Improved Frame Stability Analysis with Effective Lengths

By Jostein Hellesland[1] and Reidar Bjorhovde,[2] Fellow, ASCE

Abstract: Approximate methods for determination of effective lengths of compression members in frame systems are normally based on rotational restraints defined through so-called joint stiffness ratios or G-factors. These approaches allow the member to be considered in isolation from the rest of the frame. However, they are known to be inaccurate in many cases. A method is proposed that involves postprocessing of effective lengths from isolated column analyses to arrive at improved, weighted mean values. The approach, termed the method of means, satisfies general system instability principles, is attractively simple, and yields in general effective length predictions in excellent agreement with exact results for a wide variation of parameters. The errors are normally within a few percent of exact solutions. The method is applicable to braced and to a range of unbraced frames. It is particularly suitable for cases where drastic changes of beam or column stiffness occur, such as in the top or bottom stories of a frame, or where column stiffness, axial force level, and story height change at certain floor levels.

INTRODUCTION

Most national and international codes and standards for structural design include provisions involving effective lengths of columns for stability evaluations and approximate methods of second-order analysis. A full system instability analysis, required for exact effective length determination, may be quite involved for a frame system of some complexity. For this reason, the usefulness of approximate analyses in routine design depends to a great extent on the availability of reasonably simple and accurate methods for determining effective lengths.

Presently, the most widely used approach in the design of framed columns consists of considering the columns in isolation, with rotational end restraints defined through conventional joint stiffness ratios between the columns and beams that frame into a joint. These ratios, most frequently labelled G (or ψ), are defined by

$$G = \frac{\sum EI/L}{\sum EI_b/L_b} \tag{1}$$

where the summations are over the columns and beams (subscript b) intersecting at a joint.

In the general case, a major limitation of the methods based on such G-factors is that they do not properly recognize the contribution (positive or negative) to the rotational end restraints from columns in stories above or below (vertical interaction effects) and from adjacent columns (horizontal interaction through connecting beams). The neglect of such interaction effects, due to differences in boundary conditions, axial loads, lengths, and cross-sectional stiffness between the column in question and the other columns, may result in significant errors outside defined ranges of applicability (Hellesland and Bjorhovde 1996).

Efforts to extend the range of applicability include an improved G-factor approach proposed by Duan and Chen (1988, 1989) for continuous columns with equal flexibility parameters (α_E), and an iterative procedure for braced frames (Bridge and Fraser 1987) that explicitly reflects both positive and negative G-factors. Approximate, story-based methods for effective length prediction that account for the horizontal interaction between columns in a story have long been available for unbraced frames (Yura 1971; LeMessurier 1977).

In this paper, a new method for elastic effective length prediction is presented. The method, originally developed by Hellesland (1992b), removes some of the shortcomings of the conventional G-factor approaches, and significantly extends the range of validity. Denoted as "the method of means," the approach requires effective length factors from isolated column solutions as input. Based on basic system instability requirements, it performs a postprocessing of these input values to arrive at improved, weighted mean values.

The applicability of the method is assessed against numerical predictions for columns in both braced and unbraced frames. However, in this study, primary attention is devoted to braced frames, and in particular to the vertical interaction between columns in such frames. Different boundary conditions, as well as a wide range of different column and beam parameters, are considered.

ISOLATED COLUMN ANALYSIS

The input data for the method of means are effective length factors, K, that have been determined by traditional methods. These allow the column to be considered in isolation. For the sake of distinction and brevity, such K-values will be referred to here as "isolated values."

For instance, for isolated braced and unbraced columns with known rotational end restraints (k), elastic effective length factors can be determined from the standard alignment charts, or from the well-known transcendental equations (Galambos 1968) given by the following:

$$\frac{G_A G_B}{4}\left(\frac{\pi}{K}\right)^2 + \frac{G_A + G_B}{2}\left(1 - \frac{\pi/K}{\tan(\pi/K)}\right) + \frac{\tan(\pi/2K)}{\pi/2K} = 1 \tag{2}$$

$$\frac{G_A G_B(\pi/K)^2 - 36}{6(G_A + G_B)} = \frac{\pi/K}{\tan(\pi/K)} \tag{3}$$

respectively. Here, the joint stiffness ratios (G_A, G_B) at the ends, denoted A and B, may be defined in the general form by

$$G_i = b_0 \frac{EI/L}{k_i} \quad i = A, B \tag{4}$$

where $b_0 = 2$ for the braced column case [(2)]; and $b_0 = 6$ for the unbraced column case [(3)].

When the rotational restraint stiffness, k (k_A or k_B), at the column ends are due to beams only, then $k = k_b$, where k_b (k_{bA} or k_{bB}) is the sum of the rotational stiffnesses of all beams at the respective joint

[1]Prof. of Struct. Mech., Mech. Div., Dept. of Math., Univ. of Oslo, P.O. Box 1053—Blindern, 0316 Oslo, Norway.

[2]Prof. and Chair. of Civ. and Envir. Engrg., Univ. of Pittsburgh, Pittsburgh, PA 15261.

Note. Associate Editor: W. Samuel Easterling. Discussion open until April 1, 1997. To extend the closing date one month, a written request must be filed with the ASCE Manager of Journals. The manuscript for this paper was submitted for review and possible publication on July 5, 1995. This paper is part of the ***Journal of Structural Engineering***, Vol. 122, No. 11, November, 1996. ©ASCE, ISSN 0733-9445/96/0011-1275–1283/$4.00 + $.50 per page. Paper No. 11191.

$$k_{bi} = \left(\sum bEI_b/L_b\right)_i \quad i = A, B \tag{5}$$

For uniform beams, rigidly connected to the column, and with negligible axial force and shear deformation effects, the rotational stiffness coefficient b takes on the familiar values of b = 2, 3, 4, and 6 for beams bent in symmetrical single curvature, beams pinned at the far end, beams fixed at the far end, and beams in antisymmetrical double curvature, respectively. In cases with semirigid restraints, which are not considered here, appropriate adjustments must be made (Bjorhovde 1984).

The end restraints for a framed column may be expressed as a *restraint demand factor*, f, times the beam restraint, $k = fk_b$. With beam restraints given by (5), (4) can be restated as

$$G_i = \frac{EI/L}{\left(f \sum mEI_b/L_b\right)_i} \quad i = A, B \tag{6}$$

where $m = b/b_0$. Here, the vertical and horizontal interaction effects are reflected by f and m, respectively. In the normal design practice of frames with rigid beam-column connections, the parameter m is set equal to 1.0. This corresponds to the cases of braced columns restrained by symmetrically bent beams ($b = b_0 = 2$) and unbraced columns restrained by antisymmetrically bent beams ($b = b_0 = 6$). For cases other than these reference (datum) cases, m represents the appropriate stiffness modifier.

A number of approximate restraint demand factors for the vertical interaction of braced columns have been considered and discussed in previous studies (Hellesland 1992a; Hellesland and Bjorhovde 1996). For numerical predictions of braced columns in this study, three of these will be used, as follows:

$$f_i = \frac{EI/L}{\left(\sum EI/L\right)_i}; \quad f_i = \frac{1}{n_{ci}}; \quad f_i = \frac{\alpha_E}{\left(\sum \alpha_E\right)_i} \tag{7-9}$$

where n_{ci} = number of columns framing into joint i; and α_E = flexibility parameter = P/P_E. The summations are over all columns (n_{ci}) at the joint considered ($i = A$ or B). The relative effect of these f-factors will be discussed later.

The combination of (6) and (7) yields the conventional G-factor given by (1) when $m = 1$. Use of this factor in conjunction with alignment charts is frequently referred to as the alignment chart procedure. The term "conventional" is preferred here for effective length predictions with f according to (7), and G according to (6).

These f-factors imply that column end restraints are positive fractions of the rotational stiffness of the beams framing into the joints. Consequently, they vary between the limiting values of 0 and 1. Exact factors, which may be positive or negative, vary between much wider limits, as they reflect not only the effect of restraining beams, but also positive or negative restraining effects from other columns. This is particularly the case for small beam restraints, when vertical interaction dominates the response.

SYSTEM INSTABILITY REQUIREMENTS

The effective length L_e and the elastic effective length factor K of a compression member of length L and cross-sectional bending stiffness EI are in standard fashion expressed by $L_e = KL$ and

$$K = \sqrt{\frac{P_E}{P_{cr}}} = \sqrt{\frac{P_E}{\gamma_{cr}P}} = \sqrt{\frac{1}{\gamma_{cr}\alpha_E}} \tag{10}$$

where

$$\alpha_E = \frac{P}{P_E} \quad \text{and} \quad P_E = \frac{\pi^2 EI}{L^2} \tag{11a,b}$$

is a column flexibility parameter ("load parameter") and the Euler buckling load of a pin-ended column, respectively. The critical load (P_{cr}) is equal to the axial force in the member at overall instability of the structural system of which the member is a part, and γ_{cr} is the load factor at column buckling.

Assuming proportional loading, the effective length factors in the various compression members of a frame are interrelated through their α_E-values at system instability, since the critical load factor (γ_{cr}) is the same in the members. Considering two columns, i and j, the relationship between their effective length factors becomes

$$K_j = K_i\sqrt{Q_{ij}} \tag{12}$$

where

$$Q_{ij} = \frac{\alpha_{Ei}}{\alpha_{Ej}} = \frac{P_i}{P_j}\frac{L_i^2}{L_j^2}\frac{EI_j}{EI_i} = \frac{P_iL_i}{P_jL_j}\frac{(EI/L)_j}{(EI/L)_i} \tag{13}$$

is a nominal column flexibility ratio. The first subscript in Q refers to α_E in the numerator, and the second to α_E in the denominator. Thus, inverse values are obtained by reversing the subscripts

$$Q_{ji} = \frac{1}{Q_{ij}} = \frac{\alpha_{Ej}}{\alpha_{Ei}} \tag{14}$$

Once K is known for one column, K for the other columns can be computed using (12). Effective length predictions that do not satisfy such interrelationships violate basic system instability requirements.

This will generally be the case with isolated column predictions due to the approximations in restraint assessments. However, if such K-values are underestimated in some columns, they will be overestimated in neighboring columns. Consequently, a weighted mean of these values may give improved and realistic end results. The system instability requirement discussed may be used to establish "weighting functions" in such a procedure. This is the basis for the "method of means," presented in the following in two alternative forms.

METHOD OF MEANS

Method of Means—Alternative 1 ("Mean 1")

It is assumed that approximate isolated K-values are computed for m interacting columns, using, for instance, one of the restraint definitions presented in (7)–(9). The isolated effective length factors are denoted $K_1, K_2, K_3, \ldots, K_m$.

Using (12), it is possible, for each of the columns, to calculate an additional $m - 1$ K-estimates. For instance, the following m estimates can be found for the K-factor of column 1:

$$K_1, K_2\sqrt{Q_{21}}, K_3\sqrt{Q_{31}}, \ldots, K_m\sqrt{Q_{m1}}$$

For example, the term $K_2\sqrt{Q_{21}}$ is the value of K_1 as derived on the basis of K_2 for column 2. In an exact analysis, all of these would be equal. In approximate methods this is not likely to happen.

From these K_1-estimates the following mean effective length factor is determined for column 1:

$$\bar{K}_1 = (K_1 + K_2\sqrt{Q_{21}} + \ldots + K_m\sqrt{Q_{m1}})\frac{1}{m} \tag{15}$$

Since $Q_{11} = 1$, (15) can be written as

$$\bar{K}_1 = \frac{1}{m} \sum_{1}^{m} K_i \sqrt{Q_{i1}} \tag{16}$$

or, for an arbitrary column j, as

$$\bar{K}_j = \frac{1}{m} \sum_{i=1}^{m} K_i \sqrt{Q_{ij}} \tag{17}$$

Once the mean $\bar{K}_j$-value is computed for column j, the corresponding mean values for the other columns can be determined from this value using (12). Thus, for column i

$$\bar{K}_i = \bar{K}_j \sqrt{Q_{ji}} = \frac{\bar{K}_j}{\sqrt{Q_{ij}}} \tag{18}$$

Obviously, (17) may be used for the computation of all $\bar{K}_i$-values. However, (17) in combination with (18) will reduce the computational effort.

Method of Means—Alternative 2 ("Mean 2")

An alternative approach to that of averaging isolated effective length factors can be made on the basis of averaging isolated column stability indices. Eq. (12) can be rewritten as

$$\alpha_j = \alpha_i \tag{19}$$

where, for any i and j

$$\alpha = P/P_{cr} \quad \text{and} \quad P_{cr} = \frac{\pi^2 EI}{(KL)^2} = \frac{P_E}{K^2} \tag{20a,b}$$

i.e.

$$\alpha = K^2 \alpha_E \tag{21}$$

The ratio defined by α is often termed the "stability index." This seems like an apt term as it reflects how close to instability the frame is. Theoretically, $\alpha = 1$ in all columns at instability. However, in approximate methods based on effective length predictions for isolated columns with approximations in the end restraint assessments, this will not be the case.

An average for the various interacting columns will represent an approximation of the system stability index. Using the term $\bar{\alpha}$, this becomes

$$\bar{\alpha} = [\alpha_1 + \alpha_2 + \ldots + \alpha_m] \frac{1}{m} \tag{22}$$

where α_1, α_2, etc. are the individual isolated column data. By substituting for these from (21), (22) yields

$$\bar{\alpha}/\alpha_{E1} = (K_1^2 + K_2^2 Q_{21} + \ldots + K_m^2 Q_{m1}) \frac{1}{m} \tag{23}$$

Noting that (21) for average values gives

$$\bar{K}_j = \sqrt{\bar{\alpha}/\alpha_{E_j}} \tag{24}$$

for any j, the "mean" value of $\bar{K}_1$ now becomes equal to the square root of the mean of the weighted squares of the m K_1-estimates

$$\bar{K}_1 = \left[\frac{1}{m} (K_1^2 + K_2^2 Q_{21} + \cdots + K_m^2 Q_{m1})\right]^{1/2} = \left(\frac{1}{m} \sum_{1}^{m} K_i^2 Q_{i1}\right)^{1/2} \tag{25}$$

or, for an arbitrary column j

$$\bar{K}_j = \left(\frac{1}{m} \sum_{i=1}^{m} K_i^2 Q_{ij}\right)^{1/2} \tag{26}$$

The corresponding mean values for the other columns can be determined from this value using (18).

Lower Limits on $\bar{K}$-Values for Braced Columns

In principle, the Q_{ij} parameter may assume a value anywhere between zero and infinity. Corresponding mean effective length factors at these limits can be found from (17) and (26). For instance, in a case involving two columns, the following are the results:

$$\bar{K}_2 = K_2/a \text{ and } \bar{K}_1 \rightarrow \infty \text{ at } Q_{12} = 0$$

$$\bar{K}_1 = K_1/a \text{ and } \bar{K}_2 \rightarrow \infty \text{ at } Q_{21} = 0 \ (Q_{12} \rightarrow \infty)$$

where $a = 2$ and $\sqrt{2}$ according to (17) and (26), respectively. The input values, K_1 and K_2, computed for the isolated columns, will be close to 0.5 for a braced column that is nearly fully fixed at both ends. This implies that the lowest predicted mean values may be close to $0.5/2 = 0.25$ and $0.5/\sqrt{2} = 0.35$ for alternatives 1 and 2, respectively. First buckling mode values less than 0.5 are not physically possible. Therefore, it is necessary to impose constraints on the $\bar{K}$-predictions.

As a general constraint, predicted values should not be taken less than 0.5, or a slightly greater value, considering that columns in practice are rarely fully fixed. On this basis it is proposed that $\bar{K}$-predictions should be subjected to the following general lower limit constraint:

$$\bar{K} \geq \lim \bar{K} = 0.53 \tag{27}$$

In the theoretical case of full fixity, this constraint results in a conservative error of 6%.

Similarly, if the $\bar{K}$-prediction for a column becomes less than some other (higher) known lower limit, this can be used to improve the $\bar{K}$-predictions. For instance, for a braced column pinned at a support and interacting with the frame in question at the other end [e.g., column 4, Fig. 1(a)], exact effective lengths will always exceed 0.7. For such columns, the following lower limit constraint may be imposed:

$$\bar{K} \geq \lim \bar{K} = 0.73 \tag{28}$$

The unique relationship between $\bar{K}$-values for the various columns, as given by the system instability requirement, (18), must in any case be satisfied. Thus, all columns will be affected by a limitation in any one column. For instance, if a lower limit governs column k, i.e., if the predicted $\bar{K}_k$ of the column becomes less than lim $\bar{K}_k$, then the improved predicted effective lengths of the columns become

$$\bar{K}_i = (\lim \bar{K}_k)\sqrt{Q_{ki}} \quad i = 1, 2, \ldots, m \tag{29}$$

It is conceivable that the mean effective length factors of more than one column in a frame may be limited by a lower limit. In such a case, the calculations of (29) should be carried out based on the column (k) that yields the larger $\bar{K}_i$-values.

The $\bar{K}$-limits must not be confused with the minimum K-values that are recommended by a number of specifications or specification commentaries. In principle, the $\bar{K}$-limits are also applicable to "braced" columns in unbraced frames. Very flexible columns in an otherwise laterally stiff frame fall into this category, and may cause instability by local (braced) buck-

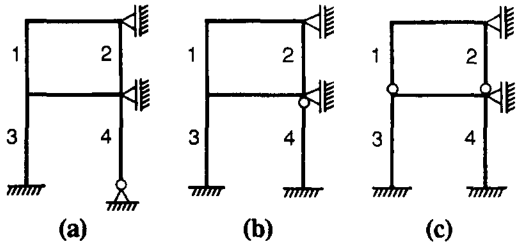

FIG. 1. Examples of Braced Frames with Different Degrees of Interaction between Columns

ling between column ends. However, for the unbraced frames discussed later in this paper, the lower $\bar{K}$-limits will not apply.

Systematic Procedure

The foregoing implies that $\bar{K}$-predictions are first determined for all columns using (17) and (18) in alternative 1, or (26) and (18) in alternative 2. Further, (29) is applied to arrive at improved values only if a lower limit constraint is applicable to one or more columns. This is probably the most practical approach.

An alternative, systematic approach to the determination of final $\bar{K}$-values would be to determine first the maximum $\bar{K}$-value for one column, the reference member, and then to use this value as the basis for the determination of $\bar{K}$-values for the other columns. Consider a braced case involving m columns, with column 1 as the reference column and with only column m pinned at a support [e.g., column 4, Fig. 1(a)]. Then, lim $\bar{K}$ is 0.73 for column m and 0.53 for the others. The calculation procedure then proceeds as follows:

1. Determine $\bar{K}_1$ for column 1 as the largest value of: $\bar{K}_1$ given by (17) in alternative 1, or (26) in alternative 2, and $\bar{K}_1 = 0.53$, $\bar{K}_1 = 0.53\sqrt{Q_{21}}, \ldots, \bar{K}_1 = 0.73\sqrt{Q_{m1}}$ [$\bar{K}_1 = (\lim \bar{K}_k)\sqrt{Q_{k1}}$ $\quad k = 1, 2, 3, \ldots, m$].
2. Determine $\bar{K}$ for the other columns from (18) based on the largest $\bar{K}_1$ found in step 1. Thus, $\bar{K}_i = \bar{K}_1\sqrt{Q_{1i}} = \bar{K}_1/\sqrt{Q_{i1}}$ $\quad i = 2, 3, \ldots, m$.

In a practical case, it will normally be apparent which one of the various expressions in step 1 it is that gives the larger value. For normal restraints and column flexibility ratios (Q), the larger $\bar{K}_1$-value will in general be that obtained in the first part of step 1. For unbraced frames subjected to the limitation given in the following section, the second part of step 1 will not govern and can be deleted.

In the summations in (17) and (26), only columns that interact with each other, either directly or through other members, should be included. Columns that do not interact with the others can be analyzed independently and proportioned such that local (braced) buckling of these do not induce premature system instability. Examples of such columns are pin-ended columns ($K = 1$) in fully braced frames and column 4 ($K = 0.7$) in Fig. 1(b). In other cases, the frame may be divided into subframes that can be analyzed separately [e.g., Fig. 1(c)].

Limitation on Applicability for Unbraced Frames

For unbraced (sway permitted) frames with both stiff and very flexible columns, the latter columns' contribution to the summations in (17) and (26) may become very dominant, and result in excessively conservative $\bar{K}$-predictions. This is particularly obvious in the extreme case of an unbraced frame that includes "leaning" (pin-ended) columns having infinite isolated (free-to-sway) K-factors. Consequently, the resulting $\bar{K}$-predictions also become infinite. This is clearly not correct.

To avoid excessive conservativeness, use of the method of means to unbraced frames should be limited to cases without excessive differences in column flexibilities. This can be achieved by limiting the application to cases where

$$K_i^2 Q_{ij} < C\bar{K}_j^2 \quad i = 1, 2, \ldots, m \tag{30}$$

holds between each term (i) in the summations and a given multiple of the mean value.

This relationship, with $C = 1.7$, has been given before (Hellesland 1995). Here, a more nuanced C-factor is suggested. It is given by

$$C = 1.05/\sqrt{d} \tag{31}$$

where d = portion of the m terms in the summations [in (17) or (26)] that are markedly greater (at least about two to three times) than the others. In other words, d represents the portion of the columns that are markedly more flexible than the others. Thus, for cases where one-third, one-half, and two-thirds of the columns are in this category (d = 1/3, 1/2, and 2/3), C becomes 1.8, 1.5, and 1.3, respectively.

With this C-definition, it is found from an examination of single-story unbraced frames that $\bar{K}$-predictions that just barely satisfy (30) normally will not exceed exact values by more than about 15%.

APPLICATION TO TWO-STORY BRACED FRAMES

The application and accuracy of the method of means is demonstrated in the following, using the two-story frames shown in Fig. 2. The results are compared to exact results of analyses using standard stability functions.

Both alternatives of the method of means are considered, although most results are obtained according to alternative 2. The input values to the method of means are the effective length factors K_1 and K_2, obtained from isolated column analyses using (2), with G-factors as shown in Fig. 3.

The conventional G-factors, according to (1) or (6) with m = 1 and f according to (7), can be written as follows:

$$G_{1A} = g_{1A} = 0; \quad G_{2C} = qg_{1C} \tag{32a,b}$$

$$G_{1B} = (1 + q)g_{1B}; \quad G_{2B} = G_{1B} \tag{33a,b}$$

where

$$q = \frac{EI_2/L_2}{EI_1/L_1}; \quad g_{1i} = \frac{EI_1/L_1}{(EI_b/L_b)_i}, \quad i = A, B, C \tag{34a,b}$$

Most results in this paper are based on these G-factors. However, for comparison, some results are also obtained based on G-factors with the restraint demand factors given by (8) and (9). With these f-factors, G_{1B} and G_{2B} can be written as

$$G_{1B} = 2g_{1B}; \quad G_{2B} = 2qg_{1B} \tag{35a,b}$$

$$G_{1B} = \left(1 + \frac{1}{Q_{12}}\right)g_{1B}; \quad G_{2B} = q(1 + Q_{12})g_{1B} \tag{36a,b}$$

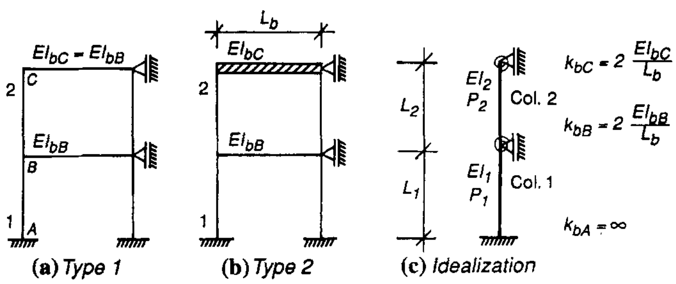

FIG. 2. Braced, Symmetrical Two-Story Frames with: (a) Unequal; (b) Equal End Restraints at the Base and Top; (c) Analysis Model

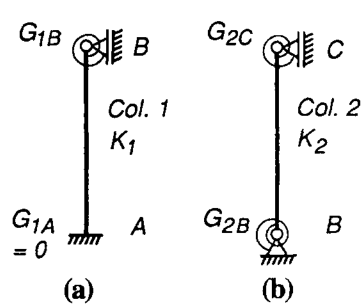

FIG. 3. Isolated Column Analysis

The factors G_{1A} and G_{2C} remain unchanged since only one column frames into each of joints A and C (i.e., $f_{1A} = 1$ and $f_{2C} = 1$).

Basis for Predictions

For convenience, the procedure outlined earlier is restated in the following for the two-story frames in question:

1. Determine $\bar{K}_1$ as the largest of the values given by

$$\bar{K}_1 = \frac{1}{2}(K_1 + K_2\sqrt{Q_{21}}) = \frac{1}{2}(K_1 + K_2/\sqrt{Q_{12}})\text{—alternative 1}$$

$$\bar{K}_1 = \left[\frac{1}{2}(K_1^2 + K_2^2Q_{21})\right]^{1/2} = \left[\frac{1}{2}(K_1^2 + K_2^2/Q_{12})\right]^{1/2}\text{—alternative 2}$$

and

$$\bar{K}_1 = 0.53 \text{ or } \bar{K}_1 = 0.53\sqrt{Q_{21}} = 0.53/\sqrt{Q_{12}}$$

2. Determine $\bar{K}_2$, based on the largest $\bar{K}_1$ found in step 1, as

$$\bar{K}_2 = \bar{K}_1\sqrt{Q_{12}}$$

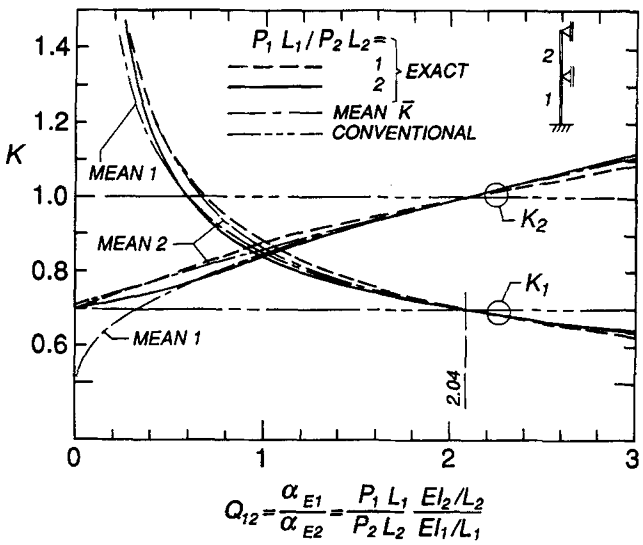

FIG. 4. Predicted versus Exact Effective Length Factors for Frame Type 1 with Negligible Beam Restraints

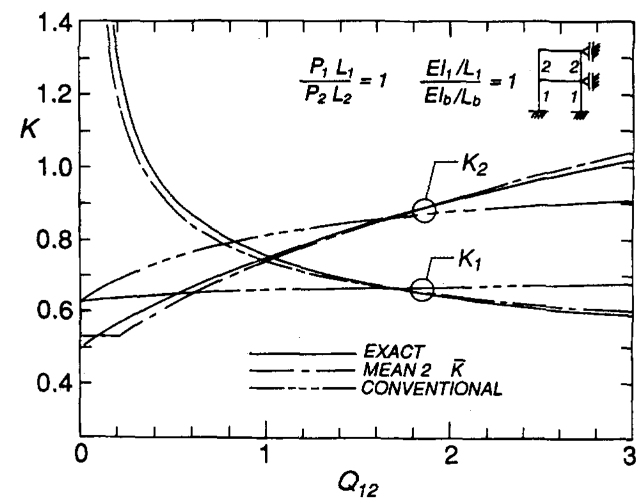

FIG. 5. Predicted versus Exact Effective Length Factors for Frame Type 1 with Intermediate Beam Restraints and $P_1L_1/P_2L_2 = 1$

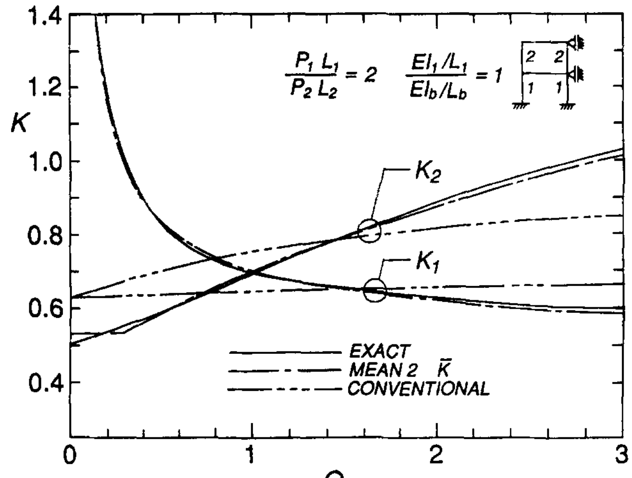

FIG. 6. Predicted versus Exact Effective Length Factors for Frame Type 1 with Intermediate Beam Restraints and $P_1L_1/P_2L_2 = 2$

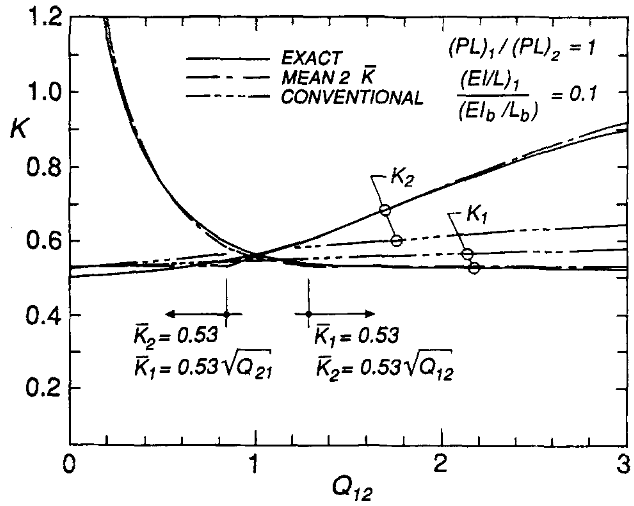

FIG. 7. Predicted versus Exact Effective Length Factors for Frame Type 1 with Strong Beam Restraints

TABLE 1. $\bar{K}/K_{exact}$-Ratios for Columns in Fig. 4

$\frac{P_1L_1}{P_2L_2}$ (1)	Alternative (2)	Q_{12} = 0 (3)	0.1 (4)	0.3 (5)	0.5 (6)	1.0 (7)	1.5 (8)	2.0 (9)	3.0 (10)	6.0 (11)	∞ (12)
1	1	0.71	0.85	0.92	0.94	0.97	0.98	1.00	1.02	1.02	1.06[a]
2	1	0.71	0.86	0.96	0.99	1.02	1.01	1.00	0.99	0.97	1.06[a]
1	2	1.01	1.01	1.00	0.99	0.98	0.99	1.00	1.03	1.05	1.06[a]
2	2	1.01	1.03	1.04	1.04	1.02	1.01	1.00	0.99	1.00	1.06[a]
	Ratio[b]	1.41	1.19	1.10	1.06	1.02	1.00	1.00	1.01	1.03	1.00

Note: $\bar{K}_1/K_{1exact} = \bar{K}_2/K_{2exact}$.

[a]The general lower $\bar{K}$-limit (0.53) governs these results. The higher limit (0.73) for the pin-ended column 2 has not been imposed in these predictions.

[b]Ratio between alternative 2 and alternative 1 results.

Results

Typical results are presented in Figs. 4–7 and Tables 1–3. Numerical details and results for additional parameter values are given in Hellesland (1992b). The figures show effective length factors versus Q_{12}. All figures include three sets of effective length factors: exact factors, predicted mean factors ($\bar{K}$), and isolated column factors.

For frame type 1 with $(EI_1/L_1)/(EI_b/L_b) = 1{,}000$, corresponding in practice to pinned beam-column connections (Fig. 4),

TABLE 2. Alternative 2 $\bar{K}$-values, Based on Various Restraint Demand Factors, versus Exact K-Values; Frame Type 1 (Fig. 2) with $(EI_1/L_1)/(EI_b/L_b) = 1$

$\frac{P_1L_1}{P_2L_2}$ (1)	Ratios R (2)	$q = (EI_2/L_2)/(EI_1/L_1)$: 0 (3)	0.1 (4)	0.3 (5)	0.5 (6)	0.7 (7)	1.0 (8)	1.5 (9)	2.0 (10)	3.0 (11)	∞ (12)
1	K_{1exact}	∞	1.69	1.08	0.91	0.83	0.75	0.69	0.64	0.59	0.50
	R1[a]	1.06[b]	0.99[b]	0.94	0.96	0.97	0.98	0.98	1.01	1.02	1.06[b]
	R2	1.06[b]	0.99[b]	0.90	0.95	0.96	0.98	0.99	1.00	1.01	1.06[b]
	R3	1.06[b]	0.99[b]	0.90[b]	0.93	0.96	0.98	0.98	1.00	1.00	1.06[b]
2	K_{1exact}	∞	1.19	0.78	0.69	0.65	0.62	0.60	0.58	0.56	0.50
	R1	1.06[b]	0.99[b]	1.01	1.01	1.00	0.99	0.98	0.98	0.98	1.06[b]
	R2	1.06[b]	0.99[b]	0.98	1.00	1.00	0.99	0.98	0.97	0.96	1.06[b]
	R3	1.06[b]	0.99[b]	0.98	1.00	1.00	0.99	0.97	0.96	0.95[b]	1.06[b]

[a] $R = \bar{K}_1/K_{1exact}$ $(=K_2/K_{2exact})$. In the R-ratios R1, R2, and R3, $\bar{K}_1$ is computed based on isolated column values with restraint demand factors according to Eqs. (7) ("conventional"), (8), and (9), respectively.

[b] The general lower $\bar{K}$-limit (0.53) governs these results.

TABLE 3. $\bar{K}_{alt.2}/K_{exact}$-Ratios for Two-Story Braced Frames (Fig. 2)

$\frac{EI_1/L_1}{(EI_b/L_b)_B}$ (1)	$\frac{P_1L_1}{P_2L_2}$ (2)	Q_{12}: 0 (3)	0.5 (4)	1.0 (5)	1.5 (6)	3.0 (7)	∞ (8)
		(a) Frame Type 1					
0.1	1	1.06[a]	1.01[a]	1.00	1.00[a]	1.02[a]	1.06[a]
	2	1.06[a]	1.03[a]	1.00	1.01[a]	1.02[a]	1.06[a]
0.5	0.5	1.06[a]	0.91	0.95	0.99	1.06	1.06[a]
	1	1.06[a]	0.95	0.99	1.00	1.02	1.06[a]
	2	1.06[a]	1.00	1.00	1.00	0.96	1.06[a]
1.0	0.5	1.06[a]	0.92	0.95	0.98	1.06	1.06[a]
	1	1.06[a]	0.96	0.98	1.00	1.02	1.06[a]
	2	1.06[a]	1.01	1.01	1.00	0.98	1.06[a]
2.0	1	1.06[a]	0.97	0.98	0.99	1.03	1.06[a]
	2	1.06[a]	1.00	1.02	1.01	0.99	1.06[a]
1,000	1	1.01	0.99	0.98	0.99	1.03	1.06[a]
	2	1.01	1.04	1.02	1.01	0.99	1.06[a]
		(b) Frame Type 2					
0.1	1	1.06[a]	1.04[a]	1.00	1.01[a]	1.02[a]	1.06[a]
	2	1.06[a]	1.04[a]	1.00	1.01[a]	1.02[a]	1.06[a]
	0.5	1.06[a]	0.96	1.00	1.03	1.05	1.06[a]
1.0	1	1.06[a]	0.99	1.00	1.00	1.00	1.06[a]
	2	1.06[a]	1.03	1.00	0.98	0.95	1.06[a]

Note: $\bar{K}$-factors are computed with "isolated" column factors based on the conventional approach.

[a] These values are due to the general lower $\bar{K}$-limit (0.53).

isolated column K-factors are constant and equal to 0.7 and 1.0, respectively, for columns 1 and 2. The predicted mean factors will therefore be independent of PL-ratios in a plot versus Q_{12}.

Alternative 1 versus Alternative 2

Comparing the $\bar{K}$-predictions based on alternatives 1 and 2, as shown in Fig. 4 and Table 1, it is seen that alternative 2 gives somewhat larger $\bar{K}$-values over the entire range of the comparison, and in particular at smaller Q_{12}-values. Furthermore, alternative 2 gives generally better agreement with exact results, as seen in Table 1. The lower limit for pin-ended columns has not been imposed on these predictions. Improved alternative 1 predictions could have been obtained in this case at small Q_{12}-values by using (29), with 0.73 as a lower limiting value for column 2 (pinned at one end).

Predictions based on alternative 1 also have been obtained for other cases. The trend is the same as discussed earlier. With increasing beam stiffnesses (i.e., decreasing column to beam stiffness ratios), the significant difference between the two alternatives at smaller Q_{12}-values becomes less pronounced. When the lower limit on $\bar{K}$ governs, it is immaterial which alternative is used.

For the overall predictions, alternative 2 gives $\bar{K}$-values closest to the exact results for the cases investigated. Also, since it is not significantly more complex in its formulation, alternative 2 would seem to be the better choice of the two, and is the one considered in the remainder of this study.

Input Values—Isolated Column Predictions

The three restraint demand factors [(7)–(9)] and the resulting G-factors [(32)–(36)] give rise to different isolated column effective length factors. As seen in Table 2, resulting mean values, $\bar{K}$, are not significantly affected by the choice of the restraint demand factor. The smallest ratios (R3) are at most 5% below the largest (R1), at $q = 0.3$ for the case considered.

In the evaluation of various restraint demand factors for braced columns (Hellesland and Bjorhovde 1996), it was concluded that the conventional G-factor gives rise to isolated column effective length factors that are the least in agreement with the exact results. It was also concluded that the use of the conventional factor reflected an incorrect influence of the major parameters. This is apparent in Figs. 5–7, where the conventional K_1-factor is seen to increase with increasing Q_{12}, whereas the exact factor decreases. In spite of this, it is of interest to note that it is the $\bar{K}$-predictions computed with isolated column values based on the conventional restraint demand factor that are closest to the exact results (R1, Table 2).

Thus, as an approach for providing input values to the present procedure, the conventional approach would not only seem to be acceptable, but would also seem to be the better approach in terms of providing improved $\bar{K}$-predictions.

Whether this also holds for frames other than those investigated here remains to be seen. However, it is likely that it will at least result in larger $\bar{K}$-predictions than the other approaches for two-story frames. This may be explained with reference to (26) as restated earlier for the two-story case (alternative 2). At smaller Q_{12}-values, conventional K_1-factors [due to (7)] are smaller than those due to (8) and (9). Consequently, for K_2-factors the opposite will be the case. At small Q_{12}-values (less than 1), the K_2-contribution to $\bar{K}_1$ will be amplified by $1/Q_{12}$ (greater than 1). Thus, at small Q_{12}-values the K_2-contribution will be the dominant one. The conventional approach, with the larger K_2-value, will therefore yield the larger $\bar{K}$-values. At larger Q_{12}-values, the conventional K_1-factor is larger and the K_2-factor smaller, than those due to the other approaches. Since $1/Q_{12}$ now is less than one, the K_1-contribution will dominate. The net result will be the same as found earlier.

The Lower Limit

The imposed general lower limit (0.53) on $\bar{K}$-predictions governs results to an increasing extent with decreasing column to beam stiffness ratios. For frame type 1 in Fig. 7, with a ratio $(EI_1/L_1)/(EI_b/L_b)$ as low as 0.1, the lower limit governs results for nearly all Q_{12}-values. The agreement with exact results is excellent. Further, at small Q_{12}-values in Figs. 5 and 6 it is also seen that 0.53 is an appropriate choice for a lower limit on predictions. For normal column to beam stiffness ratios and normal Q_{12}-values, the lower limit does not govern results.

Accuracy—Alternative 2

Results presented in Figs. 4–7, with $P_1L_1/P_2L_2 = 1$ or 2, are representative for all results obtained with such PL-ratios. With these PL-ratios, errors in the predicted results were always found to be within about ±6% of exact results. This can be seen in Table 3. For practical Q_{12}-values, believed to be commonly in the 0.5–2.0 range, errors are within ±5%, which is about the same as found for the total Q_{12}-range. With Q_{12}-values of about 1.0–1.5, errors do not exceed ±2%. With a PL-ratio of 0.5, the maximum errors increase to about −10% when combined with smaller Q_{12}-values (0.3–0.5). However, such combinations are not likely to be representative of practical situations for the frames considered.

With the foregoing qualification, it may be concluded that errors will be within about ±5% for all practical cases of the frames considered. The agreement between $\bar{K}$-predictions and exact results must in general be considered very good. It is emphasized that errors in $\bar{K}$-predictions will become equal in the various columns of a structure. This is due to the relationship imposed by (18).

APPLICATION TO UNSYMMETRICAL BRACED PANEL FRAME

The frames examined were symmetrical about the vertical centerline with respect to loading and geometry. Consequently, the frames could be modeled by a continuous two-story column with exactly defined beam restraints since axial forces (second-order effects) in the beams were neglected. The only interaction between the columns is therefore the vertical interaction. Without symmetry, horizontal interaction will take place between adjacent columns. The effect of lack of symmetry on buckling shapes is schematically illustrated for a single panel frame in Fig. 8.

The first case is representative for cases with Q_{12} close to, but greater than one (i.e., column 2 slightly stiffer than column 1). Both columns buckle into triple curvature bending. The beams deflect into single, but not symmetrical curvature. The beam restraint at column 1 becomes greater, and at column 2 smaller, than that corresponding to symmetrical beam bending.

If column 2 is markedly stiffer than column 1, it may buckle into double curvature (with strong beam restraint at one end), or into single curvature as in the second case considered (Fig. 8). In the latter case, the beams bend into double curvature, and therefore reflect increased beam restraints at the ends of column 1 and decreased (negative) restraints at the ends of column 2. In this manner, the "stronger" one (column 2) contributes to the restraint of the "weaker" one (column 1).

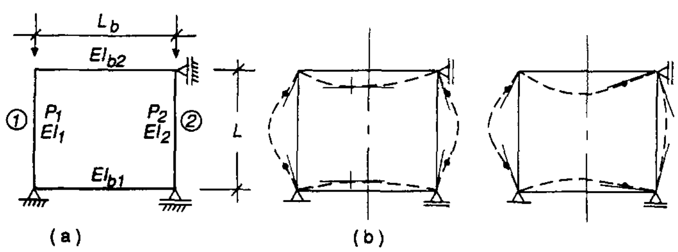

FIG. 8. Single Panel Braced Frame: (a) Frame Definition; (b) Possible Buckling Modes

TABLE 4. Effective Length Factors for Single Panel Braced Frame (Fig. 8)

Q_{12} (1)	Column (2)	Effective Length Factors: Exact (3)	Isolated (4)	Mean 2 (5)	(Isolated/Exact) (6)	(Mean 2/Exact) (7)
1	1	0.794	0.794	0.794	1.00	1.00
	2	0.794	0.794	0.794	1.00	1.00
1.1	1	0.781	0.794	0.780	1.02	1.00
	2	0.819	0.804	0.818	0.98	1.00
2	1	0.744	0.794	0.708	1.07	0.95
	2	1.052	0.864	1.001	0.82	0.95

Note: Frame data $EI_1/L{:}EI_{b1}/L_b{:}\ EI_{b2}/L_b$ = 1:0.333:1.667. $P_1 = P_2 = P$ (i.e., $Q_{12} = EI_2/EI_1$).

Exact effective length factors, computed using stability functions and neglecting axial forces in beams, are given for Q_{12} = 1, 1.1, and 2 in Table 4. Also given are conventional isolated K-values [(2) and b = 2] and predictions according to the alternative 2 of the method of means.

In the symmetrical case (Q_{12} = 1) the beams bend into symmetrical single curvature at frame buckling. Consequently, all three methods give the same effective length factors. With in-

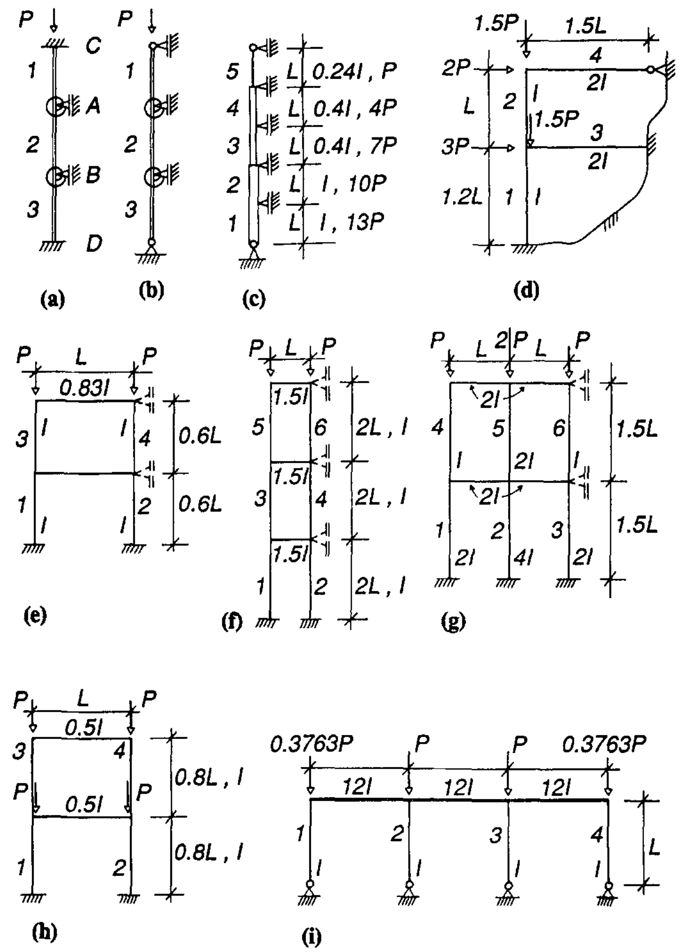

FIG. 9. Structures Considered in Other Investigations

TABLE 5. K-Factor Comparisons for Braced Continuous Three-Story Columns in Fig. 9(a, b)

G_A (1)	G_B (2)	Isolated K_1 (3)	Isolated K_2 (4)	Isolated K_3 (5)	Exact[a] K_{exact} (6)	Mean 2[a] $\bar{K}$ (7)	(Isolated/Exact) (8)	(Mean 2/Exact) (9)
(*a*) Far ends of upper and lower columns are fixed (a)								
0.1	0.3	0.524	0.588	0.563	0.571	0.559	0.92–1.03	0.98
0.2	0.6	0.546	0.649	0.599	0.616	0.600	0.89–1.05	0.97
0.3	0.4	0.563	0.644	0.578	0.606	0.596	0.93–1.06	0.98
0.4	0.8	0.578	0.701	0.615	0.647	0.633	0.89–1.08	0.98
0.5	2.0	0.590	0.765	0.656	0.693	0.674	0.85–1.10	0.97
1.0	0.1	0.626	0.654	0.524	0.634	0.604	0.83–1.03	0.95
25	50	0.695	0.988	0.697	0.809	0.805	0.86–1.22	1.00
(*b*) Far ends of upper and lower columns are hinged (b)								
0.1	0.3	0.732	0.588	0.784	0.746	0.706	0.79–1.05	0.95
0.2	0.6	0.760	0.649	0.835	0.780	0.752	0.83–1.07	0.96
0.3	0.4	0.784	0.644	0.804	0.765	0.747	0.84–1.05	0.98
0.4	0.8	0.804	0.701	0.858	0.804	0.790	0.87–1.07	0.98
0.5	2.0	0.821	0.765	0.923	0.853	0.839	0.90–1.08	0.98
1.0	0.1	0.875	0.654	0.732	0.802	0.759	0.82–1.09	0.95
25	50	0.992	0.988	0.996	0.992	0.992	1.00–1.00	1.00

[a] Q_{ij} = 1. Exact $K_1 = K_2 = K_3$, mean $\bar{K}_1 = \bar{K}_2 = \bar{K}_3 = \bar{K}$.

TABLE 6. *K*-Factor Comparisons for Braced Structures in Fig. 9 (c–g)

Frame (1)	Compression member (2)	Q_n (3)	Effective Length Factors: Exact (4)	Isolated (5)	Mean 2 (6)	(Isolated/Exact) (7)	(Mean 2/Exact) (8)
c	1	1.000	0.96	1.0	0.92	1.04	0.96
	2	0.769	1.10	1.0	1.05	0.91	0.96
	3	1.346	0.83	1.0	0.79	1.21	0.96
	4	0.769	1.10	1.0	1.05	0.91	0.96
	5	0.321	1.68	1.0	1.62	0.60	0.96
d	1	1.000	0.67	0.7	0.66	1.06	0.99
	2	0.347	1.13	1.0	1.12	0.89	0.99
	3	0.781	0.76	0.7	0.75	0.93	0.99
	4	0.521	0.93	1.0	0.92	1.10	0.99
e	1, 2	1.0	0.80	0.68	0.79	0.85	0.99
	3, 4	1.0	0.80	0.89	0.79	1.11	0.99
f	1, 2	1.0	0.68	0.61	0.67	0.90	0.99
	3, 4	1.0	0.68	0.72	0.67	1.07	0.99
	5, 6	1.0	0.68	0.68	0.67	1.00	0.99
g	1, 2, 3	1.0	0.87	0.63	0.83	0.72	0.95
	4, 5, 6	2.0	0.62	0.70	0.59	1.13	0.95

TABLE 7. *K*-Factor Comparisons for Unbraced Frames in Fig. 9 (e–i)

Frame (1)	Column (2)	Q_n (3)	Effective Length Factors: Exact (4)	Isolated (5)	Mean 2 (6)	(Isolated/Exact) (7)	(Mean 2/Exact) (8)
e	1, 2	1.0	1.65	1.44	1.62	0.87	0.99
	3, 4	1.0	1.65	1.79	1.62	1.08	0.99
f	1, 2	1.0	1.21	1.11	1.16	0.92	0.96
	3, 4	1.0	1.21	1.22	1.16	1.01	0.96
	5, 6	1.0	1.21	1.16	1.16	0.96	0.96
g	1, 2, 3	1.0	1.58	1.16	1.46	0.73	0.93
	4, 5, 6	2.0	1.12	1.21	1.03	1.09	0.93
h	1, 2	1.0	1.44	1.50	1.44	1.04	1.00
	3, 4	0.5	2.04	1.94	2.03	0.95	1.00
i	1, 4	1.0	2.77	2.05	2.74	0.74	0.99
	2, 3	2.658	1.70	2.02	1.68	1.19	0.99

creasing Q_{12}-values, isolated factors become increasingly inaccurate. For $Q_{12} = 2$ the error is -18% for column 2. The corresponding error by the method of means is -5%.

APPLICATION TO OTHER STRUCTURES

As an illustration of the broad applicability of the method of means, it has also been applied to various structures that have been analyzed by other researchers. These are shown in Fig. 9. Structures a and b were used by Duan and Chen (1988), c and d by Bridge and Fraser (1987), and e–i by Kuhn (1976). For frames e–g, braced as well as unbraced buckling is considered. Exact effective length factors have been computed using a computer program (BETA) utilizing stability functions, and developed at the University of Oslo (Frislid 1994). Except for frame d, exact factors are also given in the various studies.

Isolated conventional K-factors and $\bar{K}$-predictions (Mean 2) based on alternative 2 of the method of means are compared to exact results in Tables 5, 6, and 7. The isolated K-factors, used as input to the method of means, are obtained from (2) for the braced structures and from (3) for the unbraced structures. Both cases were based on conventional G-factor definitions, with the restraining beams in symmetrical ($b = 2$) and antisymmetrical curvature ($b = 6$), respectively. The $\bar{K}$-predictions for the unbraced cases all satisfied the suggested limitation given by (30) and (31).

Braced Structures—Comments

The three-story columns, a and b, with rotational spring restraints defined by (5) with $b = 2$, have equal flexibility parameters α_E and therefore flexibility ratios $Q_{ij} = 1$. Exact K-factors will consequently be the same for the three columns. This is also the case for the $\bar{K}$-predictions, which in this case are simply given by

$$\bar{K} = \sqrt{\frac{1}{3}(K_1^2 + K_2^2 + K_3^2)} \quad (37)$$

Except for the first case in Table 5, section b, where the suggested limit of (28) might have been adopted (i.e., replacing $\bar{K} = 0.706$ with 0.73), no $\bar{K}$-predictions are below the proposed lower limit constraints. This is also the case for the comparisons in the following.

The tiered continuous five-story column, c, and the "wharf" structure, d, consist of compression members only. Due to the absence of restraining (beam) elements, the G-factors become infinite at the member joints, and the isolated K-factors are either equal to 1.0 (c) and 1.0 or 0.7 (d).

Isolated K-predictions for the braced structures a–g are between 40% below to 22% above exact results (Tables 5 and 6). In comparison, mean $\bar{K}$-predictions (alternative 2) are all at the most 5% below the exact results. Alternative 1 predictions (not shown) are in all cases slightly below the predictions according to alternative 2.

Unbraced Structures—Comments

Structures e–i are the only unbraced ones that have been considered in this study. The four first cases, e–h, demonstrate the ability of the method of means to correct for the vertical interaction effects between columns in different stories. In the last case, i, it is the horizontal sway interaction between columns with different axial load levels that is reflected. The method of means therefore offers an alternative to the story sway (or moment) magnifier approach.

The accuracy of the mean $\bar{K}$-predictions, as shown in Table 7, is comparable to that found for braced structures. However, the structures considered in the unbraced mode do not represent a particularly wide range of influencing parameters. A wider range should be considered to verify (30) and (31), and to identify other possible limitations of applicability.

CONCLUDING REMARKS

It was found that the method of means in either of the two alternatives presented offers a simple and accurate approach for the computation of effective length factors for continuous columns and frames. Its use of isolated effective length factors obtained by standard methods makes the approach particularly attractive.

Alternative 2, with isolated factors based on the conventional approach, was found to give the better predictions. Errors were within a few percent of the exact results for the cases considered. Further, an important property of the method of means is that errors will be of the same magnitude, and in the same direction (positive or negative) for all columns. It is therefore recommended that predictions should be increased by about 5% in actual design.

The method is based on general principles and is applicable to braced frames and to a range of unbraced frames. The method is suitable to account for vertical interaction between columns on different stories of a frame and to account for horizontal interaction between columns on the same level. For unbraced frames, and in particular for such frames containing laterally very flexible columns, the range of validity of the method of means needs to be investigated further.

The method has been found to be especially useful in the analysis of frames with regions where abrupt changes of member stiffness occur. Such regions are found where there are (1) sudden changes in beam restraint conditions, such as in the top and bottom stories; (2) sudden changes in column stiffness (dimensions) at a floor level; and (3) sudden changes in axial force levels and in story heights. It is known that the approximate, isolated K-predictions are particularly inadequate for such cases.

In the applications of the method to the structures considered in this paper, all compression members were included in the summations leading to the mean $\bar{K}$-predictions. However, it is not envisioned that all compression members necessarily should be included. For instance, for large multistory frameworks partial application to a limited number of columns in each region with significant stiffness changes might possibly be adequate. For such applications, some guidelines for practical use of the method require further study.

APPENDIX I. REFERENCES

Bjorhovde, R. (1984). "Effect of end restraint on column strength: practical applications." *Engrg. J.*, AISC, 21(1), 1–13.

Bridge, R. Q., and Fraser, D. J. (1986). "Improved *G*-factor method for evaluating effective lengths of columns." *J. Struct. Engrg.*, ASCE, 113(6), 1341–1356.

Duan, L., and Chen, W. F. (1988). "Effective length factors for columns in braced frames." *J. Struct. Engrg.*, ASCE, 114(10), 2357–2370.

Duan, L., and Chen, W. F. (1989). "Effective length factor for columns in unbraced frames." *J. Struct. Engrg.*, ASCE, 115(1), 149–165.

Frislid, A. (1994). "Evaluation of simplified methods for stability analysis of frame structures," Cand.scient. thesis, Mech. Div., Dept. of Math., Univ. of Oslo, Oslo, Norway (in Norwegian).

Galambos, T. V. (1968). *Structural members and frames*. Prentice-Hall, Inc., New York, N.Y.

Hellesland, J. (1992a). "Effective lengths—an appraisal of various restraint participation factors." *Res. Rep. in Mech., No. 92-4 (May)*, Mech. Div., Dept. of Math., Univ. of Oslo, Oslo, Norway.

Hellesland, J. (1992b). "Effective lengths—simplified system instability analysis." *Res. Rep. in Mech., No. 92-5 (May)*, Mech. Div., Dept. of Math., Univ. of Oslo, Oslo, Norway.

Hellesland, J. (1995). "Simplified system instability analysis." *Proc., 4th Pacific Struct. Steel Conf. (PSSC); Vol. 1, Steel Structures*, N. E. Shanmugam and Y. S. Choo, eds., Pergamon Press, Inc., Tarrytown, N.Y., 95–102.

Hellesland, J., and Bjorhovde, R. (1996). "Restraint demand factors and effective lengths of braced columns." *J. Struct. Engrg.*, ASCE, 122(10), 1216–1224.

Kuhn, G. K. (1976). "An appraisal of the effective length alignment charts," MS thesis, Arizona State Univ., Tempe, Ariz.

LeMessurier, W. J. (1977). "A practical method of second-order analysis. 2: Rigid frames." *Engrg. J.*, AISC, 14(2), 49–67.

Yura, J. A. (1971). "Effective length of columns in unbraced frames." *Engrg. J.*, AISC, 8(2), 37–42.

APPENDIX II. NOTATION

The following symbols are used in this paper:

$b(b_0)$ = rotational beam stiffness coefficient (reference value);
C = coefficient in Eq. (30);
d = coefficient in Eq. (31);
EI, EI_b = flexural bending stiffness of column and beam section (subscript b);
f = restraint demand factor at a column end ($=k/k_b$);
G = joint stiffness ratio between columns and beams at a joint;
g = stiffness ratio between one column and the beams at a joint;
K = effective length factor;
$\bar{K}$ = weighted mean effective length factor;
k = rotational restraint stiffness at a column end;
k_b = rotational stiffness of restraining beams at a joint;
L, L_b = length of column and beam (subscript b);
L_e = effective length ($=KL$);
m = number of interacting columns in the method of means;
m = beam stiffness modifier ($=b/b_0$), Eq. (6);
n_c = number of columns meeting at a joint;
P = axial force in column (compression member);
P_E = Euler buckling load of pin-ended column;
Q_{ij} = column flexibilities ratio ($=\alpha_{Ei}/\alpha_{Ej}$) between columns i and j;
q = ratio between column EI/L-values (column 2 to column 1); and
α_E = column flexibility parameter ($=P/P_E$).