Kap 7

Oppg. 10

En stang er μ lang. La $X_1, \dots X_n$ være målinger av stangen. Anta

$$X_i \sim N(\mu, \sigma^2)$$

Vil estimere μ^2.

Forslag på estimator

$$\hat{\mu}^2 = \bar{X}^2.$$

Vil vise at $\hat{\mu}$ ikke er forventningsrett.

mai 10-10:15

$$E[\hat{\mu}^2] = E[\bar{X}^2]$$
$$= Var(\bar{X}) + E[\bar{X}]^2$$
$$= \underbrace{Var(\bar{X})}_{\neq 0} + \mu^2$$
$$\neq \mu^2$$

Så $\hat{\mu}^2$ er ikke forventningsrett.

mai 10-10:15

Foreslår en annen estimator.

$$\tilde{\mu}^2 = (\bar{X})^2 - kS^2.$$

$$E[\tilde{\mu}^2] = E[\bar{X}^2 - kS^2]$$
$$= E[\bar{X}^2] - kE[S^2]$$
$$= Var(\bar{X}) + \mu^2 - k\,Var(X)$$
$$= Var\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) + \mu^2 - k\,Var(X)$$
$$= \frac{1}{n\cancel{n}}\cancel{n}\,Var(X) + \mu^2 - k\,Var(X)$$
$$= \mu^2 \quad \underline{\text{hvis } k = \frac{1}{n}}$$

mai 10-10:23

12) Anta $X \sim N(\mu_1, \sigma^2)$

$Y \sim N(\mu_2, \sigma^2)$.

Anta vi har m obs. av X'er og n obs. av Y'er. Vil estimere σ^2.

Forslag

$$\hat{\sigma}^2 = \frac{(n_1-1)S_1^2 + (n_2-1)S_2^2}{n_1+n_2-2}.$$

Vil vise at $\hat{\sigma}^2$ er forventningsrett.

mai 10-10:30

$$E[\hat{\sigma}^2] = E\left[\frac{(n_1-1)S_1^2+(n_2-1)S_2^2}{n_1+n_2-2}\right]$$

$$= \frac{(n_1-1)E[S_1^2] + (n_2-1)E[S_2^2]}{n_1+n_2-2}$$

$$= \frac{(n_1-1)\sigma^2 + (n_2-1)\sigma^2}{n_1+n_2-2}$$

$$= \underline{\sigma^2}$$

Dermed er $\hat{\sigma}^2$ forventningsrett.

mai 10-10:34

16) $X_1, \dots, X_m \sim N(\mu, \sigma^2)$
$Y_1, \dots, Y_n \sim N(\mu, 4\sigma^2)$

Vil estimere μ.

Forslag

$$\hat{\mu} = \delta \bar{X} + (1-\delta)\bar{Y}.$$

a)
$$E[\hat{\mu}] = E[\delta \bar{X} + (1-\delta)\bar{Y}]$$
$$= \delta E[\bar{X}] + (1-\delta)E[\bar{Y}]$$
$$= \delta\mu + (1-\delta)\mu$$
$$= \mu.$$

Så $\hat{\mu}$ er forventningsrett

mai 10-10:37

b) Vi har at

$$Var(\hat{\mu}) = Var(\delta \bar{X} + (1-\delta)\bar{Y})$$
$$= Var(\delta\bar{X}) + Var((1-\delta)\bar{Y})$$
$$= \delta^2 Var(\bar{X}) + (1-\delta)^2 Var(\bar{Y})$$
$$= \delta^2 \frac{\sigma^2}{m} + (1-\delta)^2 \frac{4\sigma^2}{n}.$$

La

$$f(\delta) = \delta^2 \frac{\sigma^2}{m} + (1-\delta)^2 \frac{4\sigma^2}{n}$$

mai 10-10:42

Vi har at

$$f'(\delta) = 2\delta\frac{\sigma^2}{m} + 2(1-\delta)\frac{4\sigma^2}{n}(-1)$$
$$= 2\delta\frac{\sigma^2}{m} - \frac{8\sigma^2}{n} + \frac{8\sigma^2}{n}\delta$$
$$= 0.$$

$$\Rightarrow \quad \delta\frac{\sigma^2}{m} + \frac{4\sigma^2}{n}\delta = \frac{4\sigma^2}{n}$$
$$\delta\left(\frac{\sigma^2}{m} + \frac{4\sigma^2}{n}\right) = \frac{4\sigma^2}{n}$$

mai 10-10:48

c) $\delta = \ldots$

$$= \frac{4m}{n+4m}.$$

Så $\delta = \dfrac{4m}{n+4m}$

minim variansen til estimatoren.

mai 10-10:52

EO2

Vi har at $X_1, \ldots, X_n$ er fordelt slik at

x	-1	1
$P(X=x)$	$\frac{1}{2}$	$\frac{1}{2}$

at vi har at

$$E[X] = \tfrac{1}{2}(-1) + \tfrac{1}{2}\cdot 1 = 0$$

$$Var(X) = E[X^2] - E[X]^2$$
$$= \tfrac{1}{2}(-1)^2 + \tfrac{1}{2}(1)^2 - 0$$
$$= 1.$$

mai 10-10:55

La $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$.

$$E[\bar{X}] = E\left[\frac{1}{n}\sum_{i=1}^{n} X_i\right]$$
$$= \frac{1}{n}\, n\, E[X] = E[X] = 0$$

$$Var(\bar{X}) = Var\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right)$$
$$= \frac{1}{n^2}\, n\, Var(X) = \frac{Var(X)}{n}.$$
$$= \frac{1}{n}.$$

mai 10-11:17

b) Vi har at $\boxed{M_X(t) = E[e^{tX}]}$

$$M_X(t) = E[e^{tX}] = \frac{1}{2}e^{-t} + \frac{1}{2}e^{t}.$$
$$= \frac{1}{2}\left(e^{-t} + e^{t}\right)$$

$$M_{\bar{X}}(t) = E[e^{t\bar{X}}] = E\left[e^{t\frac{1}{n}\sum_{i=1}^{n} X_i}\right]$$
$$= E\left[e^{\frac{t}{n}X_1} e^{\frac{t}{n}X_2} \cdots e^{\frac{t}{n}X_n}\right]$$

mai 10-11:20

$$= E\left[e^{\frac{t}{n}X_1}\right] \cdots E\left[e^{\frac{t}{n}X_n}\right].$$

Vi har at

$$E\left[e^{\frac{t}{n}X}\right] = \frac{1}{2}e^{-\frac{t}{n}} + \frac{1}{2}e^{\frac{t}{n}}$$
$$= \frac{1}{2}\left(e^{-\frac{t}{n}} + e^{\frac{t}{n}}\right)$$

$$= \frac{1}{2^n}\left(e^{-\frac{t}{n}} + e^{\frac{t}{n}}\right)^n.$$

mai 10-11:24

Vi har at

$$M_X'(t) = \frac{1}{2}\left(-e^{-t} + e^{-t}\right)$$
$$M_X''(t) = \frac{1}{2}\left(e^{-t} + e^{-t}\right)$$
$$M_{\bar{X}}'(t) = \frac{1}{2^n}\cancel{n}\left(e^{-\frac{t}{n}} + e^{\frac{t}{n}}\right)^{n-1} \cdot \left(-e^{-\frac{t}{n}} + e^{-\frac{t}{n}}\right) \cdot \frac{1}{\cancel{n}}$$
$$M_{\bar{X}}''(t) = \frac{1}{2^n}\left(e^{-\frac{t}{n}} + e^{t/n}\right)^{n-1} \cdot \left(\frac{1}{n}e^{-t/n} + \frac{1}{n}e^{-t/n}\right)$$
$$+ \frac{1}{2^n}(n-1)\left(e^{-t/n} + e^{t/n}\right)^{n-2} \cdot \left(-e^{-t/n} + e^{t/n}\right)^2$$

mai 10-11:28

Vi ser at

$$E[X] = M_X'(0) = 0$$
$$V(X) = M_X''(0) - M_X'(0)^2$$
$$= 1 - 0 = 1$$
$$E[\bar{X}] = 0$$
$$V(\bar{X}) = M_{\bar{X}}''(0) - M_{\bar{X}}'(0)^2$$
$$= \frac{1}{n} + 0 = \underline{\underline{\frac{1}{n}}}$$

mai 10-11:34

c) La $n = 100$

Vi har da fra sentralgrenseteoremet at

$$\frac{\bar{X} - \mu_X}{\sigma_{\bar{X}}} \overset{\text{til}}{\sim} N(0,1).$$

Vil finne

$$P(\bar{X} > 0{,}2).$$

mai 10-11:37

Vi har at

$$P(\bar{x} > 0{,}2) = P\left(\frac{\bar{X} - \mu_x}{\sigma_{\bar{x}}} > \frac{0{,}2 - \mu_x}{\sigma_{\bar{x}}}\right)$$

$$= 1 - \Phi\left(\frac{0{,}2 - 0}{\frac{1}{\sqrt{100}}}\right)$$

$$= 0{,}0228.$$

mai 10-11:40

Kap 8

Oppg 4

La $X_1, \dots, X_n \sim N(\mu, \sigma^2)$

hvor $\sigma = 3{,}0$.

Vi vet at

$$\frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}} \sim N(0,1).$$

mai 10-11:43

Vi har at

$$P\left(-1{,}96 < \frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}} < 1{,}96\right) = 0{,}95$$

Det gir

$$P\left(\bar{X} - 1{,}96\frac{\sigma}{\sqrt{n}} < \mu < \bar{X} + 1{,}96\frac{\sigma}{\sqrt{n}}\right) = 0{,}95$$

Så et 95 % K.I. for μ er

$$\left(\bar{X} - 1{,}96\frac{\sigma}{\sqrt{n}},\ \bar{X} + 1{,}96\frac{\sigma}{\sqrt{n}}\right).$$

mai 10-11:46

a) $\bar{x} = 58{,}3$, $\sigma = 3$ og $n = 25$

Vi får da 95% K.I.

$$(57{,}1,\ 59{,}5).$$

mai 10-11:53

12) Vet ikke lenger σ.
Vi kan estimere σ med

$$s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2.$$

Hvis n er stor så er

$$\frac{\bar{X}-\mu}{\frac{s}{\sqrt{n}}} \overset{\text{til}}{\sim} N(0,1).$$

Det gir det tilnærmede 99% K.I. $\bar{X} \pm 2{,}58\,\frac{s}{\sqrt{n}}$.

mai 10-11:56

Så er tilnærmet 99% K.I. for μ er her

$$(0{,}73\ ,\ 0{,}89).$$

NB. Jeg ville ikke alltid fort oppgavene slik som her. Jeg har med mye forklaring siden det er en plenumstime.

mai 10-12:00