UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Exam in:	STK2130 — Modelling by Stochastic Processes.
Day of exam:	Wednesday June 12th 2013.
Exam hours:	14.30 – 18.30.

This examination set consists of 3 pages.

Appendices:	None.
Permitted aids:	Approved calculator. "Formelsamling" for STK1100 and STK1110

Make sure that your copy of this examination set is complete before answering.

Problem 1.

a) There are three classes

1. State 0, transient
2. States 1 and 2, recurrent
3. State 3, transient

b) When $X_0 = 1$ or $X_0 = 2$ the process starts in a recurrent and finite class with period 1, thus ergodic, and the limits $\pi_j = \lim_{n\to\infty} P_{ij}^n$ are determined by (1) $\pi_1 P_{11} + \pi_2 P_{21} = \frac{1}{2}\pi_1 + \frac{3}{4}\pi_2$ and (2) $\pi_1 + \pi_2 = 1$. The solution is $\pi_1 = 0.6, \pi_2 = 0.4$.

Since the chain will eventually reach the recurrent class also the $\lim_{n\to\infty} P_{ij}^n = \pi_j$ when $i = 0$ or $i = 3$ and $j = 1$ and 2. For other values of j the limit is zero.

c) The average converges to $\pi_1 + 2\pi_2 = 1.4$ by the law of large numbers for ergodic Markov chains. This is the expectation of X_n in a stationary state with $\pi_j = P(X_n = j)$.

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d) By a one-step argument $\nu_0 = 1 + \nu_0 P_{00} = 1 + \nu_0 \frac{1}{3}$. Thus $\nu_0 = \frac{3}{2}$. Similarly $\nu_3 = 1 + \nu_0 \frac{1}{4} + \nu_3 \frac{1}{4}$ which implies that $\nu_3 = \frac{11}{6}$.

e) If $X_0 = 0$ then A_0 has a geometric distribution on $1, 2, 3, \ldots$ with probability $p = \frac{2}{3}$, thus $\mathrm{E}[A_0|X_0 = 0] = \frac{1}{p} = \frac{3}{2} = \nu_0$ from c).

Similarly $\mathrm{E}[A_3|X_0 = 3] = \frac{1}{q} = \frac{4}{3}$ for $q = \frac{3}{4}$.

For $\mathrm{E}[A_0|X_0 = 3]$ introduce $Y = 1$ if the process ever enters state 0 and $Y = 0$ otherwise.

Then (1) $P(Y = 1|X_0 = 3) = \frac{1}{4} + (\frac{1}{4})^2 + (\frac{1}{4})^3 + \cdots = \frac{1}{3}$ and

(2) $\mathrm{E}[A_0|X_0 = 3] = \mathrm{E}\mathrm{E}[A_0|Y, X_0 = 3] = \frac{3}{2} P(Y = 1|X_0 = 3) + 0 \cdot P(Y = 0|X_0 = 3) = \frac{1}{2}$

Problem 2.

a)

$$\begin{aligned}
P_{ij}(t+s) &= \mathrm{P}(X(t+s) = j|X(0) = i) \\
&= \mathrm{P}\textstyle\bigcup_{k=1,2,3}(X(t+s) = j, X(t) = k|X(0) = i) \\
&= \textstyle\sum_{k=0}^{2} \mathrm{P}(X(t+s) = j, X(t) = k|X(0) = i) \\
&= \textstyle\sum_{k=0}^{2} \mathrm{P}(X(t+s) = j, |X(t) = k)\mathrm{P}(X(t) = k|X(0) = i) \\
&= \textstyle\sum_{k=0}^{2} P_{ik}(t) P_{kj}(s)
\end{aligned}$$

which are the Chapman-Kolmogorov equations.

b) By Chapman-Kolmogorov $\mathbf{P}(t+h) - \mathbf{P}(t) = \mathbf{P}(t)\mathbf{P}(h) - \mathbf{P}(t) = \mathbf{P}(t)(\mathbf{P}(h) - \mathbf{I}) = \mathbf{P}(t)(\mathbf{P}(h) - \mathbf{P}(0))$. Thus $\mathbf{P}'(t) = \lim_{h\to 0} \frac{1}{h}[\mathbf{P}(t+h) - \mathbf{P}(t)] = \lim_{h\to 0} \mathbf{P}(t)\frac{1}{h}(\mathbf{P}(h) - \mathbf{I}) = \mathbf{P}(t)\mathbf{R}$ which are the Kolomgorov forward equations.

c) With $v_i = q_{ij} + q_{ij'}$ for j and j' different from i we have

$$\mathbf{R} = \begin{bmatrix} -v_0 & q_{01} & q_{02} \\ q_{10} & -v_1 & q_{12} \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -3\mu & 2\mu & \mu \\ \mu & -3\mu & 2\mu \\ 0 & 0 & 0 \end{bmatrix}$$

The remaining $P_{ij}(t)$ are given by $P_{02}(t) = 1 - P_{00}(t) - P_{01}(t)$, $P_{12}(t) = 1 - P_{10}(t) - P_{11}(t)$ $P_{20}(t) = P_{21}(t) = 0$ and $P_{22}(t) = 1$.

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d) By straightforward derivation of the given expression for $P_{00}(t)$ we get

$$P'_{00}(t) = \frac{-\mu}{2}[(3-\sqrt{2})\exp(-(3-\sqrt{2})\mu t) + (3+\sqrt{2})\exp(-(3+\sqrt{2})\mu t)].$$

According to the Kolmogorov equations $P'_{00}(t) = -3\mu P_{00}(t) + \mu P_{01}(t)$ which can be expressed as

$$\begin{array}{l} \frac{-\mu}{2}[3\exp(-(3-\sqrt{2})\mu t) + 3\exp(-(3+\sqrt{2})\mu t) \\ -\sqrt{2}\exp(-(3-\sqrt{2})\mu t) + \sqrt{2}\exp(-(3+\sqrt{2})\mu t)] \end{array}$$

which after redistribution of the terms equals the straightforward derivative of $P'_{00}(t)$.

e) Let $I_1(t)$ be the indicator that $X(t) = 1$. Then $B = \int_0^\infty I_1(t)dt$. Thus $\mathrm{E}[B|X(0) = 0] = \mathrm{E}[\int_0^\infty I_1(t)dt|X(0) = 0] = \int_0^\infty \mathrm{E}[I_1(t)|X(0) = 0]dt$. But this again equals $\int_0^\infty P_{01}dt = \frac{1}{\sqrt{2}}[\frac{1}{(3-\sqrt{2})\mu} - \frac{1}{(3+\sqrt{2})\mu}] = \frac{2}{7\mu}$.

Similarly the expected time in state 0 given $X(0) = 0$ is given by $\int_0^\infty P_{00}(t)dt = \frac{3}{7\mu}$. Thus the expected total lifetime given $X_0 = 0$ equals the sum of these two terms $\frac{2}{7\mu} + \frac{3}{7\mu} = \frac{5}{7\mu}$.

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