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Repetisjonsforelesning
12. 5. 2011

<u>1. Prob.</u> : Warehouse with a capacity of c units of stock

D_n <u>demand</u> for the stock in units at time period n

$\longrightarrow$ X_n <u>residual stock</u> at the end of period n

The manager <u>restocks</u> to capacity c at the beginning of $n+1$, <u>if</u>

$$X_n \le m \qquad (m \in \{0,\dots,c-1\})$$

$$\longrightarrow \quad X_{n+1} = \begin{cases} (c - D_{n+1})^+, & X_n \le m \\ (X_n - D_{n+1})^+, & m < X_n \le c \end{cases} \qquad (1)$$

! $(a)^+ \overset{def}{=} \max(0, a)$

$D_n, n \ge 1$ i.i.d with common law D given by

$$P(D \ge i) = 3^{-i}$$

$$\Longrightarrow \quad P(D = i) = 3^{-i} - 3^{-(i+1)} \qquad (2)$$

$c = 2$

<u>Objective</u> : $a\,\bar{g}(m) + b\,\bar{f}(m) \longrightarrow \min$ w.r.t. m

for $b = 1$ \$

$$\bar{f}(m) \overset{def}{=} \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} f(X_j) \overset{\text{Ergodic theorem}}{=} \sum_{i=0}^{c} \pi_i f(i)$$

π_i: inv. distr.

$$f(i) \overset{def.}{=} E[\hat{X}_{n+1} \mid X_n = i] \qquad (3)$$

$$\hat{X}_{n+1} = \begin{cases} (D_{n+1} - c)^+, & X_n \le m \\ (D_{n+1} - X_n)^+, & m < X_n \le c \end{cases}$$

$\longrightarrow$ $\hat{X}_{n+1}$ unmet demand at $n+1$

$\longrightarrow$ $\bar{f}(m)$ <u>long-run proportion of unmet demand</u>

$$\bar{g}(m) \overset{def}{=} \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} g(X_j) \overset{\text{Ergodic theorem}}{=} \sum_{i=0}^{c} \pi_i g(i)$$

$$g(i) \overset{def}{=} 1\!\!1_{\{i \le m\}} = \begin{cases} 1, & i \le m \\ 0 & \text{else} \end{cases}$$

$\longrightarrow$ $\bar{g}(m)$ <u>long-run frequency of restocking</u>

$a \triangleq$ cost of restocking per unit stock

$b \triangleq$ (lost) profit per unit stock

② $\bar{g}(m), \bar{f}(m)$? for $m = 0, 1$ Repetisjon

$\longrightarrow$ 1. step: Computation of P_{ij} for $m = 0, 1$

$$P_{ij} \stackrel{\text{def}}{=} P(X_{n+1} = j \mid X_n = i) \underset{D_n \text{ i.i.d}}{\overset{(1)}{=}} \begin{cases} P((2-D)^+ = j), & i \le m \\ P((i-D)^+ = j), & m < i \le 2 \end{cases} \quad (4)$$

See Ex. 4.2.1

$\Rightarrow$ $m = 0$:

$$P_{00} \overset{(4)}{=} P((2-D)^+ = 0) = P(D \ge 2) \overset{(2)}{=} 3^{-2} = \frac{1}{9}$$
$$P_{01} = P((2-D)^+ = 1) = P(D = 1) \overset{(2)}{=} 3^{-1} - 3^{-2} = \frac{2}{9}$$
$$P_{02} = P((2-D)^+ = 2) = P(D = 0) = 3^{-0} - 3^{-1} = \frac{2}{3}$$
$$P_{10} \overset{(4)}{=} P((1-D)^+ = 0) = P(D \ge 1) = 3^{-1} \overset{(2)}{=} \frac{1}{3}$$
$$P_{11} \overset{(4)}{=} P((1-D)^+ = 1) = P(D = 0) = 3^{-0} - 3^{-1} = \frac{2}{3}$$
$$P_{12} \overset{(4)}{=} P((1-D)^+ = 2) = 0$$
$$P_{20} \overset{(4)}{=} P((2-D)^+ = 0) = P(D \ge 2) = 3^{-2} = \frac{1}{9}$$
$$P_{21} = P((2-D)^+ = 1) = P(D = 1) = 3^{-1} - 3^{-2} = \frac{2}{9}$$
$$P_{22} = P((2-D)^+ = 2) = P(D = 0) = 3^{-0} - 3^{-1} = \frac{2}{3}$$

$$\longrightarrow \quad P = \begin{pmatrix} 1/9 & 2/9 & 2/3 \\ 1/3 & 2/3 & 0 \\ 1/9 & 2/9 & 2/3 \end{pmatrix}$$

$m = 1$:

$$P_{00} \overset{(4)}{=} P((2-D)^+ = 0) = P(D \ge 2) \overset{(2)}{=} 3^{-2} = \frac{1}{9}$$
$$\vdots$$

$$\longrightarrow \quad P = \begin{pmatrix} 1/9 & 2/9 & 2/3 \\ 1/9 & 2/9 & 2/3 \\ 1/9 & 2/9 & 2/3 \end{pmatrix}$$

2. step: Computation of $\pi = (\pi_0, \pi_1, \pi_2)$

$m = 0$: π inv. distr. $\Longleftrightarrow$ $\pi \cdot P = \pi$ and $\pi_0 + \pi_1 + \pi_2 = 1$

$$\Longleftrightarrow \begin{cases} \pi_0 \frac{1}{9} + \pi_1 \frac{1}{3} + \pi_2 \cdot \frac{1}{9} = \pi_0 \\ \pi_0 \cdot \frac{2}{9} + \pi_1 \cdot \frac{2}{3} + \pi_2 \cdot \frac{2}{9} = \pi_1 \\ \pi_0 \frac{2}{3} + \pi_1 \cdot 0 + \pi_2 \cdot \frac{2}{3} = \pi_2 \end{cases} \quad \text{and } \pi_0 + \pi_1 + \pi_2 = 1$$

$$\Longrightarrow \pi_0 \frac{2}{3} = \frac{1}{3}\pi_2 \Longrightarrow \pi_0 = \frac{1}{2}\pi_2$$

$$\Longrightarrow \frac{1}{3}\pi_2 = \frac{1}{3}\pi_1 \Longrightarrow \pi_2 = \pi_1$$

$$\xRightarrow{\pi_0 + \pi_1 + \pi_2 = 1} \frac{1}{2}\pi_2 + \pi_2 + \pi_2 = 1 \Longrightarrow \pi_2 = \frac{2}{5}$$

$$\Longrightarrow \pi = (\pi_0, \pi_1, \pi_2) = \underline{\left(\frac{1}{5}, \frac{2}{5}, \frac{2}{5}\right)}$$

$m = 1$: $\pi = \underline{\left(\frac{1}{9}, \frac{2}{9}, \frac{2}{3}\right)}$

③ <u>3. step</u> : Comp. of $f(i)$, $i = 0,1,2$ Repetisjon

$$f(i) \underset{D_n \text{ i.i.d}}{\overset{(3)}{=}} \begin{cases} E[(D-2)^+] & , i \le m \\ E[(D-i)^+] & , m < i \le 2 \end{cases} \quad (5)$$

See Ex. 4.2.1

$$E[(D-j)^+] = \sum_{k \ge 1} \overbrace{P((D-j)^+ \ge k)}^{= P(D \ge k+j)}$$

$$= \sum_{k \ge 1} \underbrace{P(D \ge k+j)}_{\overset{(2)}{=} 3^{-(k+j)}} \overset{(2)}{=} \sum_{k \ge 1} 3^{-(k+j)} = 3^{-j} \underbrace{\left(\sum_{k \ge 1} 3^{-k}\right)}_{\frac{1}{1-\frac{1}{3}} - 1 = \frac{1}{2}}$$

$$= 3^{-j} \cdot \frac{1}{2} \quad (6)$$

<u>Recall</u> : (i) $E[Y] = \sum_{k \ge 1} P(Y \ge k)$, $Y \in \{0,1,2,\dots\}$

(ii) $\sum_{k \ge 0} q^k = \frac{1}{1-q}$, $0 \le q < 1$

<u>m = 0</u> : $f(0) \overset{(5)}{=} E[(D-2)^+] \underset{j=2}{\overset{(6)}{=}} 3^{-2} \cdot \frac{1}{2} = \frac{1}{18}$

$f(1) \overset{(5)}{=} E[(D-1)^+] \overset{(6)}{=} 3^{-1} \cdot \frac{1}{2} = \frac{1}{6}$

$f(2) \overset{(5)}{=} E[(D-2)^+] = \frac{1}{18}$

<u>m = 1</u> $f(0) = f(1) = f(2) = E[(D-2)^+] \overset{(6)}{=} 3^{-2} \cdot \frac{1}{2} = \frac{1}{18}$

<u>4. step</u> : Comp. of $\bar{f}(m)$, $\bar{g}(m)$, $m = 0,1$

<u>m = 0</u> : $\bar{f}(0) \overset{\text{def}}{=} \sum_{i=0}^{2} \pi_i f(i) \overset{2.,3.\text{step}}{=} \frac{1}{5} \cdot \frac{1}{18} + \frac{2}{5} \cdot \frac{1}{6} + \frac{2}{5} \cdot \frac{1}{18}$

$= \frac{9}{90} = \frac{1}{10}$

$\bar{g}(0) \overset{\text{def}}{=} \sum_{i=0}^{2} \pi_i \mathbb{1}_{\{i \le 0\}} = \pi_0 \overset{2.\text{step}}{=} \frac{1}{5}$

$\Longrightarrow$ $\underline{a\bar{g}(0) + b\bar{f}(0) = \frac{a}{5} + \frac{1}{10}}$

<u>m = 1</u> : $\bar{f}(1) \overset{\text{def}}{=} \sum_{i=0}^{2} \pi_i f(i) \overset{2.,3.\text{step}}{=} \frac{1}{9} \cdot \frac{1}{18} + \frac{2}{9} \frac{1}{18} + \frac{2}{3} \cdot \frac{1}{18}$

$= \frac{1}{18}$

$\bar{g}(1) \overset{\text{def}}{=} \sum_{i=0}^{2} \pi_i \mathbb{1}_{\{i \le 1\}} = \pi_0 + \pi_1 \overset{2.\text{step}}{=} \frac{1}{9} + \frac{2}{9} = \frac{1}{3}$

$\Longrightarrow$ $\underline{a\bar{g}(1) + b\bar{f}(1) = \frac{a}{3} + \frac{1}{18}}$

$\Longrightarrow$ costs become for <u>m = 0</u> minimal $\Longleftrightarrow$

$\frac{a}{5} + \frac{1}{10} \le \frac{a}{3} + \frac{1}{18} \Longleftrightarrow \frac{1}{10} - \frac{1}{18} \le \frac{a}{3} - \frac{a}{5} \Longleftrightarrow \underline{a \ge \frac{1}{3}}$ \$

for <u>m = 1</u> minimal $\Longleftrightarrow$

$\frac{a}{3} + \frac{1}{18} \le \frac{a}{5} + \frac{1}{10} \Longleftrightarrow \underline{a \le \frac{1}{3}}$ \$

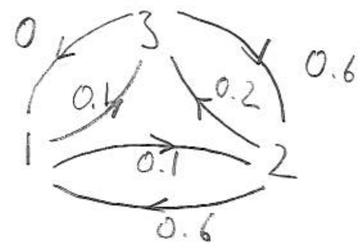
Repetisjon

(4) 2. Prob: Consider the car insurance of Ex. 3.2.7

$\longrightarrow$ $X_n \in \mathcal{I} = \{1,2,3\}$ risk category of the insured in year n

$X_n, n \geq 0$ Markov(λ, P) with

$$P = \begin{pmatrix} 0.8 & 0.1 & 0.1 \\ 0.6 & 0.2 & 0.2 \\ 0 & 0.6 & 0.4 \end{pmatrix}$$



$\underline{K_i^A \, ?}$

for $i = 1,2,3$ and $A = \{1\}$

$K_i^A \stackrel{\text{def}}{=} E[H^A \mid X_0 = i]$

$\longrightarrow$ mean time that the chain starting in i ever hits $A = \{1\}$

$H^A := \inf\{n \geq 0 : X_n \in A\}$ hitting time

$\longrightarrow$ first time that X_n hits A

$\longrightarrow$ K_i^A mean time that the customer starting in risk category i ever enters the lowest risk category

By Th. 3.2.6 K_i^A, $i \in \mathcal{I} = \{1,2,3\}$ is the min. non-neg. solution to the equations

$$\begin{cases} K_i^A = 0, & i \in A \\ K_i^A = 1 + \sum_{j \notin A} p_{ij} K_j^A, & i \notin A \end{cases}$$

$$\stackrel{A=\{1\}}{\Longleftrightarrow} \begin{cases} K_1^A = 0 \\ K_2^A = 1 + \underbrace{p_{22}}_{=0.2} K_2^A + \underbrace{p_{23}}_{=0.2} K_3^A \\ K_3^A = 1 + \underbrace{p_{32}}_{=0.6} K_2^A + \underbrace{p_{33}}_{=0.4} K_3^A \end{cases}$$

$$\Longrightarrow \quad 0.8 K_2^A = 1 + 0.2 \cdot K_3^A \quad | \cdot 3$$
$$0.6 K_3^A = 1 + 0.6 K_2^A$$

$$\Longrightarrow \quad 2.4 K_2^A = 3 + 0.6 K_3^A$$
$$0.6 K_3^A = 1 + 0.6 K_2^A$$

$$\Longrightarrow \quad 2.4 K_2^A = 3 + 1 + 0.6 K_2^A \Longleftrightarrow K_2^A = \frac{4}{1.8} \approx \underline{2.222 \text{ years}}$$

$$\Longrightarrow \quad K_3^A = \frac{1}{0.6}\left(1 + 0.6 \cdot \frac{4}{1.8}\right) \approx \underline{3.889 \text{ years}}$$

(5) period of $i = 1, 2, 3$?

$$p_{ii}^{(n)} = P(X_n = i \mid X_0 = i)$$

$$\geq P(X_0 = i, X_1 = i, X_2 = i, \ldots, X_n = i \mid X_0 = i)$$

$$\overset{\text{Th.3.1.3}}{=} \underbrace{p_{ii}}_{>0} \underbrace{p_{ii}}_{>0} \cdots \underbrace{p_{ii}}_{>0} > 0$$

for all n

$\Longrightarrow$ $i = 1, 2, 3$ aperiodic