ISBN 82-553-0929-2

Research Report in Mechanics, No. 94-1, 1994

# Approximate Effective Length Factors for Columns with Positive and Negative End Restraints

**Jostein Hellesland**

Professor, Mechanics Division, Department of Mathematics,
University of Oslo, P. O. Box 1053 – Blindern, N-0316 Oslo, Norway

## ABSTRACT

In approximate system instability analysis it is common practice to simplify by considering isolated members with end restraints that to some extent reflect the interaction with the surrounding structure. Buckling modes for such isolated members have traditionally been limited to modes that can be associated with positive end restraints. A wider range of realistic buckling modes, that also imply negative end restraints, are reviewed and discussed. Further, approximate formulas for effective length factors and inflection point locations for unbraced (free-sway) and fully braced compression members are developed. The range of applicability of the formulas are identified by comparisons with exact results for a reasonable wide range of positive and negative restraints .

## KEYWORDS

Stability, Buckling, Columns, Compression Members, Effective Length, Formulas, Critical Load, Frames

# Content

# SUMMARY AND CONCLUSIONS

In approximate system instability analysis it is common practice to simplify by considering isolated members with end restraints that to some extent reflect the interaction with the surrounding structure. Buckling modes for such isolated members have traditionally been limited to modes that can be associated with positive end restraints. A wider range of realistic buckling modes, that also imply negative end restraints, are reviewed and discussed. Further, approximate formulas for effective length factors, $\beta$, and inflection point locations, $L_A$ and $L_B$, are derived for unbraced (free-sway) and fully braced compression members. The applicability of the formulas are verified by comparison to exact results for a reasonable wide range of positive and negative restraints .

The formulas are expressed in terms of end restraints defined by so-called *degree of rotational fixity factors*, $R$, that reflects the degree of which the member ends are fixed against rotation. For a clamped end (100 % fixity), $R = 1$. For a pinned end (zero fixity), $R = 0$. These factors may also become negative, and even greater than 1.0 in some cases. The formula for the unbraced case, and for one of 5 formulas for the braced case, are given below.

**Degree of rotational fixity**:

$$R_i = \frac{1}{1 + c\frac{EI/L}{k_i}} \quad \left( = \frac{1}{1 + \frac{c}{b_o}G_i} \right) \qquad i = A, B$$

**Unbraced member**, $c = 2.4$ (and $c/b_o = 0.4$):

$$\beta = \frac{2\sqrt{R_A + R_B - R_A R_B}}{R_A + R_B} \quad ; \quad \frac{L_i}{L} = \frac{R_i}{R_A + R_B}$$

For $1 \geq R \geq 0$ , the predictions are accurate within about 0 to +2% of the exact results. In the wide range defined by $1.25 > R > -0.4$, the accuracy will be within about -3 to +4%, and generally well within these percentages.

**Braced member**, $c = 4.8$ (and $c/b_o = 2.4$):

$$\beta = \frac{2}{2 + 1.1R_{min} + 0.9R_{max}} \quad ; \quad \frac{L_i}{L} = (1 - \beta)\frac{R_i}{R_A + R_B}$$

$R_{min}$ and $R_{max}$ are the algebraically smallest and largest of $R_A$ and $R_B$, respectively. For $1 \geq R \geq 0$, the accuracy is within about –1.5 and +1% of exact results. In the wider range of $1 \geq R > -0.5$ , the accuracy is within –1.5 and +5%. Braced members with $R > 1.0$ is not considered too realistic, and have not been evaluated in any detail. The difference in the $c$–coefficients for the braced and the unbraced case is physically motivated as the formula for the braced member is based on the same physical model as that used for the unbraced one. Operating with two different $c$–coefficient is not a major inconvenience. In particular since the end restraint assessment normally will be different in the two cases anyhow. For alternative formulations, not included in this summary, see the full text.

# 1 INTRODUCTION

## 1.1 General

In design of structures that include slender compression members, it is necessary to consider stability and second order load effects. In such contexts, critical loads and corresponding effective lengths of compression members are often important parameters. In order to determine these without having to resort to a full system instability analysis using computer programs etc., it is common practice to introduce simplifications regarding end restraints so as to allow a framed compression member (column, strut, beam-column) to be considered in isolation. In such an isolated member analysis there is two major tasks.

1. The first, and most complicated one in a general case, is to make an appropriate assessment of the end restraints of the framed member that justifies it being considered in isolation. At present, the most common approach for assessing end restraints of a framed compression member in a regular structure is an approach that is often called the G-factor method. In its conventional (original) form, this approach has some shortcomings, including an implied interaction between members that is conceptually wrong. This and other aspects of the method is reviewed and discussed in detail in Hellesland (1992a) and will not be addressed further in this report. In a later study, end restraint assessments and application to special cases will be considered.

2. The second task, and the easier one, is to determine the effective length (and critical load) for *given* end restraints. For this purpose, a number of aids are available that help to simplify the process. Formulas and diagrams for a great number of special cases have been gathered and made available in various handbooks etc. (Bygg 1971, Petersen 1982, CRC of Japan; etc.). Among useful aids for isolated members, which is of main interest here, are the well known effective length nomographs or "alignment charts" (Galambos 1968, Johnston 1976) and similar diagrams (Eurocode 3 (CEN 1992) ; AS 4100 (SA 1990); etc.) for fully braced and unbraced members . Also, a number of reasonably accurate, and reasonably simple, effective length formulas are available for either braced (Newmark 1949; Burheim 1968) or unbraced members ( Mekonnen 1987), or for both (French Design Rules (Dumonteil 1992); Eurocode 3 (CEN 1992); Duan et al 1993; etc.). A review of available effective length formulas is given in Hellesland (1994).

A limitation of most of these aids is that they were developed for members with positive end restraints, and are primarily applicable to members with such restraints. For some of the aids, this limitation is to some extent due to the method of assessing end restraints of a framed compression member through the conventional G-factor method. In such cases, a generalized restraint definition would extend the applicability.

Only a few aids allow for negative end restraints. Effective length diagrams that cover both positive and negative restraints (in terms of G-factors) have been given by for Bridge and Fraser (1987). A similar diagram for an unbraced member is given by Bridge (1994). The author is not familiar with any evaluation of the applicability of existing approximate formulas in the negative restraint range. The latter is the main concern of a companion report (Hellesland 1994).

## 1.2 Object and scope

In the present study, a reasonably wide range of realistic buckling modes of compression members in a framework are identified and discussed. These include modes that also imply negative end restraints. With this as a basis, the main objective is to derive approximate formulas for effective length factors and inflection point locations that are valid over a wider range of restraints than what has been common. The study is limited to unbraced (free-sway) and fully braced compression members. It is considered important to document the applicability of the formulas by comparison to exact results. Particular attention will be paid to the restraint parameter formulation (fixity factors).

The fact that the critical load can be related directly to a readily identifiable and physical measure like the effective length, has made this length a very useful and important parameter in the analysis and design of compression members. It is believed that simple and reasonably accurate formulas for such lengths will continue to be of interest and useful in parallel with computational approaches. This has been the motivation for the present work.

# 2 BUCKLING AND EFFECTIVE LENGTHS

## 2.1 Isolated member analysis and restraints

The interaction between a compression member and the remainder of the frame or structure of which it is a part can be reflected by spring restraints at member ends. This is illustrated for member $AB$ in Fig. 1 by the rotational springs with restraint stiffnesses $k_A$ and $k_B$ at the member ends, denoted $A$ and $B$, and the lateral spring with stiffness $k_L$. The latter represents the lateral bracing of one end of the member relative to the other end.

If the lateral bracing has an infinite stiffness, there will be no relative lateral movement of the ends. Such members are often referred to as *fully braced* or *nonsway* members. If, on the other hand, the latter stiffness (bracing) is zero ($k_L = 0$), the member will have zero shear. One end of the member will then be free to sway (translate) relative to the other end. Such members are often termed *unbraced*, *zero-shear* or *free-sway* (or simply sway) members.

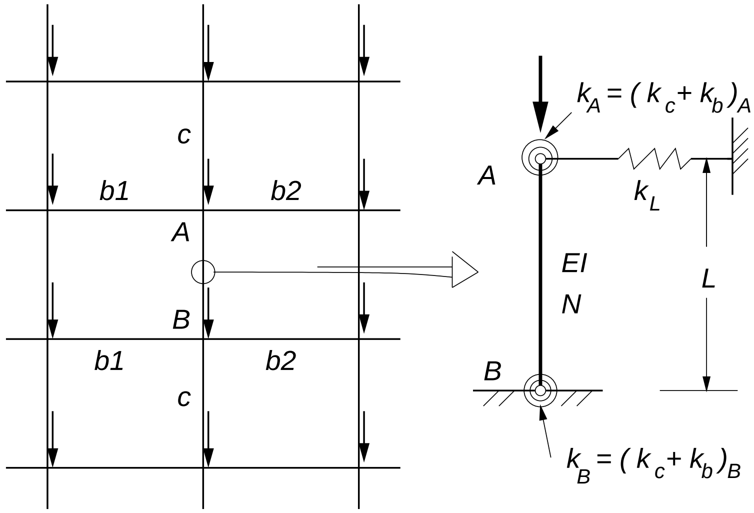


**Figure 1:** Isolated member analysis

Some intermediate or partial lateral bracing, and even negative bracing, will often be present in some columns even though the frames as such may not be braced. This is due to the lateral interaction between the various columns on the same level. This interaction is well taken care of in the so-called "$P-\Delta$" methods (see for instance Hellesland (1976), ACI (1989), AISC (1993), etc.). Even though the ideal unbraced (free-sway) member case may be rare, both in unbraced and partly braced frames, the free-sway critical load of such a member is still a very useful and important parameter in several contexts of approximate frame analysis, including in some formulations of the "$P-\Delta$" methods (Hellesland 1976, ACI 1989, etc.).

It is useful for the general understanding of the problems to be discussed, to dwell somewhat on the definition of end restraints.

In an isolated member analysis, the *rotational restraint stiffness* $k_i$ at end $i$ of a given member, is equal to the sum of the *rotational stiffnesses* of all the *other* members that frame into the considered member end. For the case of member $AB$ in Fig. 1,

$$k_i = (k_c + k_b)_i \qquad i = A, B \tag{1}$$

At end $A$, $k_c$ is the rotational stiffness of the column above joint $A$, and $k_b$ of the two beams framing into joint $A$. Similarly at joint $B$. These stiffnesses are also dependent on the respective members' far end boundary conditions, which again depend on members connected to those ends. Rotational stiffnesses $k_b$ of beams with normal far end conditions will normally be positive due negligible axial load levels. For columns and other compression members with significant axial load levels ($NL^2/EI$), $k_c$ may become negative. Even so, the sum (Eq. 1) will most often be positive in practical cases. However, if $k_c$ has a sufficiently large negative value, also the resulting restraint stiffness (Eq. 1) of the member considered will become negative. The significance of a negative restraint stiffness of the compression member considered is that it is a relatively stiff member and therefore capable of providing a positive contribution to the restraint stiffness of a neighbouring, more flexible compression member.

Rotational restraints at an end may alternatively be defined in terms of the end moments and rotations imposed at the end. For later discussions in this study, such a definition is most useful. End moments are equal to the rotational stiffness of the end restraints times the end rotations ($k\theta$). At ends with positive restraints, end moments will result in the opposite direction to that of the end rotation. Thus, they will tend to reduce, or restrain, the end rotation, and thereby strengthen the member. At ends with negative restraints, end moments will be inflicted in the same direction to that of the end rotation. In a manner, this represents a rotational disturbance that normally will tend to increase the end rotation and thereby to weaken the member. An exception will be discussed later (in conjunction with Fig. 2).

By defining end moments and end rotations as positive when they act in the same direction (e.g., clockwise), the rotational restraint stiffness is defined by

$$k_i = -\frac{M_i}{\theta_i} \qquad i = A, B \tag{2}$$

For single restrained members that do not interact with others, the end restraints will always be positive. It is only through interaction with other members in a framework that negative rotational restraints may be inflicted at the ends of a given member.

## 2.2 Selected buckling modes

Displacement curves showing selected buckling modes are illustrated schematically in Figs. 2 and 3 for unbraced and fully braced compression members, respectively. Some of these are rather well known (Fig. 2 $c-d$ and Fig. 3 $b-d$). The others are more uncommon, but can result in special cases.

The lines of axial thrust pass through the inflection points (points of contraflexure) of the displacement curves. For the unbraced members, Fig. 2, with zero shear, they stay vertical. For braced members with unequal end restraints, and thus unequal end moments, Fig. 3 , they will become inclined to the member axes due to the resulting shear forces.

For an elastic member of length $L$, constant cross-sectional bending stiffness $EI$ and constant axial compression load along the member, the buckling curves, relative to the axial thrust lines, have a sinusoidal variation along the member. The half-wave length of the curve is $L_e$, which is the so-called effective length, or buckling length, of the member. With the origin at an inflection point, the load eccentricity at an arbitrary section is then given by $v = v_o sin(\pi x/L_e)$ and the moment by $M = -EIv'' = EIv_o(\pi/L_e)^2 sin(\pi x/L_e)$. The critical compression load, $N_{\rm cr}$, often also denoted the buckling or bifurcation load, can then be

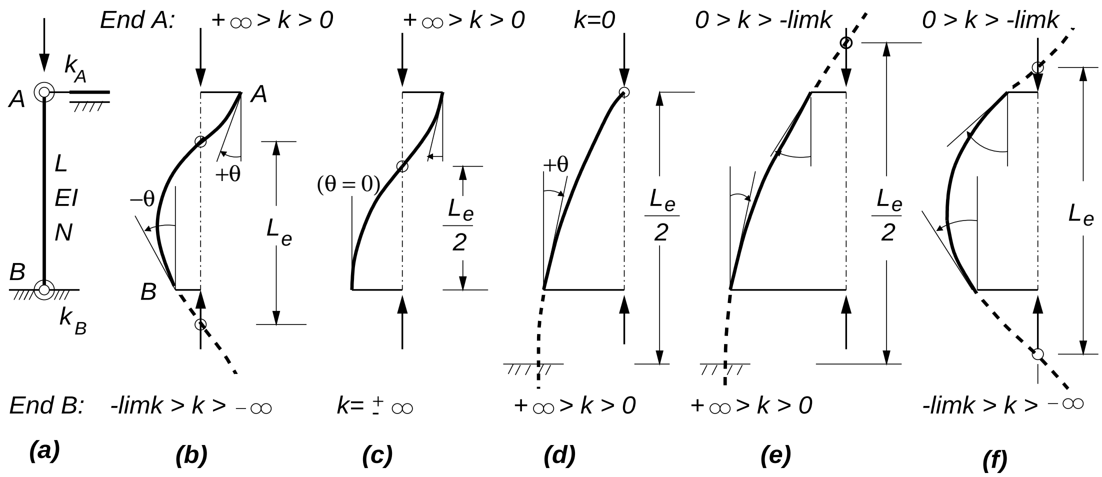


**Figure 2:** Selected buckling modes of an unbraced compression member

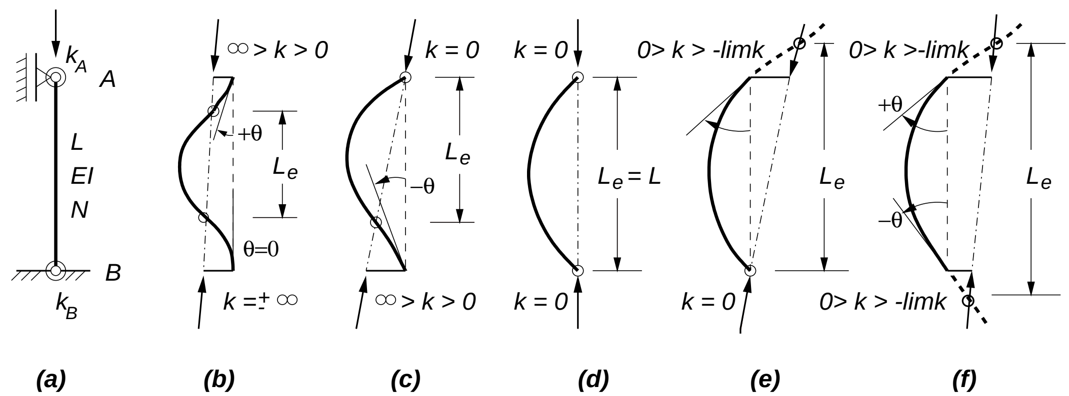


**Figure 3:** Selected buckling modes of a fully braced compression member

obtained from moment equilibrium ($M = N_{\rm cr}v$) as

$$N_{\rm cr} = \frac{M}{v} = \frac{\pi^2 EI}{(L_e)^2} \tag{3}$$

This is the well known critical load expression, in which the effective length is often expressed by

$$L_e = \beta L \tag{4}$$

where $\beta$ is the effective length factor of the member. The same expression can be used for members with varying axial load and varying stiffness. In such cases, $EI$ and $N_{\rm cr}$ are the values at a chosen reference section. Such cases are not considered in the present report.

The location of the member of length $L$ relative to the sine wave can be determined by the end moments the end restraints impose. It may be useful to dwell a bit more on this aspect.

Signs of the restraint stiffnesses for the cases illustrated in Figs. 2 and 3 are based on the definition in Eq. 2. For instance, for the case in Fig. 2$b$, $k$ must be positive at end $A$ since the moment and rotation at that end act in opposite directions (counterclockwise and clockwise, respectively). At end $B$, on the other hand, they act in the same direction (counterclockwise). As a consequence, $k$ becomes negative at that end. The restraint ranges will be discussed in later section.

The same results can be found from the sinusoidal properties of the buckling curve. In the unbraced case, an analytical expression for the restraint stiffness can readily be established from the sine curve given earlier. For instance, at end $B$ ($M_B = -M$, $\theta_B = v'$) of an unbraced member

$$k_B = -\frac{M_B}{\theta_B} = \frac{EIv''}{v'} = EI\frac{\pi}{L_e}tan\frac{\pi L_B}{L_e} \quad (5)$$

For $L_B/L_e$ less than or greater than one half, the resulting $k_B$ becomes positive or negative, respectively. This is the same as seen in Fig. 2.

For unbraced members, the sway buckling modes shown in Fig. 2 $c - d$, with inflection points between or at member ends, are well known and common in normal unbraced frames with reasonably stiff beams. Ends of columns in such frames have positive end restraints. The ends are either fully, partly or not fixed against rotations, and do not require any additional comments. Case $e$, with a negative restraint at end A, and thus with the inflection point outside the member end (on the theoretical continuation of the buckled shape), is representative of a lower story column in a frame with flexible (weak) beams. In such cases, the lower story columns, fixed at the base, are considerably stiffer than the columns above and are therefore required to provide restraint to the columns above. Due to this, negative restraints are inflicted upon the stiffer columns themselves. Such negative restraints can be said to represent rotational disturbances in that they increase the effective length. For all of these cases, the maximum point on the sine curve will be at or outside the member length.

More unusual are case $b$ and $f$, with large negative restraints at least at one end. They are not likely to result in a regular, unbraced frame. Compared to the fully fixed end $B$ in case $c$, that has a restraint stiffness (infinite ) sufficient to keep end $B$ from rotating clockwise, the restraint at end $B$ in case $b$ not only keeps the member end from rotating clockwise, but actually manages to inflict a counterclockwise rotation. In a sense, end $B$ is "more than fully fixed". As a result, the deformed member axis intersects the undeformed axis (vertical through $B$). The buckling curve will consequently have a maximum within the member length. The negative restraint tends in this case to reduce the effective length, and, thus, to rotationally stiffen the member. Thus, rather than to inflict a rotational disturbance, as in the case above, it has a rotational stabilizing effect.

The buckling shapes of braced members, Fig. 3, may have two (case $b$), one (case $c$) or no (case $d$, $e$ and $f$) inflection points between member ends. For the last two cases, with one or both end restraints being negative, effective lengths may become considerably greater than the member length ($\beta$ greater than 1). Such cases are representative of stiff (strong) columns in a frame attached to flexible beams and with flexible columns in the story above and below the considered column. The stiff column is required to restrain the more flexible ones, and will as a result of this have negative a end restraint inflicted upon itself.

The buckling curve of a braced member will, unlike that of an unbraced member, always have a maximum point within the member length. Higher buckling modes of a braced member, with two maxima within the member length, are not considered realistic and is not dealt with in this report.

## 2.3 Exact effective length factors

Exact effective length factors of an elastic compression member with given end restraints can be determined from the zero determinant condition of the stiffness matrix for the system (member plus restraints). For a member with constant stiffness and axial force along the length, zero determinant can be expressed by the transcendental equations below.

Unbraced member:

$$\frac{(\pi/\beta)^2 - \bar{k}_A\bar{k}_B}{\bar{k}_A + \bar{k}_B} = \frac{(\pi/\beta)}{tan(\pi/\beta)} \tag{6}$$

Braced member:

$$\frac{(\pi/\beta)^2}{\bar{k}_A\bar{k}} + (\frac{1}{\bar{k}_A} + \frac{1}{\bar{k}_B})(1 - \frac{(\pi/\beta)}{tan(\pi/\beta)}) + \frac{tan(\pi/2\beta)}{(\pi/2\beta)} = 1 \tag{7}$$

Here, $\bar{k}$ is defined by

$$\bar{k}_i = \frac{k_i}{(EI/L)} \qquad i = A, B \tag{8}$$

The parameter $\bar{k}$ (k with a bar) is a non-dimensional *restraint stiffness parameter*. Thus, with increasing $\bar{k}$–value, the degree of fixity at the restraint increases.

The transcendental equations above, given in terms of restraint stiffness parameters, are frequently given in terms of alternative and well known *restraint flexibility parameters* labelled $\psi$ or $G$ ( Galambos 1968, Johnston 1976, Chen and Lui 1991, etc.). The conventional $G$-factor definition, such as in the references above and in a number of codes (AISC Commentary 1993, ACI Commentary 1989, etc.), will always give positive $G$-factors, and thus imply positive end restraints. This is one of several limitations of the conventional $G$-factor definition (Hellesland 1992a).

A generalization of the conventional $G$-factor definition is given by

$$G_i = b_o\frac{(EI/L)}{k_i} \qquad i = A, B \tag{9}$$

This expression is not subject to any limitation. It may in principle become either positive or negative.

The coefficient $b_o$ can be considered to be a scaling factor, or it may be considered to be a reference (or datum) bending stiffness coefficient. It can in principle be chosen freely. With $b_o$=1, the $G$-factor definition above would simply be the spring flexibility $(1/k)$ nondimensionalized with respect to the $EI/L$ of the member itself. This would seem to be the most rational definition of the restraint flexibility. However, it has become quite common to adopt the scaling , or reference values, given by

$$b_o = 6 \qquad \text{for the unbraced case} \tag{10}$$

$$b_o = 2 \qquad \text{for the braced case}$$

For the sake of recognition, these values are also adopted here in presentations in terms of $G$-factors. It may be recalled that $b_o$=6 is the stiffness coefficient of flexural member (beam) with constant stiffness that is bent in antisymmetrical curvature, and $b_o$=2 of one in symmetrical curvature.

**Unbraced member**. Exact $\beta$-factors for the unbraced member case are given in Table A.1, in terms of $G$-factors, for a number of restraint combinations.

The buckling modes corresponding to the effective length factors within each of the four quadrants of Table A.1 can be identified in Fig. 2. The results in the upper left quadrant with positive/negative restraint combinations correspond to buckling curves of the type illustrated in the case $b$, the results in the upper right quadrant with negative/negative combinations to the last case $f$, in the lower left quadrant with positive/positive combinations to cases $c - d$ and in the lower right quadrant with positive/negative combinations to case $e$.

Since the $G$-factor is a restraint flexibility parameter, a small positive value reflects a stiffer (stronger) restraint than a larger positive value. Similarly for negative $G$-factors. A small negative value (e.g., $G_A = -7$, Table A.1) represents a stronger rotational disturbance than a larger negative value (e.g., $G_A = -20$, Table A.1). Consequently, effective length factors increase as the $G$-factors become "less negative", or in other words, with increasing rotational disturbance.

However, rather than to inflict a rotational disturbance, a strong negative restraint (e.g., $G_A = -0.5$) may have a rotational stabilizing effect as discussed previously in conjunction with end $B$ of the unbraced member in Fig. 2b. In such cases, $\beta$–values **less than 1.0** may result (!!).

As the stepped line in the lower right quadrant is approached from above, effective lengths increase towards infinity. The restraint combination at which this happens is derived below (Eq. 11)[1]. The "-lim$k$" in Fig. 2 refers to this infinity boundary. The table probably encompasses almost all results of practical interest for unbraced members. Results in the two upper quadrants are likely to be of

[1]If the stepped line is transgressed, there will be a switch to a higher buckling mode (giving $\beta$–values in the neighbourhood of 1.0). These values (below the stepped line) are not shown in Table A.1. If the upper left quadrant had been extended upwards for increasingly larger negative $G_B$– factors, these same $\beta$-values would have been obtained as the stepped line mentioned above would be reached from below. They are not of much practical interest.

interest only for description of segments of a braced member that each may can be considered unbraced (e.g., segments of a braced pin-ended member).

**Fully braced member**. Exact $\beta$-factors for the braced member case are given in Table A.2. The results in each of the four quadrants of Table A.2 can be identified in Fig. 3. Due to symmetry, the results in the lower left quadrant are not shown. The results in the upper left quadrant with positive/positive restraint combinations correspond to buckling curves of the types illustrated in cases $b-d$, in the upper right quadrant with negative/positive combinations to case $e$, and in the lower right quadrant with negative/negative combinations to case $f$. As the stepped line in the lower right quadrant is approached from above, effective lengths increase towards infinity. The restraint combination at which this happens is derived below (Eq. 13). The "-lim$k$" in Fig. 3 refers to this infinity boundary.

## 2.4 End restraint limits

The end restraint combinations giving infinite effective lengths, and critical loads equal to zero (Eq. 3), are of interest as they represent an outer limit. They can be found from the condition of the vanishing of the determinant of the first order stiffness matrix of the system (member plus end restraints).

In the simple case of a single member, they can also be obtained more directly by considering the physics of the case. For a member with no axial load, the total rotational stiffness at an end, for instance at end $A$, is equal to $k_{mA}+k_A$, where $k_{mA}$ is the first order rotational stiffness of the member itself at end $A$. Since $k_{mA}$ is different from zero, instability can only take place if the interaction with the rest of the structure of which the member is a part, reflected at $A$ through $k_A$, inflicts a restraint stiffness that is equal and opposite to $k_{mA}$.

**Unbraced member**. For a member with a free end at $A$ and a restraint stiffness $k_B$ at $B$, a moment $M_A$ at $A$ produces a first order rotation $\theta_A = M_A L/EI + M_A/k_B$. Thus, $k_{mA} = M_A/\theta_A$. The restraint stiffness combination giving infinite effective lengths for an unbraced member can then be obtained from $k_A = -k_{mA}$ as

$$\bar{k}_A = -\left(1+\frac{1}{\bar{k}_B}\right)^{-1} \qquad \text{or} \qquad \frac{1}{\bar{k}_A}+\frac{1}{\bar{k}_B} = -1 \tag{11}$$

where $\bar{k}_i$ is defined by Eq. 8. Alternatively, the limit may be written in terms of $G$-factors (with $b_o = 6$ for the unbraced case) as

$$G_A = -(6+G_B) \qquad \text{or} \qquad G_A+G_B = -6 \tag{12}$$

The limit can also readily be obtained from Eq. 6 by letting $\beta$ approach infinity.

When the $G$–factor sum is algebraically less than -6, or positive, effective lengths will be less than infinity. If one end is fully fixed, e.g. at end $B$ ($G_B = 0$), Eq. 12 yields $G_A = -6$. This is the smallest negative $G_A$-number (or the algebraically greatest negative value) that it is possible to have when end $B$ is fully fixed. For larger negative values at end $A$ (e.g., -15), effective lengths will be less than infinity.

**Fully braced member**. The restraint combination giving infinite effective lengths for a braced member can similarly be obtained directly by considering the stiffness of member with end restraints, or from the zero determinant condition, as

$$\bar{k}_A = -\frac{4(\bar{k}_B + 3)}{\bar{k}_B + 4} \qquad \text{or} \qquad \left(2 + \frac{\bar{k}_A}{2}\right)\left(2 + \frac{\bar{k}_B}{2}\right) = 1 \tag{13}$$

In terms $G$-factors (with $b_o = 2$ for the braced case), the similar relationships become

$$G_A = -\frac{1 + 2G_B}{2 + 3G_B} \qquad \text{or} \qquad \left(2 + \frac{1}{G_A}\right)\left(2 + \frac{1}{G_B}\right) = 1 \tag{14}$$

For instance, for a member fixed at end $B$ ($G_B = 0$), the smallest negative value at the other end is $G_A = -0.5$. The corresponding restraint stiffness (from Eq. 9) becomes as expected $k_A = -4EI/L$, which is just sufficient to cancel the member's own first order rotational stiffness ($4EI/L$).

The limits above are useful in the the general discussion, but are primarily of academic interest. Restraints of practical compression members will normally be well below the limits defining the infinity boundary.

# 3 APPROXIMATE UNBRACED BUCKLING

## 3.1 Displacement assumptions

In (linearized) second order theory, an elastic compression member will be in neutral equilibrium in the displaced position under the critical load corresponding to the first buckling mode. For an unbraced member $AB$, having zero shear, the equilibrium condition can be expressed for instance by

$$N_{cr} = -(M_A + M_B)/\Delta \tag{15}$$

where the parentheses gives the algebraic sum of the end moments, positive when they act in the clockwise direction, and where $\Delta$, positive when it cause a clockwise member axis rotation, is the total lateral displacement of one end relative to the other end.

Using this relationship, approximate critical loads can be established by assuming a displacement shape and calculating end moments that corresponds to a specified relative lateral displacement. One displacement assumption could be be the displacement shape due to a lateral end load due to the affinity between this displacement shape and the first buckling mode. Such assumptions have been made by several researchers (Mekonnen 1987, etc.). Alternatively, displacement assumptions believed to be better approximations of the buckled shape may be adopted. One such approach, leading to a rather simple effective length approximation, is presented here.

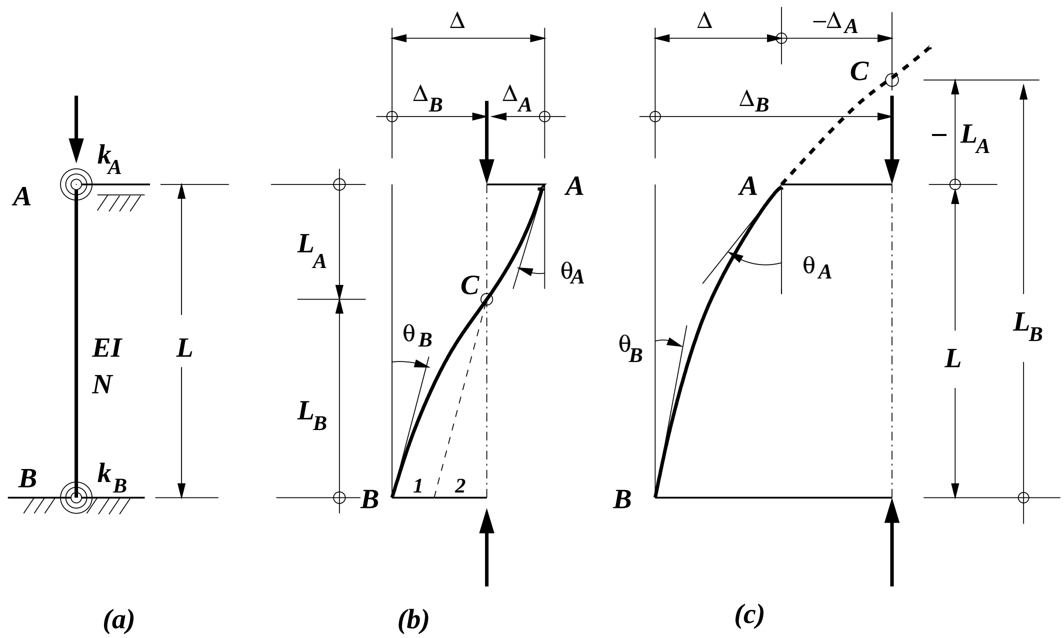


**Figure 4:** Buckling model for an unbraced compression member for two cases of end restraint combinations

Lateral displacements are shown schematically in Fig.4 for two cases of end restraints. The moment at an arbitrary section is equal to the axial force times the displacement relative to the thrust line. The real member can be represented by an upper and a lower cantilever member with the free ends at the inflection point (or point of contraflexure, $C$) that is common to both segments. The displacement of a segment is partly due to rotation at member ends, defined by $\theta_A = -M_A/k_A$ and $\theta_B = -M_B/k_B$, and partly due to bending (curvature) between ends. Here, both end rotations and end moments are defined positive when clockwise. Segment displacement, and segment lengths, are for convenience defined positive as shown in Fig.4 b). With this definition, a negative segment length would imply an inflection point outside the member length such as in Fig.4c.

The end displacement of each member segment relative to the inflection point can now be expressed by

$$\Delta_i = \frac{L_i^2}{a_i}\frac{(-M_i)}{EI} + \theta_i L_i = -\Big(\frac{L_i^2}{a_i} + \frac{EI}{k_i}L_i\Big)\frac{M_i}{EI} \qquad (16)$$

where $i = A$ or $B$. The first term in the equation represents the bending of the segment away from the tangent at the joint considered. The second term represents the segment displacement caused by the rotation at the joint. The value of the coefficient $a_i$ is dependent on which one of these two terms it is that is the dominating one.

To facilitate the discussion of which value(s) of $a_i$ to choose, it is useful to compare with the exact buckling curve. The buckled cantilever member segment will be part of half a sine wave of length $L_e$, and the exact $\Delta_i$ and $\theta_i$ for a given $L_e/L_i$ –ratio can readily be obtained. By substituting this $\theta_i$ into Eq. 16, and equating the resulting $\Delta_i$ to the exact $\Delta_i$, $a_i$ can be solved for and expressed by

$$\frac{1}{a_i} = (\frac{L_e}{\pi L_i})^2 - \frac{(L_e/\pi L_i)}{tan(\pi L_i/L_e)} \tag{17}$$

Its variation is shown in Fig. 5. For positive end restraints, which, for instance, always will be present at buckling of a free-standing cantilever, the correct value of $a_i$ is enclosed by $a = (\pi/2)^2$=2.47 and $a$=3. The former is due to a sinusoidal curvature distribution that has a maximum value at the base ( $L_e/L_i$=2). This is representative when the base is fully fixed. The latter "first order value" of $a$=3 corresponds to a linear curvature distribution and becomes increasingly representative as the base restraint becomes increasingly flexible (and the effective length approaches infinity). In cases with very flexible restraints, i.e. with large $L_e/L_i$–ratios (large $EI/L_ik_i$–values), the second term in Eq.16 becomes dominant. In such cases, results will not be too sensitive to the choice of $a_i$-values.

The greatest variation in $a_i$-values is seen to result for negative end restraints, Fig. 5, when the buckling curve maximum occurs within the segment length ( for $L_e/L_i$ less than 2), rather than at the restrained end. This may result when the member interacts with another member or structure. The lower member segment in Fig. 2b is an example of such a case. Although $a_i$-values theoretically can become quite small, as seen in the figure, practical values will not be much less than 2.4. For instance, for $L_e/L_i = 1.75$ ($L_i/L_e = 0.57$), which is believed to represent a rather extreme case in practice, $a_i = 2.3$ can be found.

In the CEB-Buckling Manual (1978), Eq.16 is used in several approximate second order analysis contexts with the approximation $a_i$=2.5. Here, values of $a_i$ closer to 2.4 is generally found to give better approximations.

## 3.2 Approximation

In order to arrive at reasonably simple expressions for the effective length and inflection point locations, it is necessary to simplify Eq.16. In the approximate treatment here, the same $a_i$-value will be chosen for both member segments. This is strictly not correct except in the case of a completely symmetrical member. Slope continuity at the inflection point will as a consequence not be completely satisfied. Furthermore, one and the same $a_i$-value will be chosen for the whole spectrum of restraints.

These simplifications, reflected by $a_A = a_B$ = constant, which will be denoted $c$, will limit the range of applicability of the developed expressions. However, for an appropriate value of the constant $c$ they may still be acceptable over a reasonably range of practical interest. In view of the previous discussion, it can be expected, in particular in the negative restraint range, that a value closer to 2.4-2.47 than to 3 would generally be the better choice. Based on a wide range

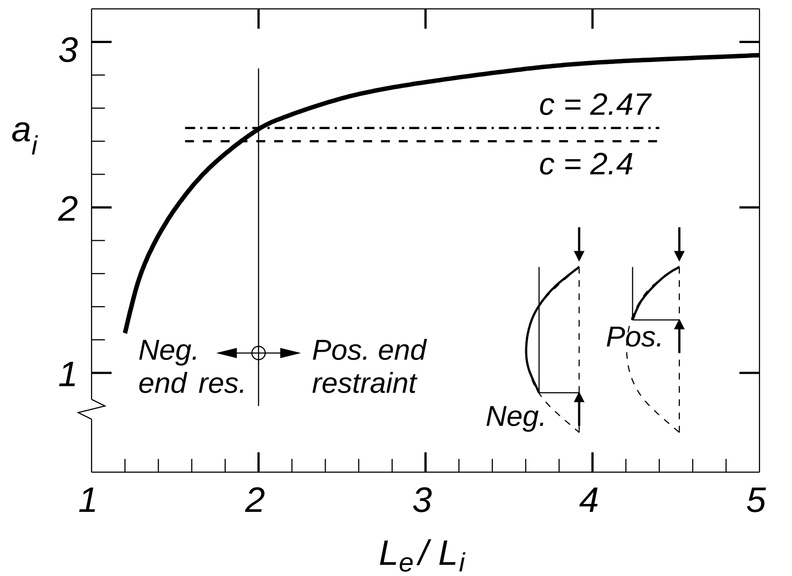


**Figure 5:** Values of the a-coefficient for a cantilever segment

of comparisons between exact results and predictions by derived expressions for inflection point locations and effective lengths, it is found that $c$– values of 2.4 and 2.5 give comparable accuracy. However, $c = 2.4$ is found to be somewhat better on the overall, and is recommended.

By equating the $\Delta_i$–expression given by Eq.16, with $a_A = a_B = c$, to the $\Delta_i$–expression obtained from moment equilibrium of each segment ($-M_i = N_{cr}\Delta_i$) as given by

$$\Delta_i = \frac{-M_i}{N_{cr}} = -\left(\frac{\beta L}{\pi}\right)^2 \frac{M_i}{EI} \tag{18}$$

the following equations,

$$\frac{1}{c}\,(L - L_B)^2 + \frac{EI}{k_A}\,(L - L_B) = \left(\frac{\beta L}{\pi}\right)^2 \tag{19}$$

$$\frac{1}{c}\,L_B^2 + \frac{EI}{k_B}\,L_B = \left(\frac{\beta L}{\pi}\right)^2 \tag{20}$$

are obtained for $i = A$ and $i = B$, respectively. In Eq. 19, $L_A$ has been replaced by $L - L_B$. From these two equations, the two unknowns $L_B$ and $\beta$ can be solved for.

## 3.3 Inflection points and degree of fixity factors

By subtracting Eq. 20 from Eq. 19, $\beta$ cancels out. So do square terms of $L_B$ due to the adopted assumption that $a_A = a_B = c$. Due to this, a rather simple expression results for the segment length $L_B$ . $L_A$ can be solved for in a similar way, or from $L_A = L - L_B$. Both segment lengths, and thus the inflection point location, may be written as

$$\frac{L_i}{L} = \frac{R_i}{R_A + R_B} \qquad i = A, B \tag{21}$$

Here, $R_A$ and $R_B$ are labelled **rotational degree of fixity factors** and defined by

$$R_i = \frac{\bar{k}_i}{\bar{k}_i + c} \qquad \text{or} \qquad R_i = \frac{1}{1 + c/\bar{k}_i} \qquad (c = 2.4) \tag{22}$$

where $i = A, B$, and where $\bar{k}$ is defined by Eq. 8. In terms of $G$-factors (Eq. 9 with $b_o = 6$ in the unbraced case), the definition becomes

$$R_i = \frac{1}{1 + (c/b_o)G_i} \qquad (c/b_o = 2.4/6 = 0.4) \tag{23}$$

These $R$-factors reflect the relative rotational degree of fixity provided by the restraints at the ends. For an end that is clamped (i.e., fully fixed, corresponding to 100% rotational fixity), $R = 1$, and for a pinned end (zero fixity), $R = 0$. For a sufficiently strong negative restraint (large negative $\bar{k}$, or small negative $G$), $R$ will become greater than 1. Such a "more than fully fixed" end has been discussed before in conjunction with end $B$ in Fig. 2b. For a small negative restraint stiffness (small negative $\bar{k}$, or large negative $G$), $R$ will become negative.

For the range of cases illustrated in Fig. 2, and Table A.1, $\bar{k}$ and $G$ become discontinuous, as restraints change from one to the other side of a fully fixed and a pinned condition, respectively. A commendable and advantageous property of the fixity factor $R$ is that it is continuous in the range of interest, which is believed to be approximately between 1.25 and -0.5.

It may be somewhat inconsistent to express a fixity factor in terms of a flexibility parameter ($G$) such as in Eq. 23. However, it is appropriate to include this form due to the widespread familiarity with and use of the $G$–factor. It should be noted that the $R$–factor is not dependent on the scaling, or reference, factor $b_o$ used in the $G$–definition (Eq. 9). In Eq. 23 the effect of the adopted $b_o$ correctly cancels out.

Eq. 21 has been compared with exact results for a reasonably wide range of restraints, and with values of $c$ of 2.4 or 2.5 (2.47). For $c = 2.4$, the accuracy was within $\pm$ 1% (and generally well within this range) for positive/positive end restraint combinations. Overall, a comparable, but slightly inferior, accuracy was obtained with $c = 2.5$.

## 3.4 Approximate effective length factor

The effective length factor can now be determined from either one of Eqs. 19 and 20. Substitution of $L_B/L$ (Eq. 21) into Eq. 20 yields

$$\beta = \frac{\pi}{\sqrt{c}} \frac{\sqrt{R_A + R_B - R_A R_B}}{R_A + R_B} \tag{24}$$

In the benchmark case of a cantilever member, free at one end and fixed at the other, e.g. at $B$, then $R_A = 0$, $R_B = 1$ and $\beta = \pi/\sqrt{c}$ result. Further, with the theoretically correct $c$-value for this case of $c = (\pi/2)^2$=2.47, as discussed above (Fig. 5), the theoretically correct $\beta$–prediction of 2 is obtained. However, if another $c$-value is adopted, the prediction will be somewhat in error. For this reason, in order to comply with a $\beta$–prediction of 2 for this benchmark case also when other $c$-values are assumed, the term $\pi/\sqrt{c}$ in Eq. 24 is rounded off to 2.

With this minor adjustment, the final effective length expression for an unbraced compression member becomes

$$\beta = \frac{2\sqrt{R_A + R_B - R_A R_B}}{R_A + R_B} \tag{25}$$

Effective length predictions using Eq. 25, and $R_i$ defined with $c$= 2.4, are compared to exact results in Fig. 6 and Table A.3a for various combinations of positive and negative end restraints.[2].

Results in the shaded area of Fig. 6 are obtained when end restraints are positive at both ends (positive/positive restraint combinations). For such cases, it is seen that the predictions (dashed lines) are slightly conservative (too large), but compare very well with exact results (full lines). For such restraint combinations ($1 > R > 0$, or $0 < G < +\infty$), the predictions are accurate within about 0 to +2% of the exact results, as can be seen from the detailed comparisons in the lower left quadrant of Table A.3a. This is considered very acceptable.

Eq. 25 provides good predictions also for a reasonably large range of combinations of positive/negative and negative/negative restraints, cfr. the upper left, the upper right and the lower right quadrant of Table A.3a. For restraints approximately in the range $1.25 > R > -0.4$ (corresponding to about $-0.5 < G < +\infty$ and $-\infty < G < -9$ ), the accuracy will be within about $\pm$ 3%. Again, this is considered very good over such a wide range of restraints. In this comparison, some cases with $\beta$-predictions in excess of about 4 were excluded. Cases with effective length factors above 3-4 are believed to be of little practical interest.

Predictions using Eq. 25 with $c$= 2.5 are compared to exact results in Table A.3b. Compared to the predictions with $c$= 2.4, use of $c$= 2.5 results in somewhat

[2]This equation in a somewhat different form, and with $c$= 2.5, was proposed by the author in 1988 for inclusion in NS 3473. In that form it was incorporated into the 1989-edition of the standard

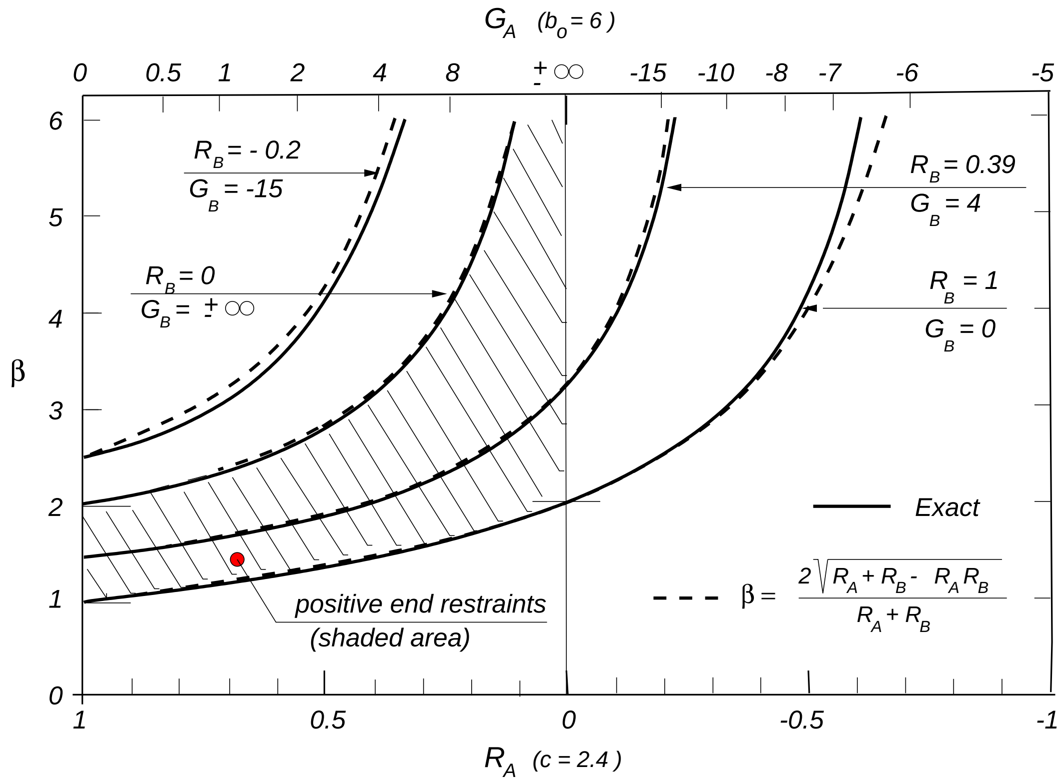


**Figure 6:** Unbraced effective length factor comparisons – Approximate vs. exact results

greater (more conservative) predictions (0 to +3%) for positive/ positive restraint combinations (lower left quadrant) and somewhat smaller predictions otherwise. The difference is not too big, but on the overall it is felt that $c$= 2.4 is to be preferred to $c$= 2.5.

Prediction errors increase as $\beta$ approaches infinity. This was to be expected, and can be seen by by comparing the limits at which effective length predictions by Eq. 25 approaches infinity to the theoretical limits established earlier (Eqs.11 and 12). Infinite $\beta$–predictions result for $R_A + R_B = 0$ in the denominator of Eq. 25, or, expressed in the similar forms previously given for the exact limits (Eqs.11 and 12), when

$$\frac{1}{\bar{k}_A} + \frac{1}{\bar{k}_B} = -\frac{1}{1.2} = -0.833 \qquad (-1) \tag{26}$$

or

$$G_A + G_B = -5 \qquad (-6) \tag{27}$$

Exact right hand values, given in parentheses, are 20 % larger than the approximate results for $c = 2.4$

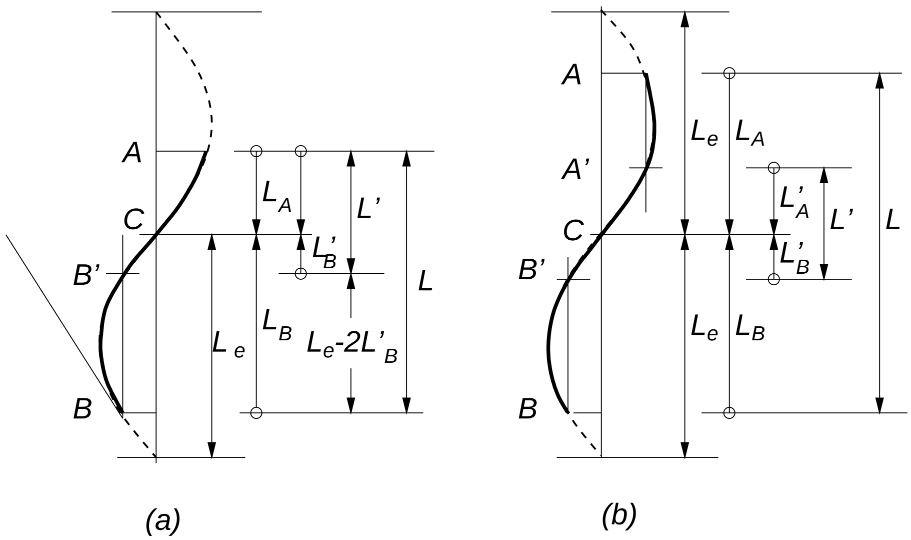


**Figure 7:** Definition of substitute member for modified application

## 3.5 Modified and extended use of unbraced buckling expression

The effective length factor formula above, Eq. 25, has a limited applicability in cases when an end has a negative restraint as well as a negative rotation, such as at joint $B$ in Fig. 2b or f. In such cases, the buckling curve will have a maximum within the member length. As the location of the joint moves away (downward in the figure) from the maximum location, the predictions become more and more inaccurate. This is due to the simple displacement assumption with a constant value for $c$, and has been discussed before.

Results in this range of restraint combinations are not believed to be of much relevance for practical structures. However, in the odd case it should be of interest with approximate $\beta$-predictions in this range, the developed $\beta$-expression may be used in a modified manner in order to obtain better results. The modified application will be described with reference to Fig. 7, where to half waves of the buckled shape is shown. First, consider Fig. 7a, where end $B$ is placed in the lower quarter of the wave length (negative restraint and negative (counter-clockwise) rotation). The vertical line through the member end $B$, intersects the buckled member at $B'$. In the exact case, with a sinusoidal buckling shape, the restraint stiffness, $k'_B$, at $B'$ will be equal, but opposite to the restraint stiffness $k_B$ at $B$. Thus, $k'_B = -k_B$. Whereas the displacement assumption may not be very adequate for the segment portion $B-C$, it is quite adequate for the segment portion $B'-C$. Then, rather than to base the effective length prediction on the real member $B-A$, of length $L$, it may be based on a substitute member defined by the portion $B'-A$, of length $L'$.

Next, consider Fig. 7b, where both ends have restraint conditions similar to that of end $B$ above. With the same reasoning as above, a substitute member may in this case be defined by the portion $B'-A'$.

The modified approach is described below, step by step.

**Step 1**. Assume a value for the ratio $L/L'$ of the real member to the substitute member length, and calculate modified restraint parameters (marked with a prime) that are applicable to the substitute member $B' - A$.

– In Fig. 7a, with modification at end $B$ only, $k'_B = -k_B$ and $k'_A = k_A$, and, in terms of $G$-factors, $G'_B = -G_B L/L'$ and $G'_A = G_A L/L'$. Here, $G_B$ and $G_A$ are the $G$-factors of the real member.

– In Fig. 7b, with modifications at both ends, $k'_B = -k_B$ and $k'_A = -k_A$, and, in terms of $G$-factors, $G'_B = -G_B L/L'$ and $G'_A = -G_A L/L'$.

Then,

$$R'_B = \frac{1}{(1 + 0.4G'_B)} \qquad R'_A = \frac{1}{(1 + 0.4G'_A)}$$

**Step 2**. Calculate $\beta'$ of the substitute member using Eq. 25, but with the $R$-values replaced by the modified $R'$-values. Then, $L_e = \beta' L'$

**Step 3**. Calculate an improved $L/L'$-ratio.

– For the case in Fig. 7a : $L/L' = (L' + L_e - 2L'_B)/L' = (1 + \beta' - 2L'_B/L')$ where $L'_B/L' = R'_B/(R'_A + R'_B)$

– For the case in Fig. 7b : $L/L' = (2L_e - L')/L' = (2\beta' - 1)$

**Step 4**. Calculate the effective length factor of the real member

$$\beta = \frac{L_e}{L} = \frac{\beta'}{(L/L')}$$

Repeat from step 1 with the $L/L'$-value in step 4 until the change in the resulting $\beta$-value in two successive iterations is small enough to neglect. Normally 2-3 iterations will be satisfactory.

**Sample calculation**: $G_A$=0, $G_B = -1.0$, $\beta_{exact} = 0.851$

Direct use of Eq. 25 gives $\beta = 0.750$ for this case. I.e. an error of -12%, which is not very accurate. Use of the modified approach explained with respect to Fig. 7a is demonstrated below. Only end $B$ is modified (sign change).

**1st** iteration:
Step 1. Set $L/L' = 1$, $G'_B = -G_B L/L' = 1.0$, $G'_A = G_A L/L' = 0$.
$R'_A = 0$, $R'_B = 1/(1 + 0.4 \cdot 1) = 0.714$.
Step 2. $\beta' = 2/(1 + R'_B) = 1.167$.

Step 3. $L'_B/L' = R'_B/(R'_A + R'_B) = 0.417, \quad L/L' = 1 + 1.167 - 2 \cdot 0.417 = 1.334$
Step 4. $\beta = L_e/L = \beta'/(L/L') = 0.875$ $(= 1.028\beta_{exact})$
The same steps are now repeated, each time with the $L/L'$–value in the previous iteration.
**2nd** iteration: $L/L' = 1.334$........... $\beta = 0.852$ $(= 1.001\beta_{exact})$
**3rd** iteration: $L/L' = 1.421$........... $\beta = 0.847$ $(= 0.995\beta_{exact})$
**4th** iteration: $L/L' = 1.443$........... $\beta = 0.845$ $(= 0.993\beta_{exact})$

The convergence is reasonably quick when $L/L'$ is used successively like here. Normally, 2-3 iterations will be more than sufficient. In this case, and in a number of other cases (about 15) investigated, results were normally within a fraction of a percent after 2 iterations.

# 4 APPROXIMATE BRACED BUCKLING

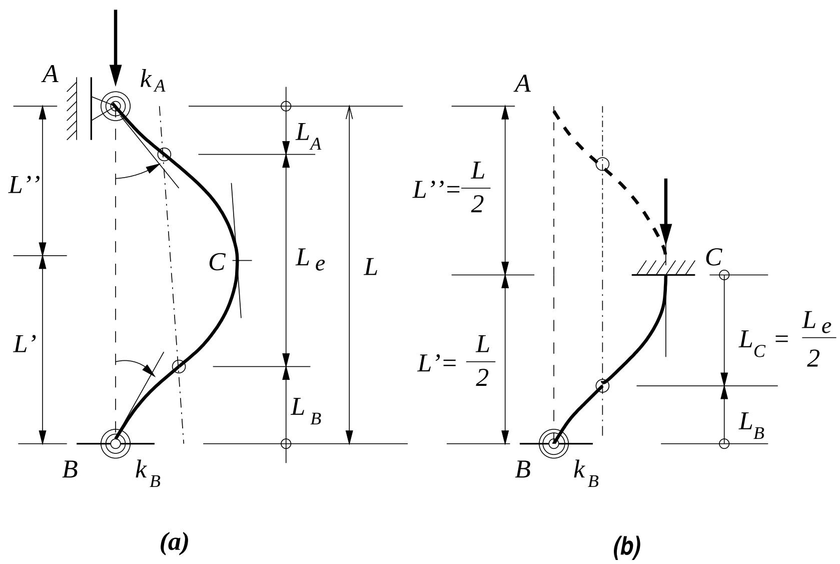


**Figure 8:** Buckling model for a braced compression member.
(a) General and (b) Symmetrical case

## 4.1 Braced effective length–Alternatives I

### Symmetrical case

A braced compression member with positive restraints at both ends will have two inflection points within the member length, as illustrated in Fig. 8 . Inflection point locations, measured from the respective member end to the nearest inflection point are again denoted $L_A$ and $L_B$ .

The symmetrical case with equal end restraints ( $k_A = k_B$) is considered first. In this case, $L_A = L_B$ and the tangent at mid-height is vertical. Then, either half of the member may be analyzed as an unbraced member of length $L' = L/2$ and that is restrained at one end and clamped at the other $(C)$. Using the expressions derived previously for an unbraced member, the fixity factors for the lower half $BC$ become $R_C$=1 and

$$R_B = \frac{1}{1 + 2.4(EI/L')/k_B} = \frac{1}{1 + 4.8(EI/L)/k_B} \tag{28}$$

The effective length may now be determined from $L_e = 2L_C$, where $L_C/L' = 1/(1 + R_B)$ according to Eq. 21. Then, with $L' = L/2$,

$$\beta = \frac{L_e}{L} = \frac{1}{1 + R_B} \tag{29}$$

This equation may be used as a base for generalization to a braced member with unequal restraints at the ends. Approaches to this end are considered below.

The real member in Fig. 8b experiences the same degree of fixity at $B$ irrespective of how it is analyzed. Thus, for the degree of fixity factors to have a physical significance, they should be the same for the full length braced member ( $AB$) and for the lower half $(BC)$ considered unbraced. Thus, it makes sense to retain the definition above, of $R$ with 4.8, rather than rewriting the $\beta$ – expression in terms of a $R$ – definition with the same $c$ used in the unbraced case $(c = 2.4)$. In the remainder of this section, $c = 4.8$ is adopted. In the next section, some alternative formulations/reformulations will be presented in terms of $c = 2.4$.

## Generalization using simple mean

By replacing $R_B$ in Eq. 29 with the mean value of the fixity factors at the two ends, i.e. by $R_m = (R_A + R_B)/2$, an approximate and very simple expression for the general braced member becomes

$$\beta = \frac{1}{1 + R_m} = \frac{2}{2 + R_A + R_B} \tag{30}$$

where, according to Eq. 28, $R_A$ and $R_B$ now are to be defined by

$$R_i = \frac{1}{1 + c/\bar{k}_i} \qquad (c = 4.8) \tag{31}$$

where, as before (Eq. 8), $\bar{k}_i = k_i/(EI/L)$ .

Predictions are shown in Fig. 9. As seen in the figure, the predictions are in reasonably good agreement with exact results over a rather large range of end restraints. Within the restraint limits

$$0.7 > R > -0.55 \qquad (\frac{1}{0.2} > \frac{1}{G} > -\frac{1}{1.2})$$

the accuracy of Eq. 30 is generally found to be well within +/− 2% of the exact results in Table A.2, and for $R$ between 0.45 ( $G = 0.5$) and -0.4 ( $G = -1.5$) within +/− 1%.

For details of selected comparisons, see Table A.4. As seen from the table, acceptable accuracy is also obtained in many cases for negative restraints outside the right hand side limits above. Outside the upper (left hand) limits, prediction errors increase to at most -5% (below exact results) in the case of a member clamped (fully fixed) at one end and flexibly restrained at the other. In a practical case, it will normally be difficult to obtain full fixity. In recognition of this, some codes recommend that effective length calculations be carried out with restraint stiffness that do not exceed an upper limit. For such cases, Eq. 30, which is attractively simple, is most suitable.

### Generalization using weighted mean

An almost equally simple and somewhat more conservative $\beta$–expression, that does not require an upper application limit as strict as above, can be obtained by replacing the simple mean in Eq. 30 by an empirically determined weighted mean $R_m = 0.55R_{min} + 0.45R_{max}$. This gives

$$\beta = \frac{2}{2 + 1.1R_{min} + 0.9R_{max}} \tag{32}$$

Here, $R_{min}$ is the algebraically smallest of $R_A$ and $R_B$, and $R_{max}$ is the algebraically largest. For instance, if $R_A = 0.1$ and $R_B = -0.4$, then $R_{min} = -0.4$ and $R_{max} = 0.1$.

For positive restraint combinations, $1 \geq R \geq 0$ ($0 \leq G \leq +\infty$), the accuracy of Eq. 32 is within about –1.5 and +1% of exact results, which is considered very adequate.

Further, the accuracy is within –1.5 and +5% subject to the limitation defined by

$$1 \geq R > -0.5 \qquad (\frac{1}{0} \geq \frac{1}{G} > -\frac{1}{1.25})$$

Predictions are compared to exact results in Fig. 9, and details of selected comparisons are given in Table A.5. For realistic braced structures, $R$ is not likely to exceed 1.0. The accuracy for fixity factors in excess of 1.0 for braced members has not been investigated in any detail.

Predictions have also been obtained for $c$= 2.5. They were found to give slightly greater predictions than use of $c$= 2.4 for combinations of positive restraints (within ± 1.5% of exact results) and somewhat smaller for combinations that include negative restraints. The difference is very small, however. On the overall, a slight preference is given to $c$= 2.4.

## Generalization using square root of product

Instead of replacing $R_B$ in Eq. 29 with the simple or weighted mean value of the fixity factors at the two ends, $1 + R_B$ may be replaced by the square root of the product $(1 + R_A)(1 + R_B)$. This gives the rather simple and attractive formulation defined by

$$\beta = \frac{1}{\sqrt{(1 + R_A)(1 + R_B)}} \tag{33}$$

As far as accuracy is concerned, this formula is comparable to the one above (Eq. 32) for positive restraint combinations, where it is accurate within -1 and +1.5% (Table A.6). However, for positive/negative combinations the accuracy is not as good. It is generally more conservative. Provided negative restraints are such that $R \geq -0.25$ ($G \leq -2.1$), the error will not exceed about +5%. For negative/negative combinations the accuracy is better, but still not as good as that of Eq. 32.

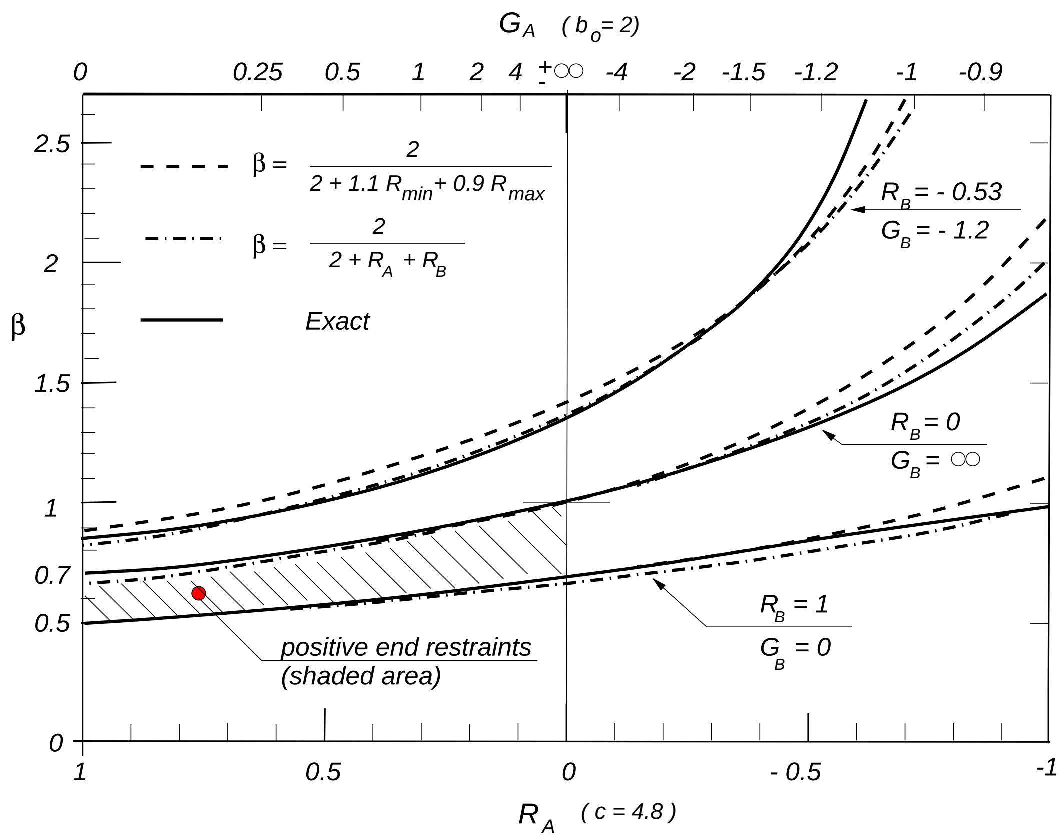


**Figure 9:** Braced effective length factor comparisons – Approximate vs. exact results

## 4.2 Inflection points

In the general case presented in Fig. 8a, $L_A + L_B = L(1-\beta)$. From this relation, the segment lengths $L_A$ and $L_B$ can be expressed by

$$\frac{L_i}{L} = (1-\beta)\,\frac{L_i}{L_A + L_B} \qquad i = A, B \tag{34}$$

Approximations of $L_A$ and $L_B$, for use in the right hand side ratio above, can be found from the unbraced member formula (Eq. 21) applied to each of the member portion $BC$ and $AC$ in Fig. 8a). However, in a more direct approach it is found by comparisons that $L_A$ and $L_B$ are approximately proportional with the rotational fixity factors defined by Eq. 31 with $c = 4.8$. Making use of this, Eq. 34 becomes

$$\frac{L_i}{L} = (1-\beta)\,\frac{R_i}{R_A + R_B} \qquad i = A, B \tag{35}$$

These lengths, that may be positive or negative, are directed from a member end (the origin) to the inflection point. A positive value (0-1) implies that the direction is from the considered end and towards the other end. A negative value implies that the direction is from the considered end and away from the other end. In the latter case the inflection point is in other words outside the member length (on the theoretical continuation of the buckling curve from the end considered).

In conjunction with exact $\beta$–factors, Eq. 35 is found to be accurate within $\pm$ 1% (and generally well within this range) for positive/positive end restraint combinations and $R$ defined with $c = 4.8$. Overall, a comparable accuracy was obtained with $c = 5.0$. When used in conjunction with approximate $\beta$–factors, the accuracy will also be affected by the accuracy of the $\beta$–prediction.

It has been argued that the twofold value of $c$ (4.8) used in the braced case as compared to the one in the unbraced case (2.4), is physically motivated. This is confirmed also with regard to Eq. 35. The accuracy of calculated inflection points using $R$–factors with $c = 2.4$ has been found to be considerably below that obtained with $c = 4.8$.

## 4.3 Braced effective length–Alternatives II

In the fixity factor definitions above, a $c$ –value resulted in the derivations for the braced member ( $c = 4.8$) that was two times the value used for the unbraced member ( $c = 2.4$). Since the braced case was derived using the results for the unbraced case, the twofold value in the braced case is physically motivated. This is a good reason for retaining the two different values. This was also discussed above in connection with the inflection point expressions, Eq. 35, and the $R$–factor as a best possible fixity measure. In the author's view, the inconvenience of using different $c$ –values in the unbraced and the braced case is minor.

Even so, alternative formulations are considered below in which the physical aspect of $R$ is relaxed in order to use the same value for $c$ used in the unbraced case , i.e. $c$= 2.4.

The relationship between fixity factors defined with $R$ equal to 2.4 and 4.8 becomes

$$R(c = 4.8) = \frac{R(c = 2.4)}{2 - R(c = 2.4)} \tag{36}$$

### Generalization using simple mean

Substitution of this $R(c = 4.8)$ into Eq. 32 does not result in a very attractive $\beta$–formulation. Instead, by way of the symmetrical case defined by Eq. 29, another and rather simple formulation is obtained. Substitution of $R(c = 4.8)$ into Eq. 29, yields

$$\beta = 1 - 0.5R_B(c = 2.4) \tag{37}$$

for the symmetrical case. In order to extend this expression to the unsymmetrical case, $R_B$ may be replaced by the simple mean or a weighted mean value.

The simple mean of the fixity factors, $R_m = (R_A + R_B)/2$, now defined with $c = 2.4$, gives

$$\beta = 1 - 0.5R_m = 1 - 0.25(R_A + R_B) \tag{38}$$

where, as before, $R_A$ and $R_B$ are defined by

$$R_i = \frac{1}{1 + c/\bar{k}_i} \qquad (c = 2.4) \tag{39}$$

For positive end restraint combinations, this equation is found to be accurate to within -0.2 and +7.1% of exact results. In this restraint range, the error increases with increasing difference between the end restraints at the two ends. Thus, for a member pinned at one end and clamped at the other, the effective length is overestimated by 7.1%. For positive/negative and negative/negative combinations the accuracy is acceptable in some cases, but is in general not very good.

### Generalization using weighted mean

The accuracy may be improved by use of an empirically determined weighted mean. The approximate weighted mean defined by $R_m = 0.4R_{min} + 0.58R_{max}$ has been found to give reasonable results. Then,

$$\beta = 1 - 0.5R_m = 1 - 0.2R_{min} - 0.29R_{max} \tag{40}$$

where, $R_{min}$ and $R_{max}$ is the algebraically smallest and greatest end restraint factor, respectively.

The accuracy of Eq. 40 is within about -1 and +2% for any combinations of positive end restraints (Table A.7), which is very acceptable. For positive/negative combinations, the accuracy is not as good. It is comparable, but generally slightly more conservative than found by Eq. 33. Provided negative restraints are such that $R \geq -0.25$ ($G \leq -4.2$), the error will not exceed about +5%. For negative/negative combinations the accuracy is better. For details, see Table A.7.

### Generalization using square root of product

Rather than to replace $1 - 0.5R_B$ in Eq. 37 by a simple or weighted mean value, it may be replaced by the square root of the product $(1 - 0.5R_A)(1 - 0.5R_B)$ in order to generalize, as discussed before. Thus,

$$\beta = \sqrt{(1 - 0.5R_A)(1 - 0.5R_B)} \qquad \text{or} \qquad \beta = \frac{1}{2}\sqrt{(2 - R_A)(2 - R_B)} \qquad (41)$$

A closer examination reveals that this is exactly the same equation that will be obtained if Eq. 36 is substituted into Eq. 33. This was to be expected since the rewriting in terms of $R_B(c = 2.4)$ for the symmetrical case took place before the generalization to unequal end restraints.

### Inflection points

Inflection point locations given in the form of Eq. 35 may still be used. However, as mentioned before, Eq. 35 is less accurate when used with $R$–factors defined with $c = 2.4$ rather than with $c = 4.8$.

## 4.4 Comments on braced effective length formulas

A total of 5 different effective length formulas have been presented for braced compression members, i.e., Eqs. 30, 32, 33, 38, 40, all derived with the same basic model as the base . The first 3 of these are defined with $R(c = 4.8)$ and the last 2 are defined with $R(c = 2.4)$. A 6th formula, Eq. 41, is simply a rewrite of Eq. 33 in terms of $R(c = 2.4)$.

Over various ranges of restraint combinations, all of the formulas give acceptable predictions. This is in particularly so for positive/positive restraint combinations. Of the various formulas, Eq. 32 is the one that is on the overall most accurate over the widest range of restraint combinations. It is recommended before the others if one formula should be selected for general use. Like the others, it is a simple formula and well suited for practical applications.

# REFERENCES

ACI. (1989). *"ACI 318-89, ACI 318R-89. Building code requirements for reinforced concrete, and Commentary"*. American Concrete Institute, Detroit, Mich.

AISC. (1993). *"Load and resistance factor design for structural steel buildings, and Commentary "*. American Institute of Steel Construction, Chicago, Ill.

Bridge, R. Q., and Fraser, D. J. (1986). "Improved G-factor method for evaluating effective lengths of columns". *Journal of Struct. Engrg.*. ASCE, Vol.**113**, No. 6, p. 1341-1356.

Bridge, R. Q. (1994). "In-plane buckling loads for members and frames". Ch. 2 in Lecture notes. Private communication.

Burheim, E. (1968). "Knekking av armerte betongsyler (Buckling of reinforced concrete columns)". Report, Institutt for betong, NTH, Trondheim. (in Norwegian).

*"Bygg 1A: Almänna grunder"*. (1971, and later editions). AB Byggmästarens Forlag, Stockholm, Sweden.

CEB. (1978). "CEB/FIP Design Manual on Buckling and Instability". *The Construction Press Ltd.*, Lancaster, England ( or CEB Bull. No. 123, dec. 1977).

CEN. (1991). "Eurocode 2, Design of concrete structures – part 1: General rules and rules for buildings". *European prestandard* ENV 1992–1–1, CEN, Brussels, Belgium.

CEN. (1992). "Eurocode 3, Design of steel structures – part 1.1: General rules and rules for buildings". *European prestandard* ENV 1993–1–1, CEN, Brussels, Belgium.

Chen, W.F., and Lui,E. M. (1991). *"Stability design of steel frames"*. Boca Raton ,CRC Press.

Column Research Committee of Japan (19??). *"Handbook of structural stability "*. Corona Publishing Company, Japan.

Duan, L., King, W.-S., and Chen, W. F. (1993). K-factor equation to alignment charts for column design. *ACI Structural Journal*, Vol.**90**, No. 3 (May-June), pp.242-248.

Dumonteil, P. (1992). Simple equations for effective length factors. *AISC Engineering Journal*, Vol.**29**, No. 3, p. 111-115.

Galambos, T. V. (1968). *"Structural members and frames"*. Prentice- Hall, Inc. New York, NY.

Hellesland, J. (1976). "Approximate second order analysis of unbraced frames". *Technical Report*, Dr. Ing. A. Aas-Jakobsen Ltd., Oslo, Norway, 43 pp.

Hellesland, J. (1992a). "Effective lengths – An appraisal of various restraint participation factors". *Research Report in Mechanics*, Mechanics Division, Dept. of Math., Univ. of Oslo, No. 92-4, 48 pp.

Hellesland, J. (1992b). "Effective lengths – Simplified system instability analysis". *Research Report in Mechanics*, Mechanics Division, Dept. of Math., Univ. of Oslo, No. 92-5, 38 pp.

Hellesland, J. (1994). "Review and evaluation of effective length formulas". *Research Report in Mechanics*, Mechanics Division, Dept. of Math., Univ. of Oslo, No. 94-2.

Johnston, B. G., ed. (1976). *"Guide to stability design criteria for metal structures"*. 3rd. ed. Wiley-Interscience, New York, NY.

Mekonnen, B. (1987). "Effective lengths and rigidity of columns". *ACI Structural Journal*, Vol.**84**, No. 4 (July-Aug.), pp.316-329.

Newmark, N. M. (1949). "A simple approximate formula for effective end-fixity of columns". J. Aeronautical Sci., Vol.**16**, No. 2.

NSF. (1973). *"NS 3472. Steel structures. Design rules. (Stålkonstruksjoner. Beregning og dimensjonering)"*. 1st. ed., Norwegian Standards Association (Norges standardiseringsforbund (NSF)), Oslo.

NSF. (1989). *"NS 3473. Concrete structures. Design rules. (Prosjektering av betongkonstruksjoner. Beregnings- og konstruksjonsregler)"*. 3rd. ed., Norwegian Standards Association (NSF), Oslo.

Petersen, Ch. (1982). *"Statik und Stabilität der Baukonstruktionen". 2nd.* ed. Vieweg, Braunschweig/Wiesbaden, Germany.

SA. (1990). *"AS4100–1990. Steel structures"*. Standards Australia (SA), Sydney, Australia.

# APPENDIX A

**TABLE A.1.** Exact effective length factors ($\beta_{\text{EXACT}}$) – Unbraced columns

| $G_B$ | $G_A$ ($\overline{k}_A$) | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 0 (∞) | 0.25 (24) | 1 (6) | 4 (1.5) | 8 (0.75) | ±∞ (±0) | −20 (-0.3) | −15 (-0.4) | −10 (-0.6) | −8 (-0.75) | −7 (-0.86) | −6.5 (-0.92) |
| −1 | 0.851 | 0.891 | 1.000 | 1.253 | 1.395 | 1.677 | 1.890 | 1.988 | 2.257 | 2.574 | 2.923 | 3.235 |
| −0.5 | 0.919 | 0.960 | 1.073 | 1.346 | 1.505 | 1.834 | 2.098 | 2.225 | 2.593 | 3.070 | 3.671 | 4.306 |
| −0.25 | 0.958 | 1.000 | 1.114 | 1.395 | 1.562 | 1.917 | 2.209 | 2.353 | 2.781 | 3.371 | 4.187 | 5.177 |
| 0 | 1.000 | 1.042 | 1.157 | 1.445 | 1.620 | 2.000 | 2.323 | 2.485 | 2.984 | 3.719 | 4.868 | 6.590 |
| 0.25 | | 1083 | 1.199 | 1.494 | 1.677 | 2.084 | 2.438 | 2.621 | 3.202 | 4.128 | 5.834 | 9.672 |
| 1 | | | 1.317 | 1.634 | 1.840 | 2.328 | 2.793 | 3.049 | 3.974 | 6.051 | ∞ | |
| 4 | | | | 2.036 | 2.332 | 3.179 | 4.315 | 5.212 | ∞ | | | |
| 8 | | | | | 2.724 | 4.073 | 7.109 | 14.85 | | | | |
| ∞ | | | | | | ∞ | | | | | | |

- Due to symmetry, results below the main diagonal are not shown.
- $\overline{k} = k/(EI/L) = 6/G$

**TABLE A.2.** Exact effective length factors ($\beta_{\text{EXACT}}$) – Braced columns

| $G_B$ | $G_A$ ($\overline{k}_A$) | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 0 (∞) | 0.25 (8) | 1 (2) | 4 (0.5) | ±∞ (±0) | −4 (-0.5) | −2 (-1) | −1.5 (-1.33) | −1.2 (-1.67) | −1 (-2) | −0.8 (-2.5) | −0.6 (-3.33) |
| 0 | 0.500 | 0.555 | 0.626 | 0.675 | 0.700 | 0.730 | 0.769 | 0.802 | 0.844 | 0.896 | 1.010 | 1.452 |
| 0.25 | | 0.611 | 0.688 | 0.744 | 0.773 | 0.809 | 0.852 | 0.901 | 0.955 | 1.027 | 1.194 | 2.13 |
| 1 | | | 0.774 | 0.840 | 0.875 | 0.921 | 0.984 | 1.041 | 1.116 | 1.222 | 1.497 | ∞ |
| 4 | | | | 0.916 | 0.956 | 1.011 | 1.087 | 1.158 | 1.257 | 1.398 | 1.824 | |
| ±∞ | | | | | 1.000 | 1.060 | 1.145 | 1.226 | 1.338 | 1.509 | 2.059 | |
| −4 | | | | | | 1.127 | 1.226 | 1.321 | 1.459 | 1.677 | 2.503 | |
| −2 | | | | | | | 1.348 | 1.469 | 1.654 | 1.974 | 3.796 | |
| −1.5 | | | | | | | | 1.624 | 1.872 | 2.352 | ∞ | |
| −1.2 | | | | | | | | | 2.259 | 3.233 | | |
| −1 | | | | | | | | | | ∞ | | |

- Due to symmetry, results below the main diagonal are not shown.
- $\overline{k} = k/(EI/L) = 2/G$

**TABLE A.3a.** $\beta_{APPROX}/\beta_{\mathrm{EXACT}}$-ratios for unbraced columns,
for $\beta_{APPROX} = 2\sqrt{R_A + R_B - R_A R_B}/(R_A + R_B)$ with $c$=**2.4** (Eq.25).

| $G_B$ | $G_A$ ($R_A$) | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | 0<br>(1) | .25<br>(.909) | 1<br>(.714) | 4<br>(.385) | $\pm\infty$<br>( $\pm$ 0) | $-20$<br>(-.143) | $-15$<br>(-.2) | $-10$<br>(-.333) | $-8$<br>(-.455) | $-7$<br>(-.556) |
| $-0.50$ | 0.967 | 0.976 | 0.982 | 0.976 | 0.975 | 0.976 | 0.976 | 0.972 | 0.956 | 0.925 |
| $-0.25$ | 0.988 | 0.995 | 0.999 | 0.991 | 0.990 | 0.993 | 0.993 | 0.991 | 0.974 | 0.931 |
| 0 | 1.000 | 1.005 | 1.008 | 1.000 | 1.000 | 1.004 | 1.006 | 1.005 | 0.986 | 0.924 |
| 0.25 | | 1.011 | 1.014 | 1.005 | 1.007 | 1.014 | 1.016 | 1.017 | 0.993 | 0.899 |
| 1 | | | 1.019 | 1.011 | 1.017 | 1.028 | 1.034 | 1.040 | 0.973 | 1) |
| 4 | | | | 1.007 | 1.014 | 1.044 | 1.063 | 1) | | |
| 8 | | | | | 1.006 | 1.062 | 1.035 | | | |

- Due to symmetry, results below the main diagonal are not shown.
- 1) Exact effective length factors are infinite.

**TABLE A.3b.** $\beta_{APPROX}/\beta_{\mathrm{EXACT}}$-ratios for unbraced columns,
for $\beta_{APPROX} = 2\sqrt{R_A + R_B - R_A R_B}/(R_A + R_B)$ with $c$=**2.5** (Eq.25).

| $G_B$ | $G_A$ ($R_A$) | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | 0<br>(1) | .25<br>(.906) | 1<br>(.706) | 4<br>(.375) | $\pm\infty$<br>( $\pm$ 0) | $-20$<br>(-.136) | $-15$<br>(-.190) | $-10$<br>(-.316) | $-8$<br>(-.429) | $-7$<br>(-.522) |
| $-0.50$ | 0.962 | 0.972 | 0.983 | 0.979 | 0.970 | 0.964 | 0.960 | 0.945 | 0.916 | 0.870 |
| $-0.25$ | 0.986 | 0.995 | 1.002 | 0.996 | 0.987 | 0.983 | 0.980 | 0.965 | 0.932 | 0.872 |
| 0 | 1.000 | 1.007 | 1.013 | 1.007 | 1.000 | 0.997 | 0.994 | 0.980 | 0.941 | 0.859 |
| 0.25 | | 1.015 | 1.021 | 1.014 | 1.008 | 1.008 | 1.005 | 0.991 | 0.945 | 0.826 |
| 1 | | | 1.028 | 1.023 | 1.023 | 1.026 | 1.026 | 1.010 | 0.908 | 1) |
| 4 | | | | 1.022 | 1.027 | 1.046 | 1.052 | 1) | | |
| 8 | | | | | 1.022 | 1.057 | 0.970 | | | |

- Due to symmetry, results below the main diagonal are not shown.
- 1) Exact effective length factors are infinite.

**TABLE A.4.** $\beta_{APPROX}/\beta_{\text{EXACT}}$-ratios for braced columns, for $\beta_{APPROX} = 1/(1+R_m)$ with **c=4.8** (Eq.30).

| $G_B$ | $G_A$ $(R_A)$ | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | 0 (1) | 0.25 (.625) | 1 (.294) | 4 (.094) | $\pm\infty$ ( $\pm$ 0) | $-4$ (-.116) | $-2$ (-.263) | $-1.5$ (-.385) | $-1.2$ (-.532) | $-1$ (-.714) | $-0.8$ (-1.09) |
| 0 | 1.000 | 0.994 | 0.970 | 0.958 | 0.952 | 0.950 | 0.950 | 0.953 | 0.960 | 0.977 | 1.035 |
| 0.25 | | 1.007 | 0.996 | 0.989 | 0.986 | 0.985 | 0.994 | 0.991 | 1.001 | 1.019 | 1.089 |
| 1 | | | 0.998 | 0.997 | 0.996 | 0.997 | 1.001 | 1.006 | 1.017 | 1.036 | 1.107 |
| 4 | | | | 0.998 | 0.999 | 1.000 | 1.005 | 1.010 | 1.018 | 1.037 | 1.088 |
| $\pm\infty$ | | | | | 1.000 | 1.002 | 1.006 | 1.010 | 1.018 | 1.031 | 1.064 |
| $-4$ | | | | | | 1.004 | 1.007 | 1.010 | 1.014 | 1.020 | 1.003 |
| $-2$ | | | | | | | 1.007 | 1.007 | 1.004 | 0.991 | 0.811 |
| $-1.5$ | | | | | | | | 1.001 | 0.986 | 0.944 | 1) |
| $-1.2$ | | | | | | | | | 0.946 | 0.821 | |

- Due to symmetry, results below the main diagonal are not shown.
- 1) Exact effective length factors are infinite.

**TABLE A.5** $\beta_{APPROX}/\beta_{\text{EXACT}}$-ratios for braced columns, for $\beta_{APPROX} = 2/(2+1.1R_{min}+0.9R_{max})$ with **c=4.8** (Eq.32).

| $G_B$ | $G_A$ $(R_A)$ | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | 0 (1) | 0.25 (.625) | 1 (.294) | 4 (.094) | $\pm\infty$ ( $\pm$ 0) | $-4$ (-.116) | $-2$ (-.263) | $-1.5$ (-.385) | $-1.2$ (-.532) | $-1$ (-.714) | $-0.8$ (-1.09) |
| 0 | 1.000 | 1.004 | 0.991 | 0.986 | 0.985 | 0.988 | 0.996 | 1.007 | 1.024 | 1.056 | 1.162 |
| 0.25 | | 1.007 | 1.007 | 1.008 | 1.010 | 1.015 | 1.033 | 1.038 | 1.059 | 1.096 | 1.225 |
| 1 | | | 0.998 | 1.005 | 1.009 | 1.016 | 1.029 | 1.043 | 1.067 | 1.107 | 1.250 |
| 4 | | | | 0.998 | 1.003 | 1.011 | 1.025 | 1.039 | 1.061 | 1.101 | 1.233 |
| $\pm\infty$ | | | | | 1.000 | 1.008 | 1.021 | 1.034 | 1.056 | 1.091 | 1.208 |
| $-4$ | | | | | | 1.004 | 1.016 | 1.028 | 1.046 | 1.075 | 1.142 |
| $-2$ | | | | | | | 1.007 | 1.016 | 1.026 | 1.036 | 0.928 |
| $-1.5$ | | | | | | | | 1.001 | 1.000 | 0.980 | 1) |
| $-1.2$ | | | | | | | | | 0.946 | 0.841 | |

- Due to symmetry, results below the main diagonal are not shown.
- 1) Exact effective length factors are infinite.

**TABLE A.6.** $\beta_{APPROX}/\beta_{\mathrm{EXACT}}$-ratios for braced columns, with $\beta_{APPROX} = 1/\sqrt{(1+R_A)(1+R_B)}$ for **c=4.8** (Eq.33).

| $G_B$ | $G_A$ ($R_A$) | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | 0 (1) | 0.25 (.625) | 1 (.294) | 4 (.094) | $\pm\infty$ ( $\pm$ 0) | $-4$ (-.116) | $-2$ (-.263) | $-1.5$ (-.385) | $-1.2$ (-.532) | $-1$ (-.714) | $-0.8$ (-1.09) |
| 0 | 1.000 | 0.999 | 0.993 | 1.001 | 1.010 | 1.030 | 1.071 | 1.124 | 1.225 | 1.476 | im.) |
| 0.25 | | 1.007 | 1.002 | 1.008 | 1.015 | 1.031 | 1.073 | 1.110 | 1.201 | 1.429 | im.) |
| 1 | | | 0.998 | 1.000 | 1.005 | 1.015 | 1.041 | 1.076 | 1.151 | 1.346 | im.) |
| 4 | | | | 0.998 | 1.000 | 1.006 | 1.024 | 1.052 | 1.112 | 1.279 | im.) |
| $\pm\infty$ | | | | | 1.000 | 1.004 | 1.017 | 1.040 | 1.092 | 1.240 | im.) |
| $-4$ | | | | | | 1.004 | 1.011 | 1.027 | 1.066 | 1.187 | im.) |
| $-2$ | | | | | | | 1.007 | 1.011 | 1.029 | 1.104 | im.) |
| $-1.5$ | | | | | | | | 1.001 | 0.995 | 1.014 | 1) |
| $-1.2$ | | | | | | | | | 0.946 | 0.846 | |

- Due to symmetry, results below the main diagonal are not shown.
- 1) Exact effective length factors are infinite.
- Identical results are obtained with $\beta_{APPROX} = 0.5\sqrt{(2-R_A)(2-R_B)}$ for **c=2.4** (Eq.41)

**TABLE A.7.** $\beta_{APPROX}/\beta_{\mathrm{EXACT}}$-ratios for braced columns, for $\beta = 1 - 0.2(R_{min} + 1.45R_{max})$ with **c=2.4** (Eq.40)

| $G_B$ | $G_A$ ($R_A$) | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | 0 (1) | 0.25 (.769) | 1 (.455) | 4 (.172) | $\pm\infty$ ( $\pm$ 0) | $-4$ (-.263) | $-2$ (-.714) | $-1.5$ (-1.25) | $-1.2$ (-2.27) | $-1$ (-5.00) | $-0.8$ (-25.0) |
| 0 | 1.020 | 1.002 | 0.989 | 1.001 | 1.014 | 1.045 | 1.109 | 1.197 | 1.380 | 1.909 | neg.) |
| 0.25 | | 1.020 | 0.997 | 0.998 | 1.005 | 1.025 | 1.080 | 1.140 | 1.290 | 1.730 | neg.) |
| 1 | | | 1.004 | 0.992 | 0.992 | 1.000 | 1.028 | 1.074 | 1.185 | 1.529 | neg.) |
| 4 | | | | 1.000 | 0.994 | 0.992 | 1.005 | 1.036 | 1.117 | 1.395 | neg.) |
| $\pm\infty$ | | | | | 1.000 | 0.993 | 0.998 | 1.020 | 1.087 | 1.325 | neg.) |
| $-4$ | | | | | | 1.002 | 0.994 | 1.004 | 1.049 | 1.238 | neg.) |
| $-2$ | | | | | | | 1.002 | 0.992 | 1.005 | 1.118 | neg.) |
| $-1.5$ | | | | | | | | 0.993 | 0.971 | 1.005 | 1) |
| $-1.2$ | | | | | | | | | 0.936 | 0.823 | |

- Due to symmetry, results below the main diagonal are not shown.
- neg.) Approximate factor is negative.