S$, represented by $h_2 \in E_\infty^{1,4}$. The class 2ν is represented by $h_0h_2 \in E_\infty^{2,5}$, and the class $4\nu = \eta^3$ is represented by $h_0^2 h_2 = h_1^3$ in $E_\infty^{3,6}$.*
5. *$\pi_4(S)_2^\wedge = 0$.*
6. *$\pi_5(S)_2^\wedge = 0$.*
7. *$\pi_6(S)_2^\wedge \cong \mathbb{Z}/2$ is generated by ν^2, represented by $h_2^2 \in E_\infty^{2,8}$.*
8. *$\pi_7(S)_2^\wedge \cong \mathbb{Z}/16$ is generated by the octonionic Hopf map $\sigma\colon S^7 \to S$, represented by $h_3 \in E_\infty^{1,8}$. The classes $2^k\sigma$ are represented by $h_0^k h_3 \in E_\infty^{k+1,k+8}$, for $0 \le k \le 3$.*

This gives the additive structure of $\pi_*(S)_2^\wedge$ for $* \le 7$. We can also determine the multiplicative structure.

Proposition 14.8. $2\eta = 0$, $\eta^3 = 4\nu$, $\eta\nu = 0$, $2\nu^2 = 0$.

Proof. These follow from the relations $h_0h_1 = 0$, $h_1^3 = h_0^2h_2$, $h_1h_2 = 0$ and $h_0h_2^2 = 0$ in $\mathrm{Ext}_{\mathscr{A}}$, together with the fact that there are no classes of higher Adams filtration, in these cases. □

Remark 14.9. By associativity, it is clear that $\eta \cdot \nu^2 = \eta\nu \cdot \nu = 0$. On the other hand, the vanishing of $h_1 \cdot h_2^2$ in $\mathrm{Ext}_{\mathscr{A}}^{3,10}(\mathbb{F}_2, \mathbb{F}_2)$ only tells us that $\eta \cdot \nu^2$ is 0 modulo classes of Adams filtration $s \ge 4$. There is one such class, namely 8σ represented by $h_0^3h_3$, but the factorization of ν^2 tells us that $\eta \cdot \nu^2$ is not equal to 8σ, but is 0.

14.2. **The Toda–Mimura range.** Toda (1962) calculated $\pi_{n+k}(S^k)$ for all $n \le 19$, Mimura and Toda (1963) extended this to $n = 20$, and Mimura (1965) carried on to $n = 21$ and $n = 22$. For k large, these computations determine the stable homotopy groups $\pi_n(S)$ for $n \le 22$. ((Maybe better to continue to $n \le 23$, to see $\nu\bar{\kappa}$.))

The Adams E_2-term in this range was originally computed by hand (by Adams (1961) for $t-s \le 17$ and Liulevicius (unpublished) for $t-s \le 23$), then by the May spectral sequence (by May (1964) for $t-s \le 42$ and Tangora (1970) for $t-s \le 70$), but can now quickly be obtained by machine computation.

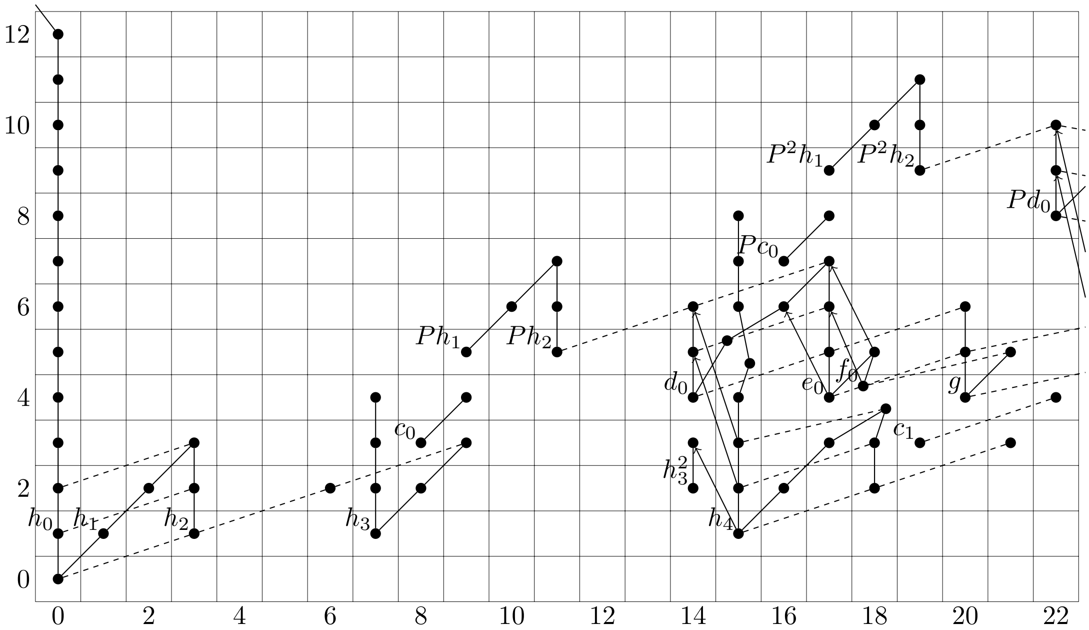


FIGURE 25. Adams spectral sequence for S, in degrees $0 \le * \le 22$

Bruner's `ext`-program yields the chart in Figure 25. The larger chart in Figure 26 was created by Christian Nassau (2001).

((Show hidden extensions: η times ρ is represented by Pc_0, η times $\eta\bar{\kappa}$ is represented by Pd_0, 2 times $2\nu\bar{\kappa}$ equals ν times $4\bar{\kappa}$ and is represented by h_1Pd_0, ν times ν^2 differs from $\eta^2\sigma$ by $\eta\epsilon$.))

With the exception of f_0, each labeled class is the unique nonzero class in its bidegree. The class f_0 is, for now, only defined modulo the decomposable class $h_1^3h_4 = h_0^2h_2h_4$. (A definite choice can be made using Steenrod operations in Ext.)

In addition to the h_0-, h_1- and h_2-multiplications shown, and the product $h_3 \cdot h_3 = h_3^2$ in $E_2^{2,16}$, there are the following nonzero h_3-multiplications:

$$
\begin{aligned}
h_3 \cdot Ph_1 &= h_1^2 d_0 \\
h_3 \cdot h_1Ph_1 &= h_2^2 Ph_2 = h_1^3 d_0 = h_0^3 e_0 \\
h_3 \cdot h_3^2 &= h_2^2 h_4 \\
h_3 \cdot e_0 &= h_1 h_4 c_0 \\
h_3 \cdot P^2h_1 &= h_1^2 Pd_0 \\
h_3 \cdot h_1P^2h_1 &= h_2^2 P^2h_2 = h_1^3 Pd_0 = h_0^3 Pe_0
\end{aligned}
$$

The last three of these land outside the displayed range of topological degrees. We omit to list the h_i-multiplications for $i \ge 4$. ((The multiplicative structure also includes relations like $c_0^2 = h_1^2 d_0$.))

The evolution of the Adams spectral sequence in this range is as follows.

Theorem 14.10. *The algebra indecomposables in topological degree $t - s \le 22$ of the Adams E_2-term are h_0, h_1, h_2, h_3 and h_4 in filtration $s = 1$, c_0 and c_1 in filtration $s = 3$, d_0, e_0, f_0 and $g = g_1$ in filtration $s = 4$, Ph_1 and Ph_2 in filtration $s = 5$, Pc_0 in filtration $s = 7$, Pd_0 in filtration $s = 8$, and P^2h_1 and P^2h_2 in filtration $s = 9$.*

The classes h_0, h_1, h_2, h_3, c_0, c_1, d_0, g, Ph_1, Ph_2, Pc_0, Pd_0, P^2h_1 and P^2h_2 are infinite cycles.

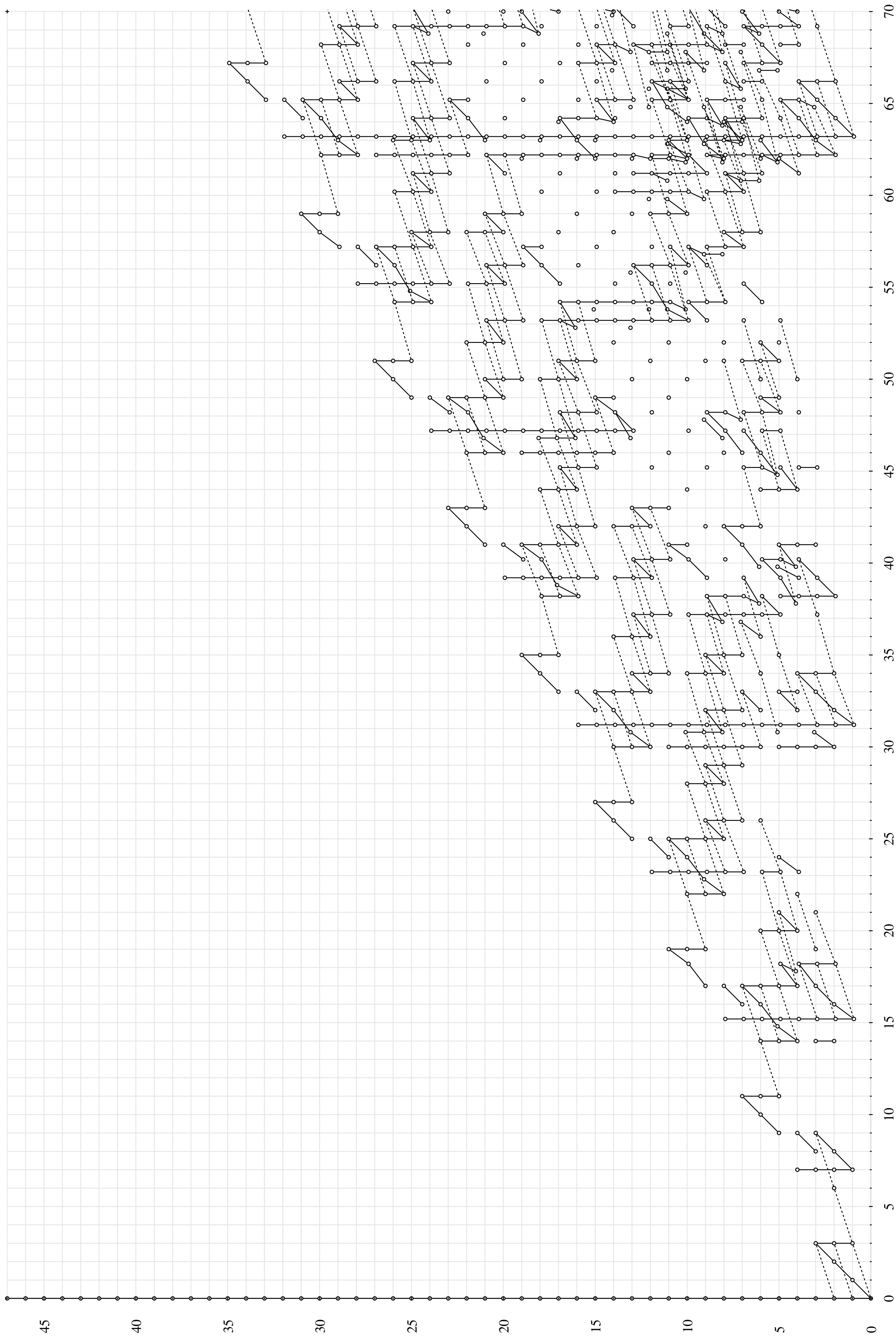


FIGURE 26. Ext over $\mathscr{A}$ by Christian Nassau (2001)

The nonzero d_2-differentials affecting this range are:

$$\begin{aligned} h_4 &\overset{d_2}{\longmapsto} h_0h_3^2 \\ e_0 &\longmapsto h_1^2d_0 \\ f_0 &\longmapsto h_0h_2d_0 = h_0^2e_0 \\ h_1e_0 = h_0f_0 &\longmapsto h_1^3d_0 = h_0^3e_0 \\ i &\longmapsto h_0Pd_0 \\ h_0i &\longmapsto h_0^2Pd_0 \end{aligned}$$

The list of algebra indecomposables of the E_3-term is as for the E_2-term, with h_4, e_0 and f_0 deleted, but with h_0h_4, h_1h_4 and h_2h_4 added. The classes h_1h_4 and h_2h_4 are infinite cycles.

The nonzero d_3-differentials are:

$$\begin{aligned} h_0h_4 &\overset{d_3}{\longmapsto} h_0d_0 \\ h_0^2h_4 &\longmapsto h_0^2d_0 \end{aligned}$$

The list of algebra indecomposables of the E_4-term is as for the E_3-term, with h_0h_4 deleted, but with $h_0^3h_4$ added. There are no further differentials, so that $E_4 = E_\infty$ in this range of topological degrees.

Sketch proof. Use graded commutativity of $\pi_*(S)$ to see that $2\sigma^2 = 0$, but $h_0h_3^2 \neq 0$ in $E_2^{3,17}$. Since $h_0h_3^2$ is an infinite cycle, it must be a boundary, so $d_2(h_4) = h_0h_3^2$.

Using the homotopy-everything structure on S, one gets a differential $d_2(f_0) = h_0^2e_0$, which implies that $d_2(h_0f_0) = h_0^3e_0$ and $d_2(e_0) = h_1^2d_0$.

Using the J-homomorphism, we known that $\pi_{15}(S)^\wedge_2$ contains $\mathbb{Z}/32$ as a direct summand. We know that $d_2(h_0h_4) = h_0^2h_3^2 = 0$. If also $d_3(h_0h_4) = 0$, then $\pi_{15}(S)^\wedge_2$ would instead contain a copy of $\mathbb{Z}/64$ (unless $d_6(h_1h_4) = h_0^7h_4$). Deduce that $d_3(h_0h_4) = h_0d_0$. □

Toda (1962) uses the following notation.

Definition 14.11. Let $\epsilon \in \pi_8(S)^\wedge_2$ be the unique class represented by $c_0 \in E_\infty^{3,11}$. Then $\eta\epsilon \in \pi_9(S)^\wedge_2$ is represented by $h_1c_0 \in E_\infty^{4,13}$. ((Claim: $\nu^3 = \eta^2\sigma + \eta\epsilon$.))

Let $\mu = \mu_9 \in \pi_9(S)^\wedge_2$ be the unique class represented by $Ph_1 \in E_\infty^{5,14}$. Then $\eta\mu = \mu_{10} \in \pi_{10}(S)^\wedge_2$ is the unique class represented by $h_1Ph_1 \in E_\infty^{6,16}$.

Let $\zeta \in \pi_{11}(S)^\wedge_2$ be a class represented by $Ph_2 \in E_\infty^{5,16}$. It is determined up to an odd multiple. Then $4\zeta = \eta^2\mu$.

The class $\sigma^2 = \theta_3$ in $\pi_{14}(S)^\wedge_2$ is decomposable. It is represented by $h_3^2 \in E_\infty^{2,16}$.

Let $\kappa \in \pi_{14}(S)^\wedge_2$ be the unique class represented by $d_0 \in E_\infty^{4,18}$. ((Then $\eta\kappa \in \pi_{15}(S)^\wedge_2$ is represented by h_1d_0, and $\nu\kappa \in \pi_{17}(S)^\wedge_2$ is represented by h_2d_0, while $\eta^2\kappa = 0$.))

Let $\rho \in \pi_{15}(S)^\wedge_2$ be a class represented by $h_0^3h_4$. It is determined up to an odd multiple. ((There is a hidden multiplicative extension: $\eta\rho$ is represented by Pc_0.))

Let $\eta^* = \eta_4 \in \pi_{16}(S)^\wedge_2$ be a class represented by h_1h_4. ((This only defines it modulo $\eta\rho$.))

Let $\nu^* \in \pi_{18}(S)^\wedge_2$ be a class represented by h_2h_4. ((This only defines it up to an odd multiple, and modulo $\eta\bar{\mu} = \mu_{18}$. Compare σ^3 to $\nu\nu^*$?))

Let $\bar{\mu} = \mu_{17} \in \pi_{17}(S)^\wedge_2$ be the unique class represented by $P^2h_1 \in E_\infty^{9,26}$. Then $\eta\bar{\mu} = \mu_{18} \in \pi_{18}(S)^\wedge_2$ is the unique class represented by $h_1P^2h_1 \in E_\infty^{10,28}$.

((Define $\bar{\sigma}$, $\bar{\zeta}$.))

Definition 14.12. It is traditional to write θ_j for a class in $\pi_{2^{j+1}-2}(S)$ represented by h_j^2 in $E_\infty^{2^{j+1},2}$, if such a class exists, and to write η_j for a class in $\pi_{2^j}(S)$ represented by $h_1h_j \in E_\infty^{2^j+2,2}$.

Remark 14.13. The classes θ_j are realized for $0 \le j \le 3$ by $2^2 = 4$, η^2, ν^2 and σ^2. It follows from the computations of Mahowald and Tangora (1967) that h_4^2 is an infinite cycle, so that $\theta_4 \in \pi_{30}(S)$ exists. It was proved by Barratt, Jones and Mahowald (1984) that h_5^2 is an infinite cycle, so that $\theta_5 \in \pi_{62}(S)$ exists. It is an open problem whether $\theta_6 \in \pi_{126}(S)$ exists. Hill, Hopkins and Ravenel (2009, to appear) showed that θ_j does not exist for $j \ge 7$.

Mahowald (Topology, 1977) proved that the η_j exist (so that h_1h_j is an infinite cycle) for all $j \ge 3$.

It is known (Mahowald and Tangora (1967), plus later calculations) that the only other classes in filtration $s = 2$ that survive to the E_∞-term are h_0h_2, h_0h_3 and h_2h_4, representing 2ν, 2σ and ν^* in $\pi_*(S)$.

Theorem 14.14. *[(a)]*

(1) $\pi_8(S)^\wedge_2 \cong (\mathbb{Z}/2)^2$ *is generated by* $\eta\sigma$ *and* ϵ*, represented by* $h_1h_3 \in E_\infty^{2,10}$ *and* $c_0 \in E_\infty^{3,11}$*, respectively.*

(2) $\pi_9(S)^\wedge_2 \cong (\mathbb{Z}/2)^3$ *is generated by* $\eta^2\sigma$*,* $\eta\epsilon$ *and* μ*, represented by* $h_1^2h_3 \in E_\infty^{3,12}$*,* $h_1c_0 \in E_\infty^{4,13}$ *and* $Ph_1 \in E_\infty^{5,14}$*, respectively.*

(3) $\pi_{10}(S)^\wedge_2 \cong \mathbb{Z}/2$ *is generated by* $\eta\mu$*, represented by* $h_1Ph_1 \in E_\infty^{6,16}$*.*

(4) $\pi_{11}(S)^\wedge_2 \cong \mathbb{Z}/8$ *is generated by* ζ*, represented by* $Ph_2 \in E_\infty^{5,16}$*. The class* 2ζ *is represented by* $h_0Ph_2 \in E_\infty^{6,17}$*, and the class* $4\zeta = \eta^2\rho$ *is represented by* $h_0^2Ph_2 = h_1^2Ph_1 \in E_\infty^{7,18}$*.*

(5) $\pi_{12}(S)^\wedge_2 = 0$*.*

(6) $\pi_{13}(S)^\wedge_2 = 0$*.*

(7) $\pi_{14}(S)^\wedge_2 \cong (\mathbb{Z}/2)^2$ *is generated by* σ^2 *and* κ*, represented by* $h_3^2 \in E_\infty^{2,16}$ *and* $d_0 \in E_\infty^{4,18}$*, respectively.*

(8) $\pi_{15}(S)^\wedge_2 \cong \mathbb{Z}/32 \oplus \mathbb{Z}/2$ *is generated by* ρ *and* $\eta\kappa$*, represented by* $h_0^3h_3 \in E_\infty^{4,19}$ *and* $h_1d_0 \in E_\infty^{5,20}$*, respectively. The classes* $2^k\rho$ *are represented by* $h_0^{k+3}h_3 \in E_\infty^{k+4,k+19}$ *for* $0 \le k \le 4$*.*

(9) $\pi_{16}(S)^\wedge_2 \cong (\mathbb{Z}/2)^2$ *is generated by* $\eta^* = \eta_4$ *and* $\eta\rho$*, represented by* $h_1h_4 \in E_\infty^{2,18}$ *and* $Pc_0 \in E_\infty^{7,23}$*, respectively. ((Note the filtration shift in* $\eta \cdot \rho$*.))*

(10) $\pi_{17}(S)^\wedge_2 \cong (\mathbb{Z}/2)^4$ *is generated by* $\eta\eta^*$*,* $\nu\kappa$*,* $\eta^2\rho$ *and* $\bar\mu = \mu_{17}$*, represented by* $h_1^2h_4 \in E_\infty^{3,20}$*,* $h_2d_0 \in E_\infty^{5,22}$*,* $h_1Pc_0 \in E_\infty^{7,24}$ *and* $P^2h_1 \in E_\infty^{9,26}$*, respectively.*

(11) $\pi_{18}(S)^\wedge_2 \cong \mathbb{Z}/8 \oplus \mathbb{Z}/2$ *is generated by* ν^* *and* $\eta\bar\mu = \mu_{18}$*, represented by* $h_2h_4 \in E_\infty^{2,20}$ *and* $h_1P^2h_1 \in E_\infty^{10,28}$*, respectively.*

(12) $\pi_{19}(S)^\wedge_2 \cong \mathbb{Z}/2 \oplus \mathbb{Z}/8$ *is generated by* $\bar\sigma$ *and* $\bar\zeta$*, represented by* $c_1 \in E_\infty^{3,22}$ *and* $P^2h_2 \in E_\infty^{9,28}$*, respectively.*

(13) $\pi_{20}(S)^\wedge_2 \cong \mathbb{Z}/8$ *is generated by* $\bar\kappa$*, represented by* $g \in E_\infty^{4,24}$*. The class* $2\bar\kappa$ *is represented by* $h_0g \in E_\infty^{5,25}$*, and the class* $4\bar\kappa = \nu^2\kappa$ *is represented by* $h_0^2g = h_2^2d_0 \in E_\infty^{6,26}$*.*

(14) $\pi_{21}(S)^\wedge_2 \cong (\mathbb{Z}/2)^2$ *((?)) is generated by* $\nu\nu^*$ *and* $\eta\bar\kappa$*, represented by* $h_2^2h_4 \in E_\infty^{3,24}$ *and* $h_1g \in E_\infty^{5,26}$*, respectively.*

(15) $\pi_{22}(S)^\wedge_2 \cong (\mathbb{Z}/2)^2$ *((?)) is generated by* $\nu\bar\sigma$ *and* $\eta^2\bar\kappa$*, represented by* $h_2c_1 \in E_\infty^{4,26}$ *and* $Pd_0 \in E_\infty^{8,30}$*, respectively. ((Note the filtration shift in* $\eta \cdot \eta\bar\kappa$*.))*

((Discuss additive splittings, by $2\eta = 0$ and associativity, and multiplicative extensions.))

Remark 14.15. There are Steenrod operations Sq^i in $E_2^{*,*} = \mathrm{Ext}_{\mathscr{A}}^{*,*}(\mathbb{F}_2, \mathbb{F}_2)$, taking $E_2^{s,t}$ to $E_2^{s+i,2t}$. In particular $Sq^0 \colon E_2^{s,t} \to E_2^{s,2t}$ is multiplicative, and maps h_i to h_{i+1} for $i \ge 0$. A sequence of elements

$$x, Sq^0(x), Sq^0(Sq^0(x)), \dots$$

is called a Sq^0-family. In the Sq^0-family $h_0, h_1, h_2, \dots$ the first four classes detect 2ι, η, ν and σ, but h_4 and all later terms are killed by the Adams differentials $d_2(h_i) = h_0h_{i-1}^2$ for $i \ge 4$.

In the Sq^0-family $h_0^2, h_1^2, h_2^2, \dots$ the first six classes detect 4ι, η^2, ν^2, σ^2, θ_4 and θ_5, but h_7^2 and all later terms are killed by (unknown) differentials. The status of h_6^2 is unknown. In the family $h_0h_2, h_1h_3, h_2h_4, \dots$ the first three classes detect 2ν, $\eta\sigma$ and ν^*, but h_3h_5 and all later terms support differentials. In the family $h_0h_3, h_1h_4, h_2h_5, \dots$ the first two classes detect 2σ and η^*, but h_2h_5 and all later terms support differentials. For each $i \ge 4$, only the term h_1h_{i+1} survives in the family $h_0h_i, h_1h_{i+1}, h_2h_{i+2}, \dots$, detecting η_{i+1}. The classes $c_0, c_1, c_2, \dots$ also form a Sq^0-family. The first two classes detect ϵ and $\bar\sigma$, but there are differentials $d_2(c_i) = h_0f_{i-1}$ for $i \ge 2$.

These results leads to the conjecture, called the "New Doomsday Conjecture" by Minami, and the "Finiteness Conjecture" by Bruner, saying that only a finite number of terms in each Sq^0-family detects nonzero homotopy classes. ((References?))

14.3. **Adams vanishing.**

Lemma 14.16 (Change of rings)**.** *Let* A *be any algebra, let* $B \subset A$ *be a subalgebra such that* A *is flat as a right* B*-module, let* M *be any left* B*-module and let* N *be any left* A*-module. There is a natural isomorphism*

$$\mathrm{Ext}_A^{s,t}(A \otimes_B M, N) \cong \mathrm{Ext}_B^{s,t}(M, N)\,.$$

Proof. Let $P_* \to M$ be a B-free resolution. Then $A \otimes_B P_* \to A \otimes_B M$ is an $\mathscr{A}$-free resolution. The isomorphism $\mathrm{Hom}_A(A \otimes_B P_*, N) \cong \mathrm{Hom}_B(P_*, N)$ induces the asserted isomorphism upon passage to cohomology. □

((TODO: Discuss compatibility of multiplicative structure(s) in Ext_A and Ext_B.))

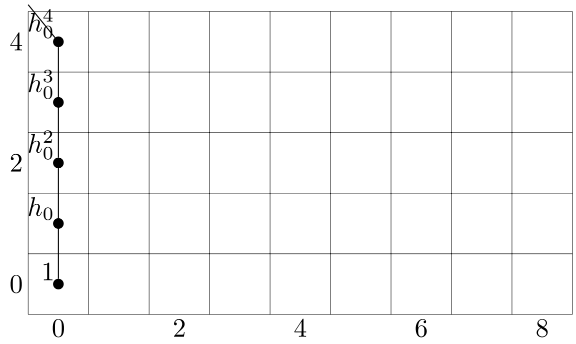


FIGURE 27. Adams spectral sequence for $H\mathbb{Z}$

Definition 14.17. Let A be an algebra and let $B \subset A$ be an augmented subalgebra, with augmentation ideal $I(B) = \ker(\epsilon)$. Let

$$A//B = A \otimes_B \mathbb{F}_2 \cong A/A \cdot I(B)\,.$$

If B is normal in A, meaning that $I(B) \cdot A = A \cdot I(B)$, then $A//B$ is a quotient algebra of A.

Recall that we write $P(x) = \mathbb{F}_2[x]$ and $E(x) = P(x)/(x^2)$ for the polynomial algebra and the exterior algebra, respectively, on a generator x. Let $A(0) = E(0) = E(Sq^1) \subset \mathscr{A}$ be the subalgebra generated by Sq^1. There are isomorphisms $H^*(H\mathbb{Z}) \cong \mathscr{A}/\mathscr{A}Sq^1 \cong \mathscr{A} \otimes_{A(0)} \mathbb{F}_2 = \mathscr{A}//A(0)$.

Proposition 14.18. *The Adams spectral sequence for $H\mathbb{Z}$ collapses at the E_2-term*

$$E_2^{*,*} = \mathrm{Ext}_{\mathscr{A}}^{*,*}(H^*(H\mathbb{Z}), \mathbb{F}_2) \cong \mathrm{Ext}_{A(0)}^{*,*}(\mathbb{F}_2, \mathbb{F}_2) \cong P(h_0)$$

where $h_0 \in E_2^{1,1}$, and converges strongly to $\pi_*(H\mathbb{Z}_2)$. The class of $2^s \in \pi_0(H\mathbb{Z}_2) = \mathbb{Z}_2$ is represented by $h_0^s \in E_\infty^{s,s}$, for each $s \ge 0$.

Proof. The Steenrod algebra $\mathscr{A}$ is free as a right $A(0)$-module, generated by the admissible monomials Sq^I for which $I = (i_1, \dots, i_\ell)$ and $i_\ell \ge 2$. (This includes the monomial $1 = Sq^{()}$.)

There is a minimal, free $A(0)$-module resolution P_* of $\mathbb{F}_2$ with $P_s = A(0)\{g_s\} = \mathbb{F}_2\{g_s, Sq^1 g_s\}$ for each $s \ge 0$, and $\partial_s(g_s) = Sq^1 g_{s-1}$ for each $s \ge 1$. Then $\mathrm{Ext}_{A(0)}^s(\mathbb{F}_2, \mathbb{F}_2) \cong \mathrm{Hom}_{A(0)}(P_s, \mathbb{F}_2) \cong \mathbb{F}_2\{\gamma_s\}$ is generated by the dual of g_s. It lifts to a chain map $\tilde\gamma_s \colon P_{*+s} \to P_*$ that takes g_{n+s} to g_n for each $n \ge 0$. These satisfy $\tilde\gamma_u \circ \tilde\gamma_s = \tilde\gamma_{u+s}$ under composition, so $\gamma_u \cdot \gamma_s = \gamma_{u+s}$ in the Yoneda product. Let $h_0 = \gamma_1$ be dual to g_1, in internal degree 1. Then $\gamma_s = h_0^s$ and we have proved that $\mathrm{Ext}_{A(0)}^*(\mathbb{F}_2, \mathbb{F}_2) \cong \mathbb{F}_2\{h_0^s \mid s \ge 0\} = P(h_0)$. □

The cofiber sequence

$$S \xrightarrow{\eta} H\mathbb{Z} \to \overline{H\mathbb{Z}}$$

induces a short exact sequence

$$0 \leftarrow \mathbb{F}_2 \xleftarrow{\eta^*} \mathscr{A}//A(0) \longleftarrow I(\mathscr{A}/\mathscr{A}Sq^1) \leftarrow 0$$

in cohomology, and a long exact sequence

$$\mathrm{Ext}_{\mathscr{A}}^{s-1,t}(I(\mathscr{A}/\mathscr{A}Sq^1), \mathbb{F}_2) \xrightarrow{\delta} \mathrm{Ext}_{\mathscr{A}}^{s,t}(\mathbb{F}_2, \mathbb{F}_2) \xrightarrow{\eta_*} \mathrm{Ext}_{A(0)}^{s,t}(\mathbb{F}_2, \mathbb{F}_2) \longrightarrow \mathrm{Ext}_{\mathscr{A}}^{s,t}(I(\mathscr{A}/\mathscr{A}Sq^1), \mathbb{F}_2)$$

of Adams E_2-terms. The map η_* is an isomorphism for $t - s = 0$, so the connecting homomorphism δ is an isomorphism for $t - s \ne 0$.

Lemma 14.19. *$I(\mathscr{A}/\mathscr{A}Sq^1)$ is free as a left $A(0)$-module, generated by the admissible Sq^I for which $I = (i_1, \dots, i_\ell)$, i_1 is even and $i_\ell \ge 2$. (This excludes the monomial $1 = Sq^{()}$.) The first few basis elements are*

$$Sq^2, Sq^4, Sq^6, Sq^4Sq^2, Sq^8, Sq^6Sq^2, Sq^6Sq^3, Sq^{10}, Sq^8Sq^2, Sq^8Sq^3, \dots\,.$$

Proof. When Sq^I ranges over the admissible monomials with i_1 even and $i_\ell \ge 2$, then Sq^I and Sq^1Sq^I range over the admissible monomials with $i_\ell \ge 2$. The only exception occurs for $I = ()$. □

Proposition 14.20. *Let M be an $\mathscr{A}$-module that is free as an $A(0)$-module, and concentrated in degrees $* \geq 0$. Let*

$$\epsilon(s) = \begin{cases} 0 & \text{for } s \equiv 0 \mod 4, \\ 1 & \text{for } s \equiv 1 \mod 4, \\ 2 & \text{for } s \equiv 2,3 \mod 4. \end{cases}$$

Then

$$Ext^{s,t}_{\mathscr{A}}(M, \mathbb{F}_2) = 0$$

for $t - s < 2s - \epsilon(s)$.

Proof. First consider the case $M = A(0)$, with the unique $\mathscr{A}$-module structure realized by $H^*(S/2)$. There is a minimal free $\mathscr{A}$-module resolution

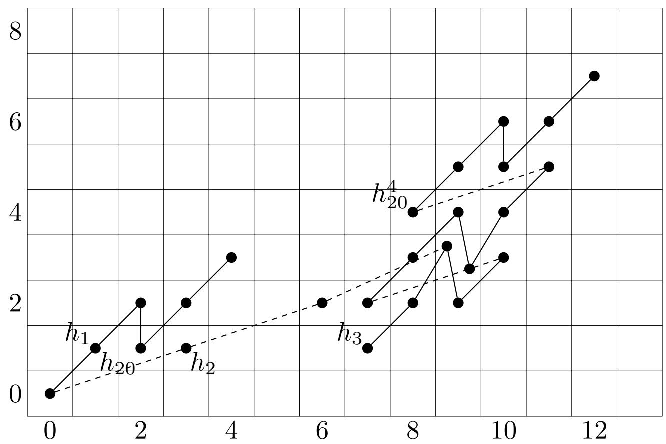
$$P_3 \xrightarrow{\partial_3} P_2 \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\epsilon} A(0) \to 0$$

with $P_0 = \mathscr{A}\{1\}$, P_1 concentrated in degrees $t \geq 2$, P_2 concentrated in degrees $t \geq 4$, P_3 concentrated in degrees $t \geq 7$, and $\Sigma^{12}K = \ker(\partial_3)$ concentrated in degrees $t \geq 12$.

This can be proved by direct calculation, or by using our previous Ext-calculations for the sphere spectrum, the cofiber sequence $S \xrightarrow{2} S \longrightarrow S/2 \longrightarrow \Sigma S = S^1$, the induced extension $0 \leftarrow \mathbb{F}_2 \leftarrow A(0) \leftarrow \Sigma\mathbb{F}_2 \leftarrow 0$ of $\mathscr{A}$-modules, and the associated long exact sequence

$$\cdots \to \mathrm{Ext}^{s-1,t-1}_{\mathscr{A}}(\mathbb{F}_2, \mathbb{F}_2) \xrightarrow{\delta} \mathrm{Ext}^{s,t}_{\mathscr{A}}(\mathbb{F}_2, \mathbb{F}_2) \longrightarrow \mathrm{Ext}^{s,t}_{\mathscr{A}}(A(0), \mathbb{F}_2) \longrightarrow \mathrm{Ext}^{s,t-1}_{\mathscr{A}}(\mathbb{F}_2, \mathbb{F}_2) \to \dots$$

in Ext. Here each connecting map δ is given by the Yoneda product with h_0, which is the class in $\mathrm{Ext}^{1,1}_{\mathscr{A}}(\mathbb{F}_2, \mathbb{F}_2)$ of the extension above. This leads to the additive structure of the following Adams chart for $\mathrm{Ext}^{*,*}_{\mathscr{A}}(A(0), \mathbb{F}_2)$:



This proves the claim for $M = A(0)$ and $0 \leq s < 4$.

Next, consider an extension $0 \to M' \to M \to M'' \to 0$ of $A(0)$-free $\mathscr{A}$-modules, all concentrated in degrees $* \geq 0$, and suppose that the claim holds for M' and M''. Then the claim follows for M, in view of the long exact sequence

$$\cdots \to \mathrm{Ext}^{s,t}_{\mathscr{A}}(M'', \mathbb{F}_2) \longrightarrow \mathrm{Ext}^{s,t}_{\mathscr{A}}(M, \mathbb{F}_2) \longrightarrow \mathrm{Ext}^{s,t}_{\mathscr{A}}(M', \mathbb{F}_2) \to \dots.$$

The claim for general $A(0)$-free M and $0 \leq s < 4$ then follows.

Since $A(0)$ and each P_s is $A(0)$-free, it follows that $\Sigma^{12}K = \ker(\partial_3)$ is $A(0)$-free, and concentrated in degrees $* \geq 12$. Thinking of P_{*+4} as a resolution of $\Sigma^{12}K$, we get an isomorphism

$$\mathrm{Ext}^{s,t}_{\mathscr{A}}(K, \mathbb{F}_2) \cong \mathrm{Ext}^{s+4,t+12}_{\mathscr{A}}(\mathbb{F}_2, \mathbb{F}_2)$$

for all $s \geq 0$. Hence the claim for $A(0)$ and $4 \leq s < 8$ follows from the one for K and $0 \leq s < 4$. The general claim for $A(0)$-free M and $4 \leq s < 8$ then follows as above. Continuing this way, the general claim follows for all $s \geq 0$. $\square$

Corollary 14.21. $\mathrm{Ext}^{s,t}_{\mathscr{A}}(\mathbb{F}_2, \mathbb{F}_2) = 0$ *for $0 < t - s < 2s - \epsilon$, where $\epsilon = 1$ for $s \equiv 1 \mod 4$, $\epsilon = 2$ for $s \equiv 2 \mod 4$ and $\epsilon = 3$ for $s \equiv 0, 3 \mod 4$.*

Proof. This follows from the isomorphisms

$$\mathrm{Ext}^{s,t}_{\mathscr{A}}(\mathbb{F}_2, \mathbb{F}_2) \cong \mathrm{Ext}^{s-1,t}_{\mathscr{A}}(I(\mathscr{A}/\mathscr{A}Sq^1), \mathbb{F}_2) \cong \mathrm{Ext}^{s-1,t-2}_{\mathscr{A}}(M, \mathbb{F}_2)$$

for $t - s > 0$, where $\Sigma^2 M = I(\mathscr{A}/\mathscr{A}Sq^1)$, and the proposition as applied to M. $\square$

This result is not quite optimal for $s \equiv 0 \mod 4$. Adams (1966) works a little harder to prove the optimal vanishing range:

Theorem 14.22 (Adams vanishing)**.** *$\mathrm{Ext}_{\mathscr{A}}^{s,t}(\mathbb{F}_2, \mathbb{F}_2) = 0$ for $0 < t - s < 2s - \epsilon$, where $\epsilon = 1$ for $s \equiv 0, 1 \mod 4$, $\epsilon = 2$ for $s \equiv 2 \mod 4$ and $\epsilon = 3$ for $s \equiv 3 \mod 4$.*

((ETC: Approximation for Ext over $A(n) \subset \mathscr{A}$.))

14.4. **Topological K-theory.**

Definition 14.23. Let ku and ko be the complex and real connective K-theory spectra, with underlying infinite loop spaces $\Omega^\infty ku = \mathbb{Z} \times BU$ and $\Omega^\infty ko = \mathbb{Z} \times BO$, respectively. These are the connective covers of the complex and real topological K-theory spectra, KU and KO, respectively.

Definition 14.24. Let bu and bsu be the 1- and 3-connected connected covers of ku, respectively, with $\Omega^\infty bu = BU$ and $\Omega^\infty bsu = BSU$. Let bo, bso and $bspin$ be the 0-, 1- and 3-connected covers of ko, respectively, with $\Omega^\infty bo = BO$, $\Omega^\infty bso = BSO$ and $\Omega^\infty bspin = BSpin$. We may also use the notations $u = \Sigma^{-1} bu$, $su = \Sigma^{-1} bsu$, $o = \Sigma^{-1} bo$, $so = \Sigma^{-1} bso$ and $spin = \Sigma^{-1} bspin$, for the desuspended spectra with infinite loop spaces U, SU, O, SO and $Spin$, respectively.

Remark 14.25. This is the notation used by Adams and May. Mahowald and Ravenel write bu and bo for "our" ku and ko.

Definition 14.26. Let $Q_1 = [Sq^1, Sq^2] = Sq^3 + Sq^2 Sq^1$. Let $E(1) = E(Sq^1, Q_1) \subset \mathscr{A}$ be the subalgebra of $\mathscr{A}$ generated by Sq^1 and Q_1, and let $A(1) = \langle Sq^1, Sq^2 \rangle \subset \mathscr{A}$ be the subalgebra generated by Sq^1 and Sq^2. Here is an additive basis for $A(1)$, with the action by Sq^1 and Sq^2 indicated by arrows:

$$\begin{tikzcd}
1 \arrow[rr] \arrow[dr] & & Sq^2 \arrow[r] \arrow[drr] & Sq^3 \arrow[rr] & & Sq^2Sq^3 \arrow[dr] & \\
& Sq^1 \arrow[rr] & & Sq^2Sq^1 \arrow[r] & Sq^3Sq^1 \arrow[rr] & & Sq^1Sq^5
\end{tikzcd}$$

For typographical reasons, we write $Sq^2 Sq^3$ in place of its admissible expansion $Sq^5 + Sq^4 Sq^1$. Note that $E(1)//A(0) \cong E(Q_1)$, $A(1)//E(Q_1) \cong E(Sq^1, Sq^2)$ and $A(1)//E(1) \cong E(Sq^2)$.

Proposition 14.27 (Stong)**.** *There are $\mathscr{A}$-module isomorphisms*

$$H^*(ku) \cong \mathscr{A}//E(1) = \mathscr{A}/\mathscr{A}\{Sq^1, Q_1\} = \mathscr{A}/\mathscr{A}\{Sq^1, Sq^3\}$$

and

$$H^*(ko) \cong \mathscr{A}//A(1) = \mathscr{A}/\mathscr{A}\{Sq^1, Sq^2\}\,.$$

Proof. By complex Bott periodicity, there is a cofiber sequence

$$\Sigma^2 ku \xrightarrow{\beta} ku \to H\mathbb{Z} \to \Sigma^3 ku\,.$$

Here $\Sigma^2 ku = bu$ is the connected cover of ku. The left hand map is a composite

$$\Sigma^2 ku = ku \wedge S^2 \xrightarrow{1 \wedge u} ku \wedge ku \xrightarrow{\phi} ku$$

where $u \in \pi_2(ku)$ is a generator and ϕ is the ring spectrum product. It is known that the mod 2 Hurewicz image of u is zero, so $\beta^* = 0$, and there is a short exact sequence of $\mathscr{A}$-modules

$$0 \leftarrow H^*(ku) \leftarrow H^*(H\mathbb{Z}) \leftarrow \Sigma^3 H^*(ku) \leftarrow 0\,.$$

The short exact sequence of $E(1)$-modules

$$0 \leftarrow \mathbb{F}_2 \leftarrow E(1)//A(0) \leftarrow \Sigma^3 \mathbb{F}_2 \leftarrow 0$$

can be induced up to a short exact sequence

$$0 \leftarrow \mathscr{A}//E(1) \leftarrow \mathscr{A}//A(0) \leftarrow \Sigma^3 \mathscr{A}//E(1) \leftarrow 0\,,$$

since $\mathscr{A}$ is free as a right $E(1)$-module.

The composite $H\mathbb{Z} \to \Sigma^3 ku \to \Sigma^3 H\mathbb{Z}$ is known to take $\Sigma^3 1$ to Q_1 in cohomology, so $ku \to H\mathbb{Z}$ takes Q_1 to 0 in cohomology. Hence there is a map of short exact sequences

$$\begin{tikzcd} 0 & \mathscr{A}//E(1) \arrow[l] \arrow[d] & \mathscr{A}//A(0) \arrow[l] \arrow[d, "\cong"'] & \Sigma^3\mathscr{A}//E(1) \arrow[l] \arrow[d] & 0 \arrow[l] \\ 0 & H^*(ku) \arrow[l] & H^*(H\mathbb{Z}) \arrow[l] & \Sigma^3 H^*(ku) \arrow[l] & 0 \arrow[l] \end{tikzcd}$$

We know that the middle map is an isomorphism, and the right hand map is the triple suspension of the left hand map. It follows by induction on the internal degree that the latter two maps are also isomorphisms.

By real Bott periodicity, there is a cofiber sequence

$$\Sigma ko \xrightarrow{\eta} ko \to ku \to \Sigma^2 ko\,.$$

The left hand map is a composite

$$\Sigma ko = ko \wedge S^1 \xrightarrow{1\wedge\eta} ko \wedge ko \xrightarrow{\phi} ko$$

where $\eta \in \pi_1(ko)$ is the image of $\eta \in \pi_1(S)$, and ϕ is the ring spectrum product. The mod 2 Hurewicz image of η is zero, so $\eta^* = 0$, and there is a short exact sequence of $\mathscr{A}$-modules

$$0 \leftarrow H^*(ko) \leftarrow H^*(ku) \leftarrow \Sigma^2 H^*(ko) \leftarrow 0\,.$$

The short exact sequence of $A(1)$-modules

$$0 \leftarrow \mathbb{F}_2 \leftarrow A(1)//E(1) \leftarrow \Sigma^2\mathbb{F}_2 \leftarrow 0$$

can be induced up to a short exact sequence

$$0 \leftarrow \mathscr{A}//A(1) \leftarrow \mathscr{A}//E(1) \leftarrow \Sigma^2\mathscr{A}//A(1) \leftarrow 0\,,$$

since $\mathscr{A}$ is free as a right $A(1)$-module.

The composite $ku \to \Sigma^2 ko \to \Sigma^2 ku$ takes $\Sigma^2 1$ to Sq^2 in cohomology, so $ko \to ku$ takes Sq^2 to 0 in cohomology. Hence there is a map of short exact sequences

$$\begin{tikzcd} 0 & \mathscr{A}//A(1) \arrow[l] \arrow[d] & \mathscr{A}//E(1) \arrow[l] \arrow[d, "\cong"'] & \Sigma^2\mathscr{A}//A(1) \arrow[l] \arrow[d] & 0 \arrow[l] \\ 0 & H^*(ko) \arrow[l] & H^*(ku) \arrow[l] & \Sigma^2 H^*(ko) \arrow[l] & 0 \arrow[l] \end{tikzcd}$$

We know that the middle map is an isomorphism, and the right hand map is the double suspension of the left hand map. It follows by induction on the internal degree that the latter two maps are also isomorphisms. □

Proposition 14.28. *The Adams spectral sequence for ku collapses at the E_2-term*

$$E_2^{*,*} = \mathrm{Ext}_{\mathscr{A}}^{*,*}(H^*(ku), \mathbb{F}_2) \cong \mathrm{Ext}_{E(1)}^{*,*}(\mathbb{F}_2, \mathbb{F}_2) \cong P(h_0, h_{20})$$

where $h_0 \in E_2^{1,1}$ and $h_{20} \in E_2^{1,3}$, and converges strongly to $\pi_*(ku_2^\wedge) = \mathbb{Z}_2[u]$. The class of $2 \in \pi_0(ku_2^\wedge)$ is represented by h_0, and the class of $u \in \pi_2(ku_2^\wedge)$ is represented by h_{20}.

Proof. We use the change of rings isomorphism $\mathrm{Ext}_{\mathscr{A}}^{*,*}(\mathscr{A}//E(1), \mathbb{F}_2) \cong \mathrm{Ext}_{E(1)}^{*,*}(\mathbb{F}_2, \mathbb{F}_2)$. ((Must justify that $\mathscr{A}$ is right free, thus flat, over $E(1)$.)) There is a Künneth isomorphism

$$\mathrm{Ext}_{E(1)}^{*,*}(\mathbb{F}_2, \mathbb{F}_2) \cong \mathrm{Ext}_{E(Sq^1)}^{*,*}(\mathbb{F}_2, \mathbb{F}_2) \otimes \mathrm{Ext}_{E(Q_1)}^{*,*}(\mathbb{F}_2, \mathbb{F}_2)$$

and $\mathrm{Ext}_{E(Q_1)}^{*,*}(\mathbb{F}_2, \mathbb{F}_2) \cong P(h_{20})$ with h_{20} dual to Q_1, by the same argument we used to show that $\mathrm{Ext}_{E(1)}^{*,*}(\mathbb{F}_2, \mathbb{F}_2) \cong P(h_0)$ with h_0 dual to Sq^1. (Another name for h_{20} is v_1.) The spectral sequence is concentrated in even columns, hence collapses for bidegree reasons. □

Proposition 14.29. *The Adams spectral sequence for ko collapses at the E_2-term*

$$\begin{aligned} E_2^{*,*} &= \mathrm{Ext}_{\mathscr{A}}^{*,*}(H^*(ko), \mathbb{F}_2) \cong \mathrm{Ext}_{A(1)}^{*,*}(\mathbb{F}_2, \mathbb{F}_2) \\ &\cong P(h_0, h_1, v, w_1)/(h_0h_1, h_1^3, h_1 v, v^2 = h_0^2 w_1) \end{aligned}$$

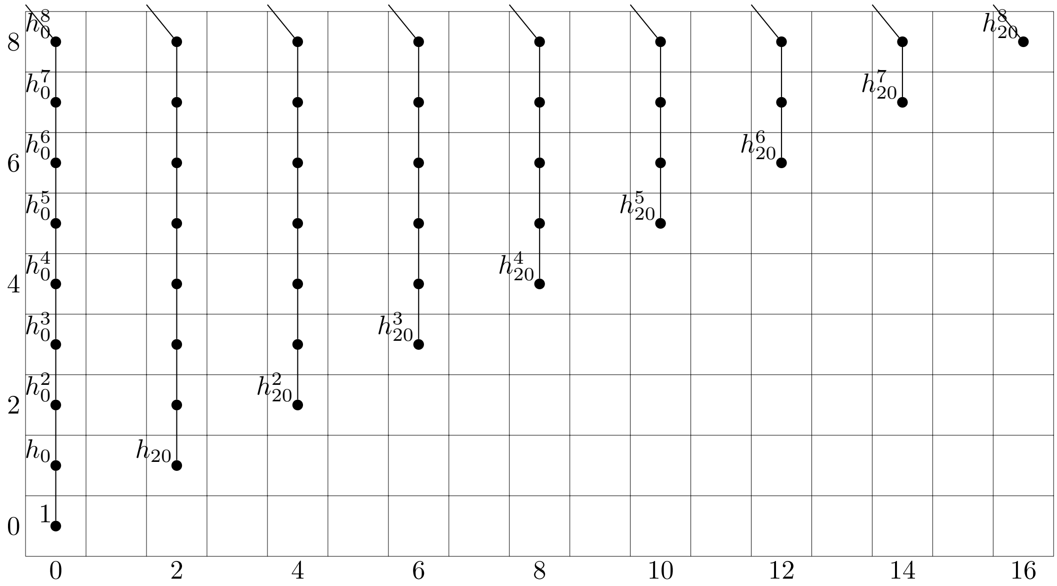


FIGURE 28. Adams spectral sequence for ku

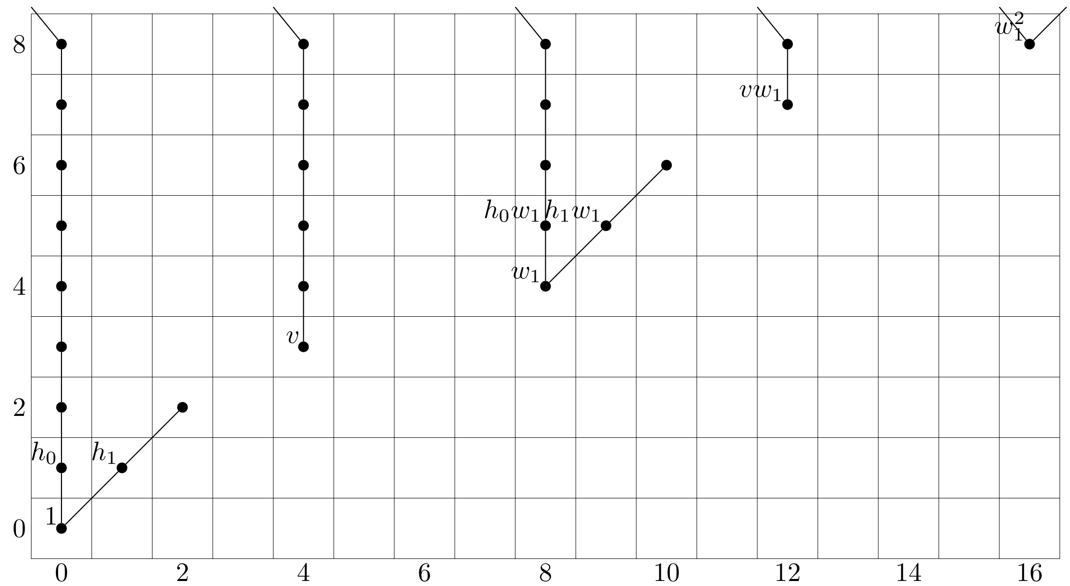


FIGURE 29. Adams spectral sequence for ko

where $h_0 \in E_2^{1,1}$, $h_1 \in E_2^{1,2}$, $v \in E_2^{3,7}$ *and* $w_1 \in E_2^{4,12}$, *and converges strongly to*

$$\pi_*(ko_2^\wedge) = \mathbb{Z}_2[\eta, \alpha, \beta]/(2\eta, \eta^3, \eta\alpha, \alpha^2 = 4\beta)\,.$$

The classes 2, η, α *and* β *are represented by* h_0, h_1, v *and* w_1, *respectively.*

Hence

$$\pi_n(ku_2^\wedge) \cong \begin{cases} \mathbb{Z}_2\{u^i\} & \text{for } n = 2i \\ 0 & \text{otherwise} \end{cases}$$

and

$$\pi_n(ko_2^\wedge) \cong \begin{cases} \mathbb{Z}_2\{\beta^i\} & \text{for } n = 8i \\ \mathbb{Z}/2\{\eta\beta^i\} & \text{for } n = 8i+1 \\ \mathbb{Z}/2\{\eta^2\beta^i\} & \text{for } n = 8i+2 \\ \mathbb{Z}_2\{\alpha\beta^i\} & \text{for } n = 8i+4 \\ 0 & \text{otherwise} \end{cases}$$

for $n \geq 0$.

((The complexification map $c\colon ko \to ku$ induces $h_0 \mapsto h_0$, $h_1 \mapsto 0$, $v \mapsto h_0 h_{20}^2$ and $w_1 \mapsto h_{20}^4$ in Ext, and similarly in homotopy.))

Remark 14.30. To compute $\mathrm{Ext}_{A(1)}^{*,*}(\mathbb{F}_2, \mathbb{F}_2)$, we can use the Cartan–Eilenberg spectral sequence (1956, Theorem XVI.6.1). If A is a connected graded algebra, $B \subset A$ is a normal subalgebra, and A is projective as a right B-module, then this is an algebra spectral sequence

$$E_2^{p,q} = \mathrm{Ext}_{A//B}^p(\mathbb{F}_2, \mathrm{Ext}_B^q(\mathbb{F}_2, \mathbb{F}_2)) \Longrightarrow \mathrm{Ext}_A^{p+q}(\mathbb{F}_2, \mathbb{F}_2)$$

of cohomological type. In the special case when $A = \mathbb{F}_2[G]$ is a group algebra, and $B = \mathbb{F}_2[N]$ is the group algebra of a normal subgroup, we have $B//A = \mathbb{F}_2[G/N]$ and the Cartan–Eilenberg spectral sequence agrees with the Lyndon–Hochschild–Serre spectral sequence

$$E_2^{p,q} = H_{gp}^p(G/N; H_{gp}^q(N; \mathbb{F}_2)) \Longrightarrow H_{gp}^{p+q}(G; \mathbb{F}_2)\,.$$

This is again a special case of the Serre spectral sequence in mod 2 singular cohomology, for the fibration $BN \to BG \to B(G/N)$.

First proof. We use the change of rings isomorphism $\mathrm{Ext}_{\mathscr{A}}^{*,*}(\mathscr{A}//A(1), \mathbb{F}_2) \cong \mathrm{Ext}_{A(1)}^{*,*}(\mathbb{F}_2, \mathbb{F}_2)$. ((Must justify that $\mathscr{A}$ is right free, thus flat, over $A(1)$.)) The subalgebra $E(Q_1) \subset A(1)$ is normal, with quotient $A(1)//E(Q_1) \cong E(Sq^1, Sq^2)$. Hence there is a Cartan–Eilenberg spectral sequence

$$E_2^{*,*} = \mathrm{Ext}_{E(Sq^1,Sq^2)}^*(\mathbb{F}_2, \mathrm{Ext}_{E(Q_1)}^*(\mathbb{F}_2, \mathbb{F}_2)) \Longrightarrow \mathrm{Ext}_{A(1)}^*(\mathbb{F}_2, \mathbb{F}_2)\,.$$

Here $\mathrm{Ext}_{E(Q_1)}^*(\mathbb{F}_2, \mathbb{F}_2) \cong P(h_{20})$. The module action of $E(Sq^1, Sq^2)$ on $P(h_{20})$ is (necessarily) trivial, so

$$E_2^{*,*} \cong P(h_0, h_1) \otimes P(h_{20})$$

with $h_0 \in E_2^{1,0}$ dual to Sq^1, $h_1 \in E_2^{1,0}$ dual to Sq^2, and $h_{20} \in E_2^{0,1}$ dual to Q_1. (We are ignoring the internal degrees here.) There is a d_2-differential $d_2(h_{20}) = h_0 h_1$, corresponding to the fact that the generator $Q_1 \in E(Q_1)$ becomes decomposable in $A(1)$. This leaves the E_3-term

$$E_3^{*,*} \cong P(h_0, h_1)/(h_0 h_1) \otimes P(h_{20}^2)\,.$$

There is a further d_3-differential $d_2(h_{20}^2) = h_1^3$. This leaves the E_3-term

$$E_4^{*,*} \cong \big(P(h_0, h_1)/(h_0 h_1, h_1^3) \oplus P(h_0)\{h_0 h_{20}^2\}\big) \otimes P(h_{20}^4)\,.$$

The spectral sequence collapses at this stage, for bidegree reasons: A d_5-differential on h_{20}^4 could only hit h_0^5, but the internal degrees do not match. ((No additive or multiplicative extensions.)) $\square$

Second proof. One might also consider the Cartan–Eilenberg spectral sequence

$$E_2^{p,q} = \mathrm{Ext}_{E(Sq^2)}^p(\mathbb{F}_2, \mathrm{Ext}_{E(1)}^q(\mathbb{F}_2, \mathbb{F}_2)) \Longrightarrow \mathrm{Ext}_{A(1)}^{p+q}(\mathbb{F}_2, \mathbb{F}_2)$$

associated to the isomorphism $A(1)//E(1) \cong E(Sq^2)$, but in this case the $E(Sq^2)$-module action on $\mathrm{Ext}_{E(1)}^*(\mathbb{F}_2, \mathbb{F}_2) = P(h_0, h_{20})$ is non-trivial, being given by $Sq^2 \cdot h_{20} = h_0$. With the usual periodic resolution for Ext over $E(Sq^2)$, this gives a d_1-differential $d_1(h_{20}) = h_0 h_1$, so that

$$E_2^{*,*} = P(h_0, h_1)/(h_0 h_1) \otimes P(h_{20}^2)\,.$$

Again there is a d_3-differential $d_3(h_{20}^2) = h_1^3$, leaving

$$E_4^{*,*} = E_\infty^{*,*} = \big(P(h_0, h_1)/(h_0 h_1, h_1^3) \oplus P(h_0)\{h_0 h_{20}^2\}\big) \otimes P(h_{20}^4)\,.$$

Note that in this case h_0, h_1 and h_{20} have bigradings $(p, q) = (0, 1)$, $(1, 0)$ and $(0, 1)$, respectively. $\square$

Third proof. For a proof without the Cartan–Eilenberg spectra sequence, we may construct a minimal resolution of $\mathbb{F}_2$ by "almost free" $A(1)$-modules. Some interesting examples of indecomposable modules appear along the way. There is an exact sequence

$$0 \to \Sigma^{12}\mathbb{F}_2 \to \Sigma^7 A(1)//A(0) \xrightarrow{\partial_3} \Sigma^4 A(1) \xrightarrow{\partial_2} \Sigma^2 A(1) \xrightarrow{\partial_1} A(1)//A(0) \xrightarrow{\epsilon} \mathbb{F}_2 \to 0$$

of $A(1)$-modules. The kernel of the augmentation ϵ from $A(1)//A(0) = A(1)/A(1)Sq^1$:

$$1 \longrightarrow Sq^2 \longrightarrow Sq^3 \longrightarrow Sq^2Sq^3$$

is the "question mark module"

$$\begin{tikzcd} Sq^2 \arrow[r] & Sq^3 \arrow[rr, bend right] & & Sq^2Sq^3 \end{tikzcd}$$

which is isomorphic to $\Sigma^2(A(1)/A(1)Sq^2)$. Here $1 \otimes \epsilon \colon \mathscr{A}//A(0) \to \mathscr{A}//A(1)$ is induced by the zeroth Postnikov section $ko \to H\mathbb{Z}$, with homotopy fiber bo, so $\Sigma H^*(bo) \cong \mathscr{A} \otimes_{A(1)} \ker(\epsilon)$ and $H^*(bo) \cong \Sigma(\mathscr{A}/\mathscr{A}Sq^2)$.

The kernel of $\partial_1 \colon \Sigma^2 A(1) \to \ker(\epsilon)$, taking $\Sigma^2 1$ to Sq^2, is the double suspension of the "joker module"

$$\begin{tikzcd} Sq^2 \arrow[r] \arrow[rr, bend left] & Sq^3 \arrow[rr, bend right] & Sq^3Sq^1 \arrow[rr, bend left] & Sq^2Sq^3 \arrow[r] & Sq^1Sq^5 \end{tikzcd}$$

which is isomorphic to $\Sigma^4(\mathscr{A}/\mathscr{A}Sq^3)$. Here $1 \otimes \partial_1 \colon \Sigma^2\mathscr{A} \to \mathscr{A} \otimes_{A(1)} \ker(\epsilon)$ is induced by the Postnikov section $bo \to \Sigma H$, with homotopy fiber bso, so $\Sigma^2 H^*(bso) \cong \mathscr{A} \otimes_{A(1)} \ker(\partial_1)$ and $H^*(bso) \cong \Sigma^2(\mathscr{A}/\mathscr{A}Sq^3)$.

The kernel of $\partial_2 \colon \Sigma^4 A(1) \to \ker(\partial_1)$, taking $\Sigma^4 1$ to $\Sigma^2 Sq^2$, is the fourfold suspension of the "inverted question mark module"

$$\begin{tikzcd} Sq^3 \arrow[r, bend left] & Sq^2Sq^3 \arrow[r] & Sq^1Sq^5 \end{tikzcd}$$

which is isomorphic to $\Sigma^3(\mathscr{A}/\mathscr{A}\{Sq^1, Sq^2Sq^3\})$. Here $1 \otimes \partial_2 \colon \Sigma^4\mathscr{A} \to \mathscr{A} \otimes_{A(1)} \ker(\partial_1)$ is induced by the Postnikov section $bso \to \Sigma^2 H$, with homotopy fiber $bspin \cong \Sigma^4 ksp$, so $\Sigma^3 H^*(bspin) \cong \mathscr{A} \otimes_{A(1)} \ker(\partial_2)$ and $H^*(bspin) \cong \Sigma^4(\mathscr{A}/\mathscr{A}\{Sq^1, Sq^2Sq^3\})$.

The kernel of $\partial_3 \colon \Sigma^7 A(1)//A(0) \to \ker(\partial_2)$, taking $\Sigma^7 1$ to $\Sigma^4 Sq^3$, is the sevenfold suspension of the trivial module

$$Sq^2Sq^3$$

which is isomorphic to $\Sigma^5\mathbb{F}_2$. Here $1 \otimes \partial_3 \colon \Sigma^7\mathscr{A}//A(0) \to \mathscr{A} \otimes_{A(1)} \ker(\partial_2)$ is induced by the Postnikov section $bspin \to \Sigma^4 H\mathbb{Z}$, with homotopy fiber $\Sigma^8 ko$, so $\Sigma^4 H^*(\Sigma^8 ko) \cong \mathscr{A} \otimes_{A(1)} \ker(\partial_3)$ and $H^*(\Sigma^8 ko) \cong \Sigma^8(\mathscr{A}//A(1))$, which we already knew.

From the exact sequence of $A(1)$-modules, we get short exact sequences

$$\begin{aligned}
0 \to \operatorname{Ext}^{s-1,t}_{A(1)}(\ker(\epsilon), \mathbb{F}_2) &\xrightarrow{\delta} \operatorname{Ext}^{s,t}_{A(1)}(\mathbb{F}_2, \mathbb{F}_2) \longrightarrow \operatorname{Ext}^{s,t}_{A(0)}(\mathbb{F}_2, \mathbb{F}_2) \to 0 \\
0 \to \operatorname{Ext}^{s-2,t}_{A(1)}(\ker(\partial_1), \mathbb{F}_2) &\xrightarrow{\delta} \operatorname{Ext}^{s-1,t}_{A(1)}(\ker(\epsilon), \mathbb{F}_2) \longrightarrow \operatorname{Ext}^{s-1,t}_{\mathbb{F}_2}(\Sigma^2\mathbb{F}_2, \mathbb{F}_2) \to 0 \\
0 \to \operatorname{Ext}^{s-3,t}_{A(1)}(\ker(\partial_2), \mathbb{F}_2) &\xrightarrow{\delta} \operatorname{Ext}^{s-2,t}_{A(1)}(\ker(\partial_1), \mathbb{F}_2) \longrightarrow \operatorname{Ext}^{s-2,t}_{\mathbb{F}_2}(\Sigma^4\mathbb{F}_2, \mathbb{F}_2) \to 0 \\
0 \to \operatorname{Ext}^{s-4,t}_{A(1)}(\Sigma^{12}\mathbb{F}_2, \mathbb{F}_2) &\xrightarrow{\delta} \operatorname{Ext}^{s-3,t}_{A(1)}(\ker(\partial_2), \mathbb{F}_2) \longrightarrow \operatorname{Ext}^{s-3,t}_{A(0)}(\Sigma^7\mathbb{F}_2, \mathbb{F}_2) \to 0
\end{aligned}$$

This determines $\operatorname{Ext}^{*,*}_{A(1)}(\mathbb{F}_2, \mathbb{F}_2)$. □

Corollary 14.31. *There are $\mathscr{A}$-module isomorphisms:*

$$\begin{aligned}
H^*(bo) &\cong \Sigma(\mathscr{A}/\mathscr{A}Sq^2) \\
H^*(bso) &\cong \Sigma^2(\mathscr{A}/\mathscr{A}Sq^3) \\
H^*(bspin) &\cong \Sigma^4(\mathscr{A}/\mathscr{A}\{Sq^1, Sq^2Sq^3\})
\end{aligned}$$

((Also $k(1) = ku/2$, $ko/2$.))

15. The dual Steenrod algebra

15.1. **Hopf algebras.** Let G be a topological group with $H_*(G)$ of finite type. Then the cohomology cross product

$$H^*(G) \otimes H^*(G) \xrightarrow{\times} H^*(G \times G)$$

is an isomorphism. The (cocommutative) diagonal map $\Delta\colon G \to G \times G$, and the augmentation $G \to *$ induce a pairing

$$\phi\colon H^*(G) \otimes H^*(G) \cong H^*(G \times G) \xrightarrow{\Delta^*} H^*(G)$$

and a unit map

$$\eta\colon \mathbb{F}_p \longrightarrow H^*(G)$$

that make $H^*(G)$ a (graded commutative) algebra. The group multiplication $m\colon G \times G \to G$ and the inclusion $\{e\} \to G$ induce homomorphisms

$$\psi\colon H^*(G) \xrightarrow{m^*} H^*(G \times G) \cong H^*(G) \otimes H^*(G)$$

and

$$\epsilon\colon H^*(G) \longrightarrow \mathbb{F}_p$$

that make $H^*(G)$ a commutative Hopf algebra, and the group inverse $i\colon G \to G$ induces a homomorphism

$$\chi\colon H^*(G) \xrightarrow{i^*} H^*(G)$$

that makes $H^*(G)$ a commutative Hopf algebra with conjugation, according to the following definitions. It is connected if and only if G is path connected as a topological space.

Dually, the Pontryagin product $\phi = m_*\colon H_*(G) \otimes H_*(G) \to H_*(G)$, unit inclusion $\eta\colon \mathbb{F}_p \to H_*(G)$, diagonal coproduct $\psi = \Delta_*\colon H_*(G) \to H_*(G) \otimes H_*(G)$, augmentation $\epsilon\colon H_*(G) \to \mathbb{F}_p$ and conjugation $\chi = i_*\colon H_*(G) \to H_*(G)$ make $H_*(G)$ a cocommutative Hopf algebra with conjugation.

Let k be any field, and write $\otimes$ for $\otimes_k$.

Definition 15.1. A k-algebra is a graded k-module A equipped with homomorphisms $\phi\colon A \otimes A \to A$ and $\eta\colon k \to A$, such that the diagrams

$$\begin{tikzcd} A \otimes A \otimes A \arrow[r, "\phi\otimes 1"] \arrow[d, "1\otimes\phi"'] & A \otimes A \arrow[d, "\phi"] \\ A \otimes A \arrow[r, "\phi"] & A \end{tikzcd}$$

(associativity) and

$$\begin{tikzcd} k \otimes A \arrow[r, "\eta\otimes 1"] \arrow[dr, "\cong"'] & A \otimes A \arrow[d, "\phi"] & A \otimes k \arrow[l, "1\otimes\eta"'] \arrow[dl, "\cong"] \\ & A & \end{tikzcd}$$

(unitality) commute. It is commutative if the diagram

$$\begin{tikzcd} A \otimes A \arrow[rr, "\gamma", "\cong"'] \arrow[dr, "\phi"'] & & A \otimes A \arrow[dl, "\phi"] \\ & A & \end{tikzcd}$$

commutes, where $\gamma(a \otimes b) = (-1)^{|a||b|} b \otimes a$. A k-algebra homomorphism $f\colon A \to B$ is a degree-preserving k-module homomorphism such that the diagram

$$\begin{tikzcd} A \otimes A \arrow[r, "\phi"] \arrow[d, "f\otimes f"'] & A \arrow[d, "f"'] & k \arrow[l, "\eta"'] \arrow[d, "="] \\ B \otimes B \arrow[r, "\phi"] & B & k \arrow[l, "\eta"'] \end{tikzcd}$$

commutes.

Definition 15.2. A k-coalgebra is a graded k-module A equipped with homomorphisms $\psi\colon A \to A \otimes A$ and $\epsilon\colon A \to k$, such that the diagrams

$$\begin{tikzcd} A \arrow[r, "\psi"] \arrow[d, "\psi"'] & A \otimes A \arrow[d, "1\otimes\psi"] \\ A \otimes A \arrow[r, "\psi\otimes 1"] & A \otimes A \otimes A \end{tikzcd}$$

(coassociativity) and

$$\begin{tikzcd} & A \arrow[dl, "\cong"'] \arrow[d, "\psi"] \arrow[dr, "\cong"] & \\ k \otimes A & A \otimes A \arrow[l, "\epsilon\otimes 1"] \arrow[r, "1\otimes\epsilon"'] & A \otimes k \end{tikzcd}$$

(counitality) commute. It is cocommutative if the diagram

$$\begin{tikzcd} & A \arrow[dl, "\psi"'] \arrow[dr, "\psi"] & \\ A \otimes A \arrow[rr, "\gamma", "\cong"'] & & A \otimes A \end{tikzcd}$$

commutes. A k-coalgebra homomorphism $f\colon A \to B$ is a degree-preserving k-module homomorphism such that the diagram

$$\begin{tikzcd} k \arrow[d, "="'] & A \arrow[l, "\epsilon"'] \arrow[r, "\psi"] \arrow[d, "f"'] & A \otimes A \arrow[d, "f\otimes f"] \\ k & B \arrow[l, "\epsilon"'] \arrow[r, "\psi"] & B \otimes B \end{tikzcd}$$

commutes.

Definition 15.3. A k-algebra A is connected if the underlying graded k-module is zero in negative degrees and $\eta\colon k \to A$ is an isomorphism in degree 0. A k-coalgebra A is connected if it is zero in negative degrees and $\epsilon\colon A \to k$ is an isomorphism in degree 0.

Definition 15.4. An augmented k-algebra is a k-algebra A with a k-algebra homomorphism $\epsilon\colon A \to k$. Let $I(A) = \ker(\epsilon)$ be the augmentation ideal, and let

$$Q(A) = I(A)/I(A)^2 = k \otimes_A I(A)$$

be the indecomposable quotient module.

$$\begin{tikzcd} I(A) \otimes I(A) \arrow[r] \arrow[d] & I(A) \arrow[r, two heads] \arrow[d, tail] & Q(A) \\ A \otimes A \arrow[r, "\phi"] & A \arrow[d, two heads, "\epsilon"] & \\ & k & \end{tikzcd}$$

A homomorphism of augmented algebras is an algebra homomorphism that commutes with the augmentations.

(We make sense of the tensor product over A in the next subsection.)

Proposition 15.5 (Milnor–Moore)**.** *Let $f\colon A \to B$ be a homomorphism of augmented algebras, with B connected. Then f is surjective if and only if $Q(f)\colon Q(A) \to Q(B)$ is surjective.*

Definition 15.6. A coaugmented k-coalgebra is a k-coalgebra A with a k-coalgebra homomorphism $\eta\colon k \to A$. Let $J(A) = \mathrm{cok}(\eta)$ be the coaugmentation coideal, and let

$$P(A) = \{x \in A \mid \psi(x) = x \otimes 1 + 1 \otimes x\} = k \,\square_A\, J(A)$$

be the submodule of primitives.

$$\begin{tikzcd}
J(A) \otimes J(A) & J(A) \arrow[l] & P(A) \arrow[l, tail] \\
A \otimes A \arrow[u] & A \arrow[l, "\psi"'] \arrow[u, two heads] \\
 & k \arrow[u, tail, "\eta"']
\end{tikzcd}$$

A homomorphism of coaugmented coalgebras is a coalgebra homomorphism that commutes with the coaugmentations.

(We make sense of the cotensor products under A in the next subsection.)

Proposition 15.7 (Milnor–Moore)**.** *Let $f\colon A \to B$ be a homomorphism of coaugmented coalgebras, with A connected. Then f is injective if and only if $P(f)\colon P(A) \to P(B)$ is injective.*

Definition 15.8. A Hopf algebra (over k) is a k-algebra structure (ϕ, η) and a k-coalgebra structure (ψ, ϵ) on the same graded k-module A, such that ψ and ϵ are algebra homomorphisms and ϕ and η are coalgebra homomorphisms. This means that the diagrams

$$\begin{tikzcd}
A \otimes A \arrow[r, "\phi"] \arrow[d, "\psi\otimes\psi"'] & A \arrow[r, "\psi"] & A \otimes A \\
A \otimes A \otimes A \otimes A \arrow[rr, "\cong", "1\otimes\gamma\otimes1"'] & & A \otimes A \otimes A \otimes A \arrow[u, "\phi\otimes\phi"']
\end{tikzcd}$$

and

$$\begin{tikzcd}
A \otimes A \arrow[r, "\epsilon\otimes\epsilon"] \arrow[d, "\phi"'] & k \otimes k \arrow[d, "\cong"] \\
A \arrow[r, "\epsilon"] & k
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
k \arrow[r, "\eta"] \arrow[d, "\cong"'] & A \arrow[d, "\psi"] \\
k \otimes k \arrow[r, "\eta\otimes\eta"] & A \otimes A
\end{tikzcd}$$

commute. A homomorphism if Hopf algebras is an algebra homomorphism that is simultaneously a coalgebra homomorphism.

Definition 15.9. A Hopf algebra with conjugation is a Hopf algebra A with a homomorphism $\chi\colon A \to A$ such that the diagram

$$\begin{tikzcd}
A \arrow[r, "\epsilon"] \arrow[d, "\psi"'] & k \arrow[r, "\eta"] & A \\
A \otimes A \arrow[rr, "1\otimes\chi"] & & A \otimes A \arrow[u, "\psi"']
\end{tikzcd}$$

commutes. A homomorphism of Hopf algebras with conjugation is a Hopf algebra homomorphism that commutes with the conjugation.

Definition 15.10. Let A be a k-algebra, and let $B \subset A$ be a subalgebra with an augmentation $\epsilon\colon B \to k$, making k a B-module. Then we let

$$A//B = A \otimes_B k = A/A \cdot I(B)$$

and

$$B\backslash\backslash A = k \otimes_B A = A/I(B) \cdot A\,.$$

If $A \cdot I(B) = I(B) \cdot A$ we say that B is normal in A. Then $A//B$ is a k-algebra, and the canonical map $A \to A//B$ is an algebra homomorphism.

Theorem 15.11 (Milnor–Moore)**.** *Let A be a connected Hopf algebra and $B \subset A$ a Hopf subalgebra. Then there is an isomorphism $A \cong A//B \otimes B$ of right B-modules, and an isomorphism $A \cong B \otimes B\backslash\backslash A$ of left B-modules, so A is free as a left B-module and as a right B-module.*

This is part of Theorem 4.4 in Milnor–Moore (1965). More concretely, let $i\colon B \to A$ be the inclusion and let $s\colon A//B \to A$ be any k-linear section to the projection $A \to A//B$. Then the composite

$$A//B \otimes B \xrightarrow{s\otimes i} A \otimes A \xrightarrow{\phi} A$$

is an isomorphism of right B-modules. It is not usually true that A is free as a B-B-bimodule.

15.2. **Actions and coactions.**

Definition 15.12. Let A be a k-algebra. A left A-module is a graded k-module M with a pairing $\lambda\colon A \otimes M \to M$ such that the diagrams

$$\begin{tikzcd} A \otimes A \otimes M \arrow[r, "1\otimes\lambda"] \arrow[d, "\phi\otimes 1"'] & A \otimes M \arrow[d, "\lambda"] \\ A \otimes M \arrow[r, "\lambda"] & M \end{tikzcd} \qquad \begin{tikzcd} k \otimes M \arrow[r, "\eta\otimes 1"] \arrow[dr, "\cong"'] & A \otimes M \arrow[d, "\lambda"] \\ & M \end{tikzcd}$$

commute. A right A-module is a graded k-module N with a pairing $\rho\colon N \otimes A \to N$ such that the diagrams

$$\begin{tikzcd} N \otimes A \otimes A \arrow[r, "\rho\otimes 1"] \arrow[d, "1\otimes\phi"'] & N \otimes A \arrow[d, "\rho"] \\ N \otimes A \arrow[r, "\rho"] & N \end{tikzcd} \qquad \begin{tikzcd} N \otimes k \arrow[r, "1\otimes\eta"] \arrow[dr, "\cong"'] & N \otimes A \arrow[d, "\rho"] \\ & N \end{tikzcd}$$

commute. The tensor product $N \otimes_A M$ is the coequalizer in the diagram

$$\begin{tikzcd} N \otimes A \otimes M \arrow[r, shift left, "1\otimes\lambda"] \arrow[r, shift right, "\rho\otimes 1"'] & N \otimes M \arrow[r, two heads] & N \otimes_A M \end{tikzcd}$$

Definition 15.13. Let A be a k-coalgebra. A left A-comodule is a graded k-module M with a pairing $\lambda\colon M \to A \otimes M$ such that the diagrams

$$\begin{tikzcd} M \arrow[r, "\lambda"] \arrow[d, "\lambda"'] & A \otimes M \arrow[d, "1\otimes\lambda"] \\ A \otimes M \arrow[r, "\psi\otimes 1"'] & A \otimes A \otimes M \end{tikzcd} \qquad \begin{tikzcd} M \arrow[d, "\lambda"'] \arrow[dr, "\cong"] & \\ A \otimes M \arrow[r, "\epsilon\otimes 1"'] & k \otimes M \end{tikzcd}$$

commute. A right A-comodule is a graded k-module N with a pairing $\rho\colon N \to N \otimes A$ such that the diagrams

$$\begin{tikzcd} N \arrow[r, "\rho"] \arrow[d, "\rho"'] & N \otimes A \arrow[d, "\rho\otimes 1"] \\ N \otimes A \arrow[r, "1\otimes\psi"'] & N \otimes A \otimes A \end{tikzcd} \qquad \begin{tikzcd} N \arrow[d, "\rho"'] \arrow[dr, "\cong"] & \\ N \otimes A \arrow[r, "1\otimes\epsilon"'] & N \otimes k \end{tikzcd}$$

commute. The cotensor product $N \,\square_A\, M$ is the equalizer in the diagram

$$\begin{tikzcd} N \,\square_A\, M \arrow[r, tail] & N \otimes M \arrow[r, shift left, "1\otimes\lambda"] \arrow[r, shift right, "\rho\otimes 1"'] & N \otimes A \otimes M \end{tikzcd}$$

Lemma 15.14. *Let M be a left A-module, with action $a \cdot m = \lambda(a \otimes m)$ for $a \in A$ and $m \in M$. Then the linear dual $M^* = \mathrm{Hom}(M, k)$ is a right A-module, with action $\mu \cdot a = \rho(\mu \otimes a)$ given by $\mu \cdot a\colon m \mapsto \mu(a \cdot m)$, for $\mu\colon M \to k$ in M^*. Likewise, if N is a right A-module then N^* is a left A-module.*

Proof. $\mu \cdot a\colon m \mapsto \mu(a \cdot m)$, so $(\mu \cdot a) \cdot b\colon m \mapsto (\mu \cdot a)(b \cdot m) = \mu(a \cdot b \cdot m) = \mu(ab \cdot m)$ equals $\mu \cdot ab$. $\square$

Lemma 15.15. *Let A be a k-algebra, bounded below and of finite type. Then $A^* = \mathrm{Hom}(A, k)$ is a k-coalgebra with coproduct $\psi = \phi^*\colon A^* \to (A \otimes A)^* \cong A^* \otimes A^*$ and counit $\epsilon = \eta^*\colon A^* \to k$. Conversely, if A is a k-coalgebra then A^* is a k-algebra. If A was bounded below and of finite type, then so is A^*, and $A \cong (A^*)^*$.*

Lemma 15.16. *Let A be an augmented k-algebra, bounded below and of finite type. Then A^* is a coaugmented k-coalgebra, $J(A^*) \cong I(A)^*$ and $P(A^*) \cong Q(A)^*$.*

Lemma 15.17. *Let A be a k-algebra, M a left A-module and N a right A-module, all bounded below and of finite type. Then M^* is a left A^*-comodule with coaction $\lambda = \lambda^*\colon M^* \to (A \otimes M)^* \cong A^* \otimes M^*$, and N^* is a right A^*-comodule with coaction $\rho = \rho^*\colon N^* \to (N \otimes A)^* \cong N^* \otimes A^*$.*

Conversely, let A be a k-coalgebra, M a left A-comodule and N a right A-comodule. Then M^* is a left A^*-module with action $\lambda\colon A^* \otimes M^* \to (A \otimes M)^* \to M^*$, and N^* is a right A^*-module with action $\rho\colon N^* \otimes A^* \to (N \otimes A)^* \to N^*$.

Definition 15.18. Let A be an augmented k-algebra and let M be a left A-module. The A-module indecomposables in M is the quotient k-module $k \otimes_A M = M/I(A) \cdot M$.

Definition 15.19. Let A be a coaugmented k-coalgebra and let M be a left A-comodule. The A-comodule primitives in M is the k-submodule $k \,\square_A\, M = \{m \in M \mid \lambda(m) = 1 \otimes m\}$.

Lemma 15.20. *Let A be an augmented k-algebra and M left A-module, both bounded below and of finite type. Let M^* be the dual left A^*-comodule. Then there are natural isomorphisms*

$$\mathrm{Hom}_A(M, k) \cong \mathrm{Hom}(k \otimes_A M, k) \cong k \,\square_{A^*}\, M^*$$

that are compatible with the inclusions into $\mathrm{Hom}(M, k) = M^*$.

See Boardman (1982) for more on left/right algebra/coalgebra actions/coactions.

Definition 15.21. Let A be a Hopf algebra, and let M and N be left A-modules. Then $M \otimes N$ is a left A-module, with the action $\lambda\colon A \otimes M \otimes N$ defined as the composite

$$A \otimes M \otimes N \xrightarrow{\psi\otimes 1\otimes 1} A \otimes A \otimes M \otimes N \underset{\cong}{\xrightarrow{1\otimes\gamma\otimes 1}} A \otimes M \otimes A \otimes N \xrightarrow{\lambda\otimes\lambda} M \otimes N\,.$$

Likewise for right A-modules.

Conversely, let M and N be left A-comodules. Then $M \otimes N$ is a left A-comodule, with the coaction $\lambda\colon M \otimes N \to A \otimes M \otimes N$ defined as the composite

$$M \otimes N \xrightarrow{\lambda\otimes\lambda} A \otimes M \otimes A \otimes N \underset{\cong}{\xrightarrow{1\otimes\gamma\otimes 1}} A \otimes A \otimes M \otimes N \xrightarrow{\phi\otimes 1\otimes 1} A \otimes M \otimes N\,.$$

Likewise for right A-comodules.

15.3. **The coproduct.** Let Y and Z be spectra. If Y and Z are bounded below with $H_*(Y)$ and $H_*(Z)$ of finite type, then the cohomology smash product

$$H^*(Y) \otimes H^*(Z) \xrightarrow{\wedge} H^*(Y \wedge Z)$$

is an isomorphism. The Cartan formula

$$Sq^k(y \wedge z) = \sum_{i+j=k} Sq^i(y) \wedge Sq^j(z)$$

implies the more general formula

$$Sq^K(y \wedge z) = \sum_{I+J=K} Sq^I(y) \wedge Sq^J(z)$$

for sequences $K = (k_1, \dots, k_\ell)$ of non-negative integers, where the sum is over pairs of sequences $I = (i_i, \dots, i_\ell)$ and $J = (j_1, \dots, j_\ell)$ of non-negative integers, such that $k_u = i_u + j_u$ for all $1 \le u \le \ell$. Milnor proved that the rule

$$Sq^K \longmapsto \sum_{I+J=K} Sq^I \otimes Sq^J$$

respects the Adem relations, in the sense that it gives a well-defined algebra homomorphism

$$\psi\colon \mathscr{A} \longrightarrow \mathscr{A} \otimes \mathscr{A}\,.$$

Since $\mathscr{A}$ is connected, there is a unique homomorphism

$$\chi\colon \mathscr{A} \longrightarrow \mathscr{A}$$

with $\chi(1) = 1$ and $\sum a'\chi(a'') = 0$ for all $a \in I(\mathscr{A})$ with $\psi(a) = \sum a' \otimes a''$. Then $\chi(ab) = \chi(b)\chi(a)$ and χ^2 is the identity.

Theorem 15.22 (Milnor (1958))**.** *The Steenrod algebra $\mathscr{A}$, with the composition coproduct ϕ, the coproduct ψ and the conjugation χ, is a cocommutative Hopf algebra with conjugation.*

Definition 15.23. Let the dual Steenrod algebra $\mathscr{A}_* = \mathrm{Hom}(\mathscr{A}, \mathbb{F}_2)$ be the linear dual of the Steenrod algebra. Since $\mathscr{A}$ is of finite type, there is a natural isomorphism $\mathscr{A} \cong \mathrm{Hom}(\mathscr{A}_*, \mathbb{F}_2)$. The algebra structure maps $\phi\colon \mathscr{A} \otimes \mathscr{A} \to \mathscr{A}$ and $\eta\colon \mathbb{F}_2 \to \mathscr{A}$ dualize to coalgebra structure maps $\psi\colon \mathscr{A}_* \to \mathscr{A}_* \otimes \mathscr{A}_*$ and $\epsilon\colon \mathscr{A}_* \to \mathbb{F}_2$. The cocommutative coalgebra structure maps $\psi\colon \mathscr{A} \to \mathscr{A} \otimes \mathscr{A}$ and $\epsilon\colon \mathscr{A} \to \mathbb{F}_2$ dualize to commutative algebra structure maps $\phi\colon \mathscr{A}_* \otimes \mathscr{A}_* \to \mathscr{A}_*$ and $\eta\colon \mathbb{F}_2 \to \mathscr{A}_*$. The conjugation $\chi\colon \mathscr{A} \to \mathscr{A}$ dualizes to a conjugation $\chi\colon \mathscr{A}_* \to \mathscr{A}_*$. With these structure maps, $\mathscr{A}_*$ is a commutative Hopf algebra.

Remark 15.24. The isomorphism $\mathscr{A} \cong H^*(H)$ is dual to an isomorphism $\mathscr{A}_* \cong H_*(H)$. This may justify why we write $\mathscr{A}_*$ instead of $\mathscr{A}^*$ for the dual Steenrod algebra, thinking of the star as a homological grading rather than as the symbol for dualization. The ring spectrum product $\mu\colon H \wedge H \to H$ induces the product $\phi\colon \mathscr{A}_* \otimes \mathscr{A}_* \cong H_*(H) \otimes H_*(H) \cong H_*(H \wedge H) \to H_*(H) \cong \mathscr{A}_*$ in homology, and the counit $\epsilon\colon \mathscr{A}_* = \pi_*(H \wedge H) \to \pi_*(H) = \mathbb{F}_2$ in homotopy. The ring spectrum unit $\eta\colon S \to H$ induces a map $H \cong S \wedge H \to H \wedge H$ that induces the coproduct $\psi\colon \mathscr{A}_* = H_*(H) \to H_*(H \wedge H) \cong H_*(H) \otimes H_*(H) \cong \mathscr{A}_* \otimes \mathscr{A}_*$ in homology. The two maps $H \cong S \wedge H \to H \wedge H$ and $H \cong H \wedge S \to H \wedge H$ both induce the unit $\eta\colon \mathbb{F}_2 \to \mathscr{A}_*$ in homotopy. The twist map $\gamma\colon H \wedge H \to H \wedge H$ induces the conjugation $\chi\colon \mathscr{A}_* \to \mathscr{A}_*$. ((Reference?))

By definition, $\psi\colon \mathscr{A} \to \mathscr{A} \otimes \mathscr{A}$ makes the diagram

$$\begin{tikzcd}
\mathscr{A} \otimes \mathscr{A} \otimes H^*(Y) \otimes H^*(Z) \arrow[d, "1\otimes\gamma\otimes 1"', "\cong"] & \mathscr{A} \otimes H^*(Y) \otimes H^*(Z) \arrow[l, "\psi\otimes 1\otimes 1"'] \arrow[r, "1\otimes\wedge"] \arrow[d, dashed, "\lambda"'] & \mathscr{A} \otimes H^*(Y \wedge Z) \arrow[d, "\lambda"] \\
\mathscr{A} \otimes H^*(Y) \otimes \mathscr{A} \otimes H^*(Z) \arrow[r, "\lambda\otimes\lambda"] & H^*(Y) \otimes H^*(Z) \arrow[r, "\wedge"] & H^*(Y \wedge Z)
\end{tikzcd}$$

commute, where $\lambda\colon \mathscr{A} \otimes H^*(Y) \to H^*(Y)$ denotes the left $\mathscr{A}$-module action. We defined the $\mathscr{A}$-module action on the tensor product $H^*(Y) \otimes H^*(Z)$ by the dashed composite in this diagram, so that the Künneth homomorphism $\wedge$ is an $\mathscr{A}$-module homomorphism.

By the Hom-tensor adjunction, the diagram can be reformulated as follows:

$$\begin{tikzcd}
\mathrm{Hom}(\mathscr{A}, H^*(Y)) \otimes \mathrm{Hom}(\mathscr{A}, H^*(Z)) \arrow[d, "\otimes"'] & H^*(Y) \otimes H^*(Z) \arrow[l, "\tilde\lambda\otimes\tilde\lambda"'] \arrow[r, "\wedge"] \arrow[d, dashed, "\tilde\lambda"'] & H^*(Y \wedge Z) \arrow[d, "\tilde\lambda"] \\
\mathrm{Hom}(\mathscr{A} \otimes \mathscr{A}, H^*(Y) \otimes H^*(Z)) \arrow[r, "\psi^*"] & \mathrm{Hom}(\mathscr{A}, H^*(Y) \otimes H^*(Z)) \arrow[r, "\wedge_*"] & \mathrm{Hom}(\mathscr{A}, H^*(Y \wedge Z))
\end{tikzcd}$$

where $\tilde\lambda\colon H^*(Y) \to \mathrm{Hom}(\mathscr{A}, H^*(Y))$ takes y to the homomorphism $a \mapsto a(y)$, etc. If we add the assumption that $H^*(Y)$ is bounded above, so that $H_*(Y)$ is (totally) finite, then there is a natural isomorphism

$$H^*(Y) \otimes \mathscr{A}_* \cong \mathrm{Hom}(\mathscr{A}, H^*(Y))$$

taking $y \otimes \alpha$ to $a \mapsto \alpha(a)y$, with $y \in H^*(Y)$, $\alpha \in \mathscr{A}_*$ and $a \in \mathscr{A}$. We also assume that $H_*(Z)$ is (totally) finite. Then we can rewrite the diagram as:

$$\begin{tikzcd}
H^*(Y) \otimes \mathscr{A}_* \otimes H^*(Z) \otimes \mathscr{A}_* \arrow[d, "1\otimes\gamma\otimes 1"', "\cong"] & H^*(Y) \otimes H^*(Z) \arrow[l, "\rho\otimes\rho"'] \arrow[r, "\wedge"] \arrow[d, dashed, "\rho"'] & H^*(Y \wedge Z) \arrow[d, "\rho"] \\
H^*(Y) \otimes H^*(Z) \otimes \mathscr{A}_* \otimes \mathscr{A}_* \arrow[r, "1\otimes 1\otimes\phi"] & H^*(Y) \otimes H^*(Z) \otimes \mathscr{A}_* \arrow[r, "\wedge\otimes 1"] & H^*(Y \wedge Z) \otimes \mathscr{A}_*
\end{tikzcd}$$

where ϕ is the algebra structure on $\mathscr{A}_*$, dual to the coproduct ψ on $\mathscr{A}$, and $\rho\colon H^*(Y) \to H^*(Y) \otimes \mathscr{A}_*$ is the right $\mathscr{A}_*$-comodule coaction on $H^*(Y)$, corresponding to $\tilde\lambda$ via the isomorphism above. We defined the $\mathscr{A}_*$-coaction on the tensor product $H^*(Y) \otimes H^*(Z)$ by the dashed composite. Hence the Künneth morphism $\wedge$ is an $\mathscr{A}_*$-comodule homomorphism.

Proposition 15.25 (Milnor)**.** *Let X be a space with $H_*(X)$ (totally) finite. The right $\mathscr{A}$-comodule coaction*

$$\rho\colon H^*(X) \to H^*(X) \otimes \mathscr{A}_*$$

is an algebra homomorphism, where $H^*(X)$ has the cup product and $\mathscr{A}_*$ has the product dual to the coproduct ψ on $\mathscr{A}$.

Proof. Let $Y = Z = \Sigma^\infty(X_+)$. Then the diagonal $\Delta\colon X \to X \times X$ induces the commutative diagram

$$\begin{tikzcd}
H^*(X) \otimes \mathscr{A}_* \otimes H^*(X) \otimes \mathscr{A}_* \arrow[d, "1\otimes\gamma\otimes 1"', "\cong"] & H^*(X) \otimes H^*(X) \arrow[l, "\rho\otimes\rho"'] \arrow[r, "\cup"] \arrow[d, dashed, "\rho"'] & H^*(X) \arrow[d, "\rho"] \\
H^*(X) \otimes H^*(X) \otimes \mathscr{A}_* \otimes \mathscr{A}_* \arrow[r, "1\otimes 1\otimes\phi"] & H^*(X) \otimes H^*(X) \otimes \mathscr{A}_* \arrow[r, "\cup\otimes 1"] & H^*(X) \otimes \mathscr{A}_*
\end{tikzcd}$$

which says that the cup product $\cup$ is an $\mathscr{A}_*$-comodule homomorphism, or equivalently, that the coaction ρ is an algebra homomorphism. ☐

This results encodes the Cartan formula for the Steenrod algebra action on the cohomology of a product of spaces, in terms of the coaction of the dual Steenrod algebra, in a very convenient form.

15.4. **The Milnor generators.** Without appealing to the conjugation χ, we have the following four left and right actions and coactions on the homology and cohomology of a space X with $H_*(X)$ finite:

$$\begin{aligned} \lambda &\colon \mathscr{A} \otimes H^*(X) \longrightarrow H^*(X) \\ \rho &\colon H_*(X) \otimes \mathscr{A} \longrightarrow H_*(X) \\ \rho &\colon H^*(X) \longrightarrow H^*(X) \otimes \mathscr{A}_* \\ \lambda &\colon H_*(X) \longrightarrow \mathscr{A}_* \otimes H_*(X) \end{aligned}$$

We specialize to the test object $X = \mathbb{R}P^N \subset \mathbb{R}P^\infty = H_1$, with $H^*(X) = P(x)/(x^{N+1})$ and $H_*(X) = \mathbb{F}_2\{\gamma_j \mid 0 \le j \le N\}$, where x^j is dual to γ_j. We are interested in the limit as $N \to \infty$, when $\lim_N H^*(\mathbb{R}P^N) = P(x)$ and $\operatorname{colim}_N H_*(\mathbb{R}P^N) = \mathbb{F}_2\{\gamma_j \mid j \ge 0\}$. The limiting right coaction

$$\rho\colon P(x) \longrightarrow P(x) \mathbin{\widehat{\otimes}} \mathscr{A}_*$$

was just seen to be an algebra homomorphism, hence is determined by the single value

$$\rho(x) = \sum_{j\ge1} x^j \otimes \alpha_j$$

where $\alpha_j \in \mathscr{A}_*$ has degree $(j-1)$, for each $j \ge 1$.

Lemma 15.26. *There are well-defined classes $\xi_i \in \mathscr{A}_*$ such that*

$$\rho(x) = \sum_{i\ge0} x^{2^i} \otimes \xi_i \,.$$

Here $\xi_0 = 1$, and ξ_i has degree $2^i - 1$, for each $i \ge 0$.

Proof. There is a pairing $m\colon \mathbb{R}P^\infty \times \mathbb{R}P^\infty \to \mathbb{R}P^\infty$ that represents the tensor product of real line bundles, or comes from the loop structure on $H_1 \simeq \Omega H_2$. It induces a homomorphism

$$m^*\colon P(x) = H^*(\mathbb{R}P^\infty) \to H^*(\mathbb{R}P^\infty \times \mathbb{R}P^\infty) = P(x_1, x_2)$$

with $m^*(x) = x_1 + x_2$, where $x_1 = x \times 1$ and $x_2 = 1 \times x$. By naturality of the right $\mathscr{A}_*$-coaction ρ, we have that

$$m^*(\rho(x)) = \sum_{j\ge1} (x_1 + x_2)^j \otimes \alpha_j$$

is equal to

$$\rho(m^*(x)) = \rho(x_1 + x_2) = \rho(x_1) + \rho(x_2) = \sum_{j\ge1} x_1^j \otimes \alpha_j + \sum_{j\ge1} x_2^j \otimes \alpha_j$$

in $P(x_1, x_2) \mathbin{\widehat{\otimes}} \mathscr{A}_*$. The product formula for binomial coefficients mod 2 implies that $(x_1+x_2)^j \ne x_1^j + x_2^j$ for all j not of the form $j = 2^i$, $i \ge 0$, hence $\alpha_j = 0$ for all such j. We let $\xi_i = \alpha_{2^i}$ for $i \ge 0$. Counitality of the coaction implies that $\xi_0 = 1$. □

Let $P(\xi_i \mid i \ge 1) = P(\xi_1, \xi_2, \xi_3, \dots)$ be the polynomial algebra generated by the classes ξ_i for $i \ge 0$, only subject to the relation $\xi_0 = 1$.

Theorem 15.27 (Milnor)**.** *The canonical homomorphism*

$$P(\xi_i \mid i \ge 1) \xrightarrow{\cong} \mathscr{A}_*$$

is an algebra isomorphism.

See Milnor (1958) Theorem 2 or Steenrod–Epstein (1962) Theorem 2.2 for the proof. Surjectivity of $P(\xi_i \mid i \ge 1) \to \mathscr{A}_*$ follows by the detection results for $\mathscr{A}$. A count of dimensions then proves isomorphism.

Theorem 15.28 (Milnor)**.** *The Hopf algebra coproduct $\psi\colon \mathscr{A}_* \to \mathscr{A}_* \otimes \mathscr{A}_*$ is given by*

$$\psi(\xi_k) = \sum_{i+j=k} \xi_i^{2^j} \otimes \xi_j$$

where $i, j \ge 0$ and $\xi_0 = 1$. Hence the conjugation $\chi\colon \mathscr{A}_* \to \mathscr{A}_*$ is determined by

$$\sum_{i+j=k} \xi_i^{2^j} \chi(\xi_j) = 0$$

for all $k \geq 1$.

Proof. The coassociativity of the right coaction tells us that

$$(\rho \otimes 1)\rho(x) = (\rho \otimes 1)(\sum_{j\geq 0} x^{2^j} \otimes \xi_j) = \sum_{j \geq 0} \rho(x)^{2^j} \otimes \xi_j = \sum_{i,j\geq 0} x^{2^{i+j}} \otimes \xi_i^{2^j} \otimes \xi_j$$

is equal to

$$(1 \otimes \psi)\rho(x) = \sum_{k\geq 0} x^{2^k} \otimes \psi(\xi_k)\,.$$

□

These formulas for the coproduct in $\mathscr{A}_*$ are often more manageable than the Adem relations for the product in $\mathscr{A}$. Here is list of $\psi(\xi_k)$ and $\chi(\xi_k)$ for small k:

$$\begin{aligned}
\psi(\xi_1) &= \xi_1 \otimes 1 + 1 \otimes \xi_1 \\
\psi(\xi_2) &= \xi_2 \otimes 1 + \xi_1^2 \otimes \xi_1 + 1 \otimes \xi_2 \\
\psi(\xi_3) &= \xi_3 \otimes 1 + \xi_2^2 \otimes \xi_1 + \xi_1^4 \otimes \xi_2 + 1 \otimes \xi_3 \\
\psi(\xi_4) &= \xi_4 \otimes 1 + \xi_3^2 \otimes \xi_1 + \xi_2^4 \otimes \xi_2 + \xi_1^8 \otimes \xi_3 + 1 \otimes \xi_4
\end{aligned}$$

$$\begin{aligned}
\chi(\xi_1) &= \xi_1 \\
\chi(\xi_2) &= \xi_2 + \xi_1^3 \\
\chi(\xi_3) &= \xi_3 + \xi_1\xi_2^2 + \xi_1^4\xi_2 + \xi_1^7 \\
\chi(\xi_4) &= \xi_4 + \xi_1\xi_3^2 + \xi_1^8\xi_3 + \xi_2^5 + \xi_1^3\xi_2^4 + \xi_1^9\xi_2^2 + \xi_1^{12}\xi_2 + \xi_1^{15}
\end{aligned}$$

We note that $\xi_1^{2^i}$ is primitive for each $i \geq 0$, and that $\chi(\xi_k) \equiv \xi_k$ modulo decomposables.

We now make the Milnor classes $\xi_i \in \mathscr{A}_*$ a little more explicit. Dualizing the formula for $\rho(x)$, the right action

$$\rho\colon H_*(\mathbb{R}P^\infty) \otimes \mathscr{A} \longrightarrow H_*(\mathbb{R}P^\infty)$$

is given in total degree 1 by

$$\gamma_j \otimes a \longmapsto \begin{cases} \langle a, \xi_i\rangle \gamma_1 & \text{for } j = 2^i \\ 0 & \text{otherwise.} \end{cases}$$

Here $a \in \mathscr{A}$ has degree $(j-1)$ and $\langle -,-\rangle\colon \mathscr{A} \otimes \mathscr{A}_* \to \mathbb{F}_2$ is the evaluation pairing. Likewise, the left action

$$\lambda\colon \mathscr{A} \otimes P(x) \longrightarrow P(x)$$

is given on $\mathscr{A} \otimes \mathbb{F}_2\{x\}$ by

$$a \otimes x \longmapsto a(x) = \sum_{i\geq 0} \langle a, \xi_i\rangle x^{2^i}\,.$$

Lemma 15.29. *For admissible sequences* I,

$$Sq^I(x) = \begin{cases} x^{2^i} & \textit{for } I = (2^{i-1}, 2^{i-2}, \dots, 2, 1),\ i \geq 0 \\ 0 & \textit{otherwise.} \end{cases}$$

Hence

$$\langle Sq^I, \xi_i\rangle = \begin{cases} 1 & \textit{for } I = (2^{i-1}, 2^{i-2}, \dots, 2, 1) \\ 0 & \textit{otherwise.} \end{cases}$$

In other words, ξ_i *is dual to* $Sq^{2^{i-1}} Sq^{2^{i-2}} \dots Sq^2 Sq^1$ *when we give* $\mathscr{A}$ *the admissible basis.*

The identification of $\mathbb{R}P^\infty$ with the first space H_1 in the Eilenberg–Mac Lane spectrum H leads to a stable map $f\colon \Sigma^\infty H_1 \to \Sigma H$. The induced $\mathscr{A}$-module homomorphism

$$f^*\colon \Sigma\mathscr{A} = H^*(\Sigma H) \longrightarrow \tilde{H}^*(H_1) \subset P(x)$$

takes the generator $\Sigma 1$ to x, hence agrees with the $\mathscr{A}$-module homomorphism $\mathscr{A} \otimes \mathbb{F}_2\{x\} \to P(x)$ taking $a \otimes x$ to

$$a(x) = \sum_{i\geq 0} \langle a, \xi_i\rangle x^{2^i}\,,$$

via the isomorphism $\Sigma\mathscr{A} \cong \mathscr{A} \otimes \mathbb{F}_2\{x\}$. Dually, it follows that the $\mathscr{A}_*$-comodule homomorphism

$$f_* \colon \tilde{H}_*(H_1) \longrightarrow H_*(\Sigma H) \cong \Sigma\mathscr{A}_*$$

is the linear dual mapping

$$\gamma_j \longmapsto \begin{cases} \Sigma\xi_i & \text{for } j = 2^i,\ i \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 15.30. *The map $f\colon \Sigma^\infty \mathbb{R}P^\infty \to \Sigma H$ induces a homomorphism $\tilde{H}_{*+1}(\mathbb{R}P^\infty) \to \mathscr{A}_*$ taking $\gamma_j \in \tilde{H}_j(\mathbb{R}P^\infty)$ to ξ_i if $j = 2^i$, $i \geq 0$, and to 0 otherwise.*

Definition 15.31. The dual Steenrod algebra $\mathscr{A}_* \cong P(\xi_k \mid k \geq 1)$ has a basis $\{\xi^R\}_R$ given by the monomials

$$\xi^R = \xi_1^{r_1}\xi_2^{r_2}\cdots\xi_\ell^{r_\ell}$$

where $R = (r_1, \dots, r_\ell)$ ranges over all finite sequences of non-negative integers, with $r_\ell \geq 1$ if $\ell \geq 1$. The Milnor basis $\{Sq^R\}_R$ for the Steenrod algebra $\mathscr{A}$ is the dual basis, defined so that

$$\langle Sq^R, \xi^S\rangle = \begin{cases} 1 & \text{for } R = S \\ 0 & \text{otherwise.} \end{cases}$$

Hence $|Sq^R| = |\xi^R| = \sum_{u=1}^{\ell} r_u(2^u - 1)$. The coproduct is given by $\psi(Sq^T) = \sum_{R+S=T} \psi^R \otimes \psi^S$.

Remark 15.32. One should not confuse the notations Sq^I and Sq^R. We let I, J and K range over admissible sequences, and let Sq^I, Sq^J and Sq^K denote the corresponding admissible composites of Steenrod squares. We let R, S and T range over finite sequences of non-negative integers, and let Sq^R, Sq^S and Sq^T denote the corresponding elements in the Milnor basis.

Example 15.33. It is clear that $Sq^{()} = 1$, $Sq^{(1)} = Sq^1$ and $Sq^{(2)} = Sq^2$. In degree 3, we have $\langle Sq^3, \xi_2\rangle = 0$, $\langle Sq^2Sq^1, \xi_2\rangle = 1$, $\langle Sq^3, \xi_1^3\rangle = 1$ and $\langle Sq^2Sq^1, \xi_1^3\rangle = 1$. For example,

$$\begin{aligned}\langle Sq^2Sq^1, \xi_1^3\rangle &= \langle Sq^2Sq^1, \phi(\xi_1 \otimes \xi_1^2)\rangle = \langle \psi(Sq^2Sq^1), \xi_1 \otimes \xi_1^2\rangle \\ &= \langle (Sq^2 \otimes 1 + Sq^1 \otimes Sq^1 + 1 \otimes Sq^2)(Sq^1 \otimes 1 + 1 \otimes Sq^1), \xi_1 \otimes \xi_1^2\rangle \\ &= \langle Sq^1 \otimes (Sq^2 + Sq^1Sq^1), \xi_1 \otimes \xi_1^2\rangle = \langle Sq^1, \xi_1\rangle\langle Sq^2, \xi_1^2\rangle = 1\,.\end{aligned}$$

Hence $Sq^{(3)} = Sq^3$ and $Sq^{(0,1)} = Sq^3 + Sq^2Sq^1 = Q_1$.

Lemma 15.34. *The Milnor basis element $Sq^{(r)}$ equals the Steenrod operation Sq^r, for each $r \geq 1$.*

Proof. Let $S = (s_1, \dots, s_\ell)$ be a finite sequence of non-negative integers, with $s_\ell \geq 1$. We must prove that $\langle Sq^r, \xi^S\rangle$ equals 1 for $S = (r)$ and 0 otherwise. Let Φ be the $\sum_{u=1}^{\ell} s_u$-fold product on $\mathscr{A}_*$, and let Ψ be the $\sum_{u=1}^{\ell} s_u$-fold coproduct on $\mathscr{A}$. Writing $\xi^S = \Phi(\xi_a \otimes \cdots \otimes \xi_\ell)$ with $a \leq \cdots \leq \ell$, we must compute $\langle Sq^r, \xi^S\rangle = \langle Sq^r, \Phi(\xi_a \otimes \cdots \otimes \xi_\ell)\rangle = \langle \Psi(Sq^r), \xi_a \otimes \cdots \otimes \xi_\ell\rangle$. Here $\Psi(Sq^r)$ is a sum of tensor products of factors of the form Sq^j. We have $\langle Sq^{2^i-1}, \xi_i\rangle$ equals 1 for $i = 1$ and 0 for $i \geq 2$. Hence $\langle \Psi(Sq^r), \xi_a \otimes \cdots \otimes \xi_\ell\rangle = 0$ if $\ell \geq 2$. Furthermore, $\langle \Psi(Sq^r), \xi_a \otimes \cdots \otimes \xi_\ell\rangle = 1$ if $S = (r)$ and $a = \cdots = \ell = 1$, since $\Psi(Sq^r)$ contains the summand $Sq^1 \otimes \cdots \otimes Sq^1$ that evaluates to 1 on $\xi_1 \otimes \cdots \otimes \xi_1$. □

Theorem 15.35 (Milnor)**.** *For each infinite matrix of non-negative integers (almost all zero)*

$$X = \begin{bmatrix} * & x_{01} & x_{02} & \cdots \\ x_{10} & x_{11} & x_{12} & \cdots \\ x_{20} & x_{21} & x_{22} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

let $R(X) = (r_1, r_2, \dots)$, $S(X) = (s_1, s_2, \dots)$ and $T(X) = (t_1, t_2, \dots)$ be given by the sums

$$\begin{aligned} r_i &= \sum_j 2^j x_{ij} && \textit{(weighted row sum)}, \\ s_j &= \sum_i x_{ij} && \textit{(column sum)}, \\ t_k &= \sum_{i+j=k} x_{ij} && \textit{(diagonal sum)}. \end{aligned}$$

Then

$$Sq^R \cdot Sq^S = \sum_X b(X) Sq^T$$

where X ranges over the matrices with $R(X) = R$ and $S(X) = S$, with $T = T(X)$ and

$$b(X) = \prod_k t_k! / \prod_{i,j} x_{ij}! \,.$$

See Milnor (1958) Theorem 4b. To prove this, one must count how often $\xi^R \otimes \xi^S \in \mathscr{A}_* \otimes \mathscr{A}_*$ occurs as a summand in $\psi(\xi^T) = \psi(\xi_1)^{t_1} \cdot \dots \cdot \psi(\xi_\ell)^{t_\ell}$.

Example 15.36. Let $k \geq 2$, $R = (2^k)$ and $S = (0, \dots, 0, 1)$ with $(k-1)$ zeroes. Then $Sq^R \cdot Sq^S$ is a sum of terms $b(X)Sq^T$, where X ranges over the matrices (x_{ij}) with $x_{00} = 0$, $\sum_j 2^j x_{1j} = 2^k$, $\sum_j 2^j x_{ij} = 0$ for $i \geq 2$, $\sum_i x_{ik} = 1$ and $\sum_i x_{ij} = 0$ for $1 \leq j \leq k-1$ and for $j \geq k+1$. There are only two possible matrices X, namely X' with $x'_{1k} = 1$ and the remaining terms zero, and X'' with $x''_{0k} = 1$, $x''_{10} = 2^k$ and the remaining terms zero. The corresponding sequences are $T' = T(X') = (0, \dots, 0, 1)$ with k zeroes, and $T'' = T(X'') = (2^k, 0, \dots, 0, 1)$ with $(k-2)$ zeroes. The coefficients $b(X')$ and $b(X'')$ are 1, so

$$Sq^{(2^k)} \cdot Sq^{(0,\dots,0,1)} = Sq^{(0,\dots,0,0,1)} + Sq^{(2^k,0,\dots,0,1)} \,.$$

On the other hand, $Sq^S \cdot Sq^R$ is the sum of a single term $b(X)Sq^T$, where X has $x_{01} = 2^k$, $x_{k0} = 1$ and the remaining terms are zero. Again $b(X) = 1$, so

$$Sq^{(0,\dots,0,1)} \cdot Sq^{(2^k)} = Sq^{(2^k,0,\dots,0,1)} \,.$$

Hence the commutator

$$[Sq^{(2^k)}, Sq^{(0,\dots,0,1)}] = Sq^{(2^k)} \cdot Sq^{(0,\dots,0,1)} + Sq^{(0,\dots,0,1)} \cdot Sq^{(2^k)}$$

($(k-1)$ zeroes each time) equals the Milnor element $Sq^{(0,\dots,0,0,1)}$, now with k zeroes.

15.5. **Subalgebras of the Steenrod algebra.**

Definition 15.37. A Hopf ideal in a Hopf algebra A is a two-sided ideal $I \subset A$ such that $\psi(I) \subset A \otimes I + I \otimes A$ and $\epsilon(I) = 0$:

$$\begin{tikzcd}
0 & I \arrow[l] \arrow[r] \arrow[d, tail] & A \otimes I + I \otimes A \arrow[d, tail] \\
k \arrow[d, "="'] & A \arrow[l, "\epsilon"'] \arrow[r, "\psi"] \arrow[d, two heads] & A \otimes A \arrow[d, two heads] \\
k & A/I \arrow[l, "\bar\epsilon"'] \arrow[r, "\bar\psi"] & A/I \otimes A/I
\end{tikzcd}$$

Then ψ and ϵ induce a coproduct $\bar\psi \colon A/I \to A/I \otimes A/I$ and a counit $\bar\epsilon \colon A/I \to k$ that make A/I a Hopf algebra, and the canonical surjection $A \to A/I$ is a Hopf algebra homomorphism. Dually, $(A/I)^* \to A^*$ is a Hopf subalgebra.

Definition 15.38. For each $k \geq 0$, let $Q_k = Sq^{(0,\dots,0,1)}$ (k zeroes) denote the Milnor basis element in $\mathscr{A}$ that is dual to ξ_{k+1}, in degree $2^{k+1} - 1$.

These classes are known as the Milnor primitives; see the next lemma. By the sample calculation above, these classes can also be recursively defined by $Q_0 = Sq^1$ and $[Sq^{2^k}, Q_{k-1}] = Q_k$ for all $k \geq 1$. The first few Milnor primitives are:

$$\begin{aligned}
Q_0 &= Sq^1 \\
Q_1 &= Sq^{(0,1)} = Sq^3 + Sq^2 Sq^1 \\
Q_2 &= Sq^{(0,0,1)} = Sq^7 + Sq^6 Sq^1 + Sq^5 Sq^2 + Sq^4 Sq^2 Sq^1 \\
Q_3 &= Sq^{(0,0,0,1)}
\end{aligned}$$

Lemma 15.39. *The Q_k are primitive elements, and they generate an exterior Hopf subalgebra*

$$E = E(Q_k \mid k \geq 0) \subset \mathscr{A}$$

of the Steenrod algebra. In symbols, $\psi(Q_k) = Q_k \otimes 1 + 1 \otimes Q_k$, $Q_k^2 = 0$ and $Q_i Q_j = Q_j Q_i$ for all $i, j, k \geq 0$. The conjugation is trivial: $\chi(Q_k) = Q_k$.

Proof. First note that if $A = E(\xi)$ is the primitively generated exterior algebra on one generator, viewed as a bicommutative Hopf algebra, then the dual Hopf algebra $A^* = E(Q)$ is also a primitively generated exterior algebra, with 1 and Q dual to 1 and ξ, respectively.

Now consider the quotient algebra $E_* = \mathscr{A}_*/(\xi_k^2 \mid k \geq 1)$ of the dual Steenrod algebra. The ideal $J = (\xi_k^2 \mid k \geq 1) \subset \mathscr{A}_*$ is a Hopf ideal, since $\psi(\xi_k^2) = \sum_{i+j=k} \xi_i^{2^{j+1}} \otimes \xi_j^2$ lies in $\mathscr{A}_* \otimes J + J \otimes \mathscr{A}_*$, and $\epsilon(\xi_k^2) = 0$. Hence $\mathscr{A}_* \to E_*$ is a Hopf algebra surjection. The generators ξ_k are primitive in E_*, since

$$\psi(\xi_k) \equiv \xi_k \otimes 1 + 1 \otimes \xi_k$$

modulo $A \otimes J + J \otimes A$. It follows that $\chi(\xi_k) \equiv \xi_k$ modulo J. Hence $E_* = E(\xi_k \mid k \geq 1) = \bigotimes_{k\geq 1} E(\xi_k)$ is a primitively generated exterior Hopf algebra.

Passing to duals, we have a Hopf algebra injection $E = (E_*)^* \to \mathscr{A}$. Here $E = E(Q_k \mid k \geq 0) = \bigotimes_{k\geq 0} E(Q_k)$ is also primitively generated, with Q_k dual to ξ_{k+1} in the monomial basis for E_*. Since I is generated by monomials, it follows that the inclusion maps $Q_k \in E$ to $Q_k \in \mathscr{A}$. Hence the Q_k are primitive in $\mathscr{A}$. □

Lemma 15.40. *$Q(\mathscr{A}) \cong \mathbb{F}_2\{Sq^{2^i} \mid i \geq 0\}$, $P(\mathscr{A}_*) \cong \mathbb{F}_2\{\xi_1^{2^i} \mid i \geq 0\}$, $Q(\mathscr{A}_*) \cong \mathbb{F}_2\{\xi_{i+1} \mid i \geq 0\}$ and $P(\mathscr{A}) \cong \mathbb{F}_2\{Q_i \mid i \geq 0\}$.*

Definition 15.41. For each $n \geq 0$, let $E(n) = E(Q_0, \dots, Q_n) \subset \mathscr{A}$ be the exterior subalgebra generated by the Milnor primitives $Q_0, \dots, Q_n$. It is a Hopf subalgebra with conjugation. The dual of $E(n)$ is the quotient Hopf algebra $E(n)_* = \mathscr{A}_*/J(n)$ of $\mathscr{A}_*$ by the Hopf ideal

$$J(n) = (\xi_1^2, \dots, \xi_{n+1}^2, \xi_k \mid k \geq n+2)\,.$$

Definition 15.42. For each $n \geq 0$, let $A(n) = \langle Sq^1, \dots, Sq^{2^n}\rangle \subset \mathscr{A}$ be the subalgebra generated by the Steenrod squares $Sq^1, \dots, Sq^{2^n}$. It is a Hopf subalgebra with conjugation.

Lemma 15.43. *The dual of $A(n)$ is the quotient Hopf algebra $A(n)_* = \mathscr{A}_*/I(n)$ of $\mathscr{A}_*$ by the Hopf ideal*

$$I(n) = (\xi_1^{2^{n+1}}, \xi_2^{2^n}, \dots, \xi_n^{2^2}, \xi_{n+1}^2, \xi_k \mid k \geq n+2)\,.$$

Proof. The ideal $I(n)$ is generated by the classes $\xi_s^{2^t}$ with $s \geq 1$ and $s + t \geq n+2$. It is a Hopf ideal since

$$\psi(\xi_s^{2^t}) = \sum_{i+j=s} \xi_i^{2^{j+t}} \otimes \xi_j^{2^t}$$

is a sum of terms in $\mathscr{A} \otimes I(n)$ (for $i = 0$) and in $I(n) \otimes \mathscr{A}$ (for $1 \leq i \leq s$). Hence $\mathscr{A}_*/I(n)$ is a finite commutative Hopf algebra, and the dual is a finite cocommutative Hopf subalgebra of $\mathscr{A}$.

We claim that $Sq^k \in A(n)$ for all $0 \leq k < 2^{n+1}$. Equivalently, we must prove that $\langle Sq^k, \xi\rangle = 0$ for all $\xi \in I(n)$. By induction, we may assume that this holds for all smaller values of k. The ideal $I(n)$ is additively generated by products $\xi_s^{2^t} \cdot \xi^R$ with $s \geq 1$ and $s + t \geq n+2$, and

$$\langle Sq^k, \xi_s^{2^t} \cdot \xi^R\rangle = \langle Sq^k, \phi(\xi_s^{2^t} \otimes \xi^R)\rangle = \langle \psi(Sq^k), \xi_s^{2^t} \otimes \xi^R\rangle = \sum_{i+j=k} \langle Sq^i, \xi_s^{2^t}\rangle\langle Sq^j, \xi^R\rangle\,.$$

By the inductive hypothesis, this equals $\langle Sq^k, \xi_s^{2^t}\rangle \cdot \langle 1, \xi^R\rangle$, which is 0 for $k < 2^{n+1}$ since $|\xi_s^{2^t}| \geq 2^{n+1}$ when $s \geq 1$ and $s + t \geq n+2$. ((It remains to prove that the Sq^k for $k \leq 2^n$, or for $k < 2^{n+1}$, generate all of the dual of $A(n)_*$.)) □

Corollary 15.44. *$\mathscr{A} = \operatorname{colim}_{n\geq 0} A(n)$ is a countable union of finite algebras. Hence each element in positive degree of $\mathscr{A}$ is nilpotent.*

Remark 15.45. Steenrod and Epstein (1962) write $\mathscr{A}_h$ for our $A(h+1)$. Adams (Math. Proc. Camb. Phil. Soc., 1966) writes A_r for our $A(r)$. Clearly $E(0) = A(0)$, and $E(n) \subset A(n)$ for $n \geq 1$. This can also be seen from the inclusion $I(n) \subset J(n)$.

((Write $P_s^t = Sq^{(0,\dots,0,2^t)}$ for the dual of $\xi_s^{2^t}$, so that $P_1^t = Sq^{2^t}$ and $P_{s+1}^0 = Q_s$? Review Adams–Margolis classification of Hopf ideals in $\mathscr{A}_*$ and Hopf subalgebras of $\mathscr{A}$, in terms of profile functions.))

15.6. **Spectral realizations.**

Definition 15.46. Brown and Peterson (Topology, 1966) construct a spectrum BP such that $H^*(BP) \cong \mathscr{A}//E$ as an $\mathscr{A}$-module. Johnson and Wilson (Topology, 1973) construct spectra $BP\langle n\rangle$ such that $H^*(BP\langle n\rangle) \cong \mathscr{A}//E(n)$, for each $n \geq 0$. As a convention, one may define $BP\langle -1\rangle = H$.

The connective cover $k(n)$ of the n-th Morava K-theory spectrum $K(n)$ has cohomology $H^*(k(n)) \cong \mathscr{A}//E(Q_n)$, for each $n \geq 1$. By convention, $k(0) = H\mathbb{Z}_{(2)}$ and $K(0) = H\mathbb{Q}$.

Remark 15.47. Baker and Jeanneret (HHA, 2002), using methods of Lazarev (K-Theory, 2001), show that there is a diagram

$$BP \to \cdots \to BP\langle n\rangle \to \cdots \to BP\langle 0\rangle \to H$$

of S-algebras, or equivalently, of A_∞ ring spectra, inducing the surjections

$$\mathscr{A} \to \mathscr{A}//E(0) \to \cdots \to \mathscr{A}//E(n) \to \cdots \to \mathscr{A}$$

in cohomology. Naumann and Lawson (J. Topology, 2011) prove (for $p = 2$ only) that $BP\langle 2\rangle$ can be realized as a commutative S-algebra, or equivalently as an E_∞ ring spectrum, like the realizations $BP\langle 0\rangle^\wedge_2 \simeq H\mathbb{Z}_2$ and $BP\langle 1\rangle^\wedge_2 \simeq ku^\wedge_2$. It is an open problem whether BP can be realized as a commutative S-algebra.

Baas and Madsen (Math. Scand., 1972) realize $k(n)$. Angeltveit (Compos. Math., 2011) proves that $K(n)$ has a unique S-algebra structure. For $n = 1$ (and $p = 2$) one can take $k(1) = ku/2$ and $K(1) = KU/2$. None of the $k(n)$ for $n \geq 1$ admit commutative S-algebra structures, since the map $k(n) \to H$ induces a homomorphism $H_*(k(n)) \to \mathscr{A}_*$ that cannot commute with the Dyer–Lashof operations in the target.

Proposition 15.48. *The Adams spectral sequence for BP collapses at the E_2-term*

$$E_2^{*,*} \cong \mathrm{Ext}_E^{*,*}(\mathbb{F}_2, \mathbb{F}_2) \cong P(v_k \mid k \geq 0)$$

to the abutment

$$\pi_*(BP^\wedge_2) \cong \mathbb{Z}_2[v_k \mid k \geq 1]\,,$$

where v_k in degree $2^{k+1}-2$ is detected in $E_\infty^{1,2^{k+1}-1}$ by the dual of $Q_k \in E$.

Similarly, the Adams spectral sequence for $BP\langle n\rangle$ collapses at

$$E_2^{*,*} \cong \mathrm{Ext}_{E(n)}^{*,*}(\mathbb{F}_2, \mathbb{F}_2) \cong P(v_0, \dots, v_n)$$

to the abutment

$$\pi_*(BP\langle n\rangle) = \mathbb{Z}_2[v_1, \dots, v_n]\,,$$

and the Adams spectral sequence for $k(n)$ collapses at

$$E_2^{*,*} \cong \mathrm{Ext}_{E(Q_n)}^{*,*}(\mathbb{F}_2, \mathbb{F}_2) \cong P(v_n)$$

to the abutment

$$\pi_*(k(n)) = \mathbb{F}_2[v_n]\,.$$

Proof. The E_2-term can be computed using change-of-rings:

$$\mathrm{Ext}_{\mathscr{A}}^{*,*}(H^*(BP), \mathbb{F}_2) \cong \mathrm{Ext}_{\mathscr{A}}^{*,*}(\mathscr{A}//E, \mathbb{F}_2) \cong \mathrm{Ext}_E^{*,*}(\mathbb{F}_2, \mathbb{F}_2) \cong P(v_k \mid k \geq 0)$$

where v_k is dual to the indecomposable $Q_k \in E$. In particular, $v_0 = h_0$ is dual to $Q_0 = Sq^1$. Since the E_2-term is concentrated in even total degrees, there is no room for differentials. There is also no room for other multiplicative extensions than the h_0-towers, since $\mathbb{Z}_2[v_k \mid k \geq 1]$ is free as a graded commutative algebra. ((This presumes that $\pi_*(BP)$ is commutative.)) □

Remark 15.49. Let MU be the complex bordism spectrum. Milnor (Ann. Math., 1960) and Novikov ((ref?)) shows that $H^*(MU)$ is a direct sum of suspensions of copies of $H^*(BP) = \mathscr{A}//E$. Brown and Peterson (Topology, 1966) showed that $MU_{(p)}$ splits as a wedge sum of suspensions of BP. One finds that $\pi_*(MU) \cong \mathbb{Z}[x_k \mid k \geq 1]$ with $|x_k| = 2k$. Quillen (Bull. Amer. Math. Soc., 1969) relates $\pi_*(MU)$ to formal group laws, in such a way that $\pi_*(BP)$ corresponds to p-typical formal group laws. The introduction of spectra like $BP\langle n\rangle$, $E(n)$, $k(n)$ and $K(n)$ is then motivated by the classification of formal group laws according to height, which in turn leads to the chromatic perspective on stable homotopy theory, which seeks to organize the homotopy groups of S and related spectra in periodic families of varying wave-lengths.

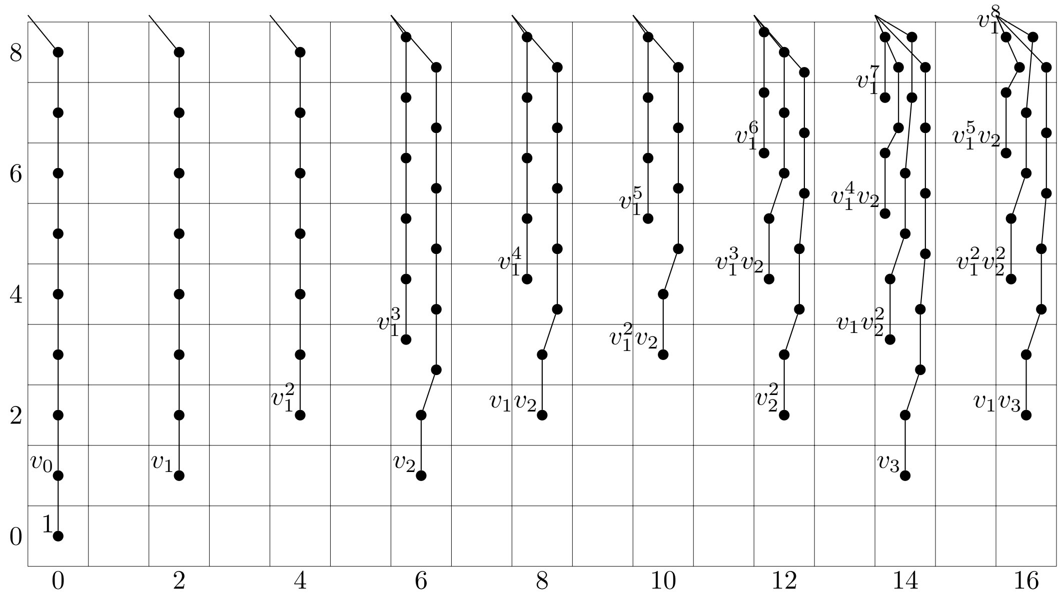


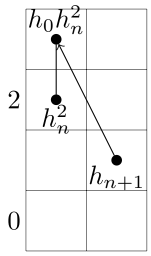
FIGURE 30. Adams spectral sequence for BP

Remark 15.50. Starting with the Hopkins–Miller obstruction theory for A_∞ ring structures, continued by Goerss–Hopkins–Miller and Lurie for E_∞ ring structures, Hopkins and Mahowald (preprint, 1994) produce a connective E_∞ ring spectrum tmf with $H^*(tmf) \cong \mathscr{A}//A(2)$. We have already discussed the realizations $H^*(ko) \cong \mathscr{A}//A(1)$ and $H^*(H\mathbb{Z}) \cong \mathscr{A}//A(0)$. (The Davis–Mahowald proof of the non-realizability of $\mathscr{A}//A(2)$ (Amer. J. Math., 1982) contains an error.)

There is no spectrum with cohomology $H^*(X) \cong \mathscr{A}//A(n)$ for $n \geq 3$, since the unit map $S \to X$ would induce a map of Adams spectral sequences

$$E_2^{*,*}(S) = \mathrm{Ext}_{\mathscr{A}}^{*,*}(\mathbb{F}_2, \mathbb{F}_2) \longrightarrow \mathrm{Ext}_{A(n)}^{*,*}(\mathbb{F}_2, \mathbb{F}_2) = E_2^{*,*}(X)$$

mapping $h_n \mapsto h_n$ and $h_{n+1} \mapsto 0$. This contradicts the Adams differential $d_2(h_{n+1}) = h_0 h_n^2$, since $h_0 h_n^2 \neq 0$ on the right hand side for $n \geq 3$. ((Elaborate?))



(($B_* = A_*/(\xi_1^4, \xi_2^2, \xi_3^2, \xi_4, ...)$ has dual $B = A(1) \otimes E(Q_2)$ and Ext_B is $Ext_{A(1)} \otimes P(v_2)$.))

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