STK2130 Spring 2015 – Mandatory assignment

Deadline Thursday March 19th, 14:30

You are allowed to collaborate and discuss the problems with other students, but each student has to formulate her or his own answers. You should give the names of the students you collaborate with, so that it is possible to compare the written solutions.

The assignment consists of 2 Problems over 3 pages. Make sure you have the complete assignment.

The answer to the exam project should be handed in on paper on the 7th floor in NHA (Niels Henrik Abels hus). You may deliver a handwritten or Latex/Word-processed answer to the project in English, Norwegian or any other Scandinavian language

Problem 1

A Markov chain $X_0, X_1, X_2, \ldots$ on the states $\{0, 1, 2, 3, 4\}$ is defined by the transition matrix

$$\mathbf{P} = \begin{bmatrix} \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ \frac{2}{5} & \frac{1}{5} & 0 & \frac{1}{5} & \frac{1}{5} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

a) The chain has three classes, $\mathcal{C}_0 = \{0, 1\}, \mathcal{C}_1 = \{2, 3\}, \mathcal{C}_2 = \{4\}$ where $\mathcal{C}_0$ is transient, $\mathcal{C}_1$ is closed and $\mathcal{C}_2$ is absorbing (and closed). Explain why this is so.

   Which of the classes are recurrent?

b) Let T be the time until the chain enters one of the closed classes and define $\mu_i = \mathrm{E}[T|X_0 = i]$ for $i \in \mathcal{C}_0$.

   Explain why the μ_i satisfy the following two equations

$$(1) \qquad \mu_0 = (\mu_0 + 1)\tfrac{1}{5} + (\mu_1 + 1)\tfrac{1}{5} + \tfrac{3}{5}$$

$$(2) \qquad \mu_1 = (\mu_0 + 1)\tfrac{2}{5} + (\mu_1 + 1)\tfrac{1}{5} + \tfrac{2}{5}$$

   Solve the equations (1) and (2) to obtain μ_0 and μ_1.

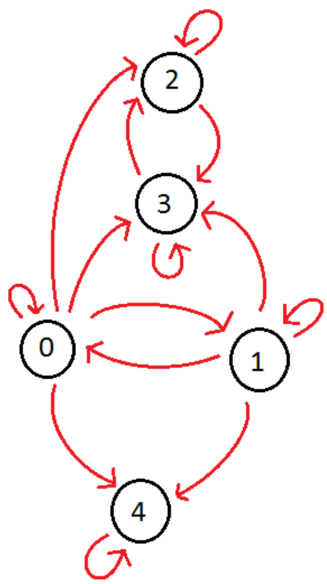
c) Let q_i be the probability that the chain ends up in state 4 conditional on $X_0 = i$. Find and explain equations for obtaining q_0 and q_1. Solve the equations.

d) Let s_{ij} denote the expected number of visits to states $j = 0$ and $j = 1$ conditional on $X_0 = i$. Find and solve equations for determining the s_{ij}.

Solution:

(a) States 0 and 1 communicate with each other but there is positive probability of leaving any of them with no chance to return, so $\mathcal{C}_0$ is a transient class. States 2 and 3 also communicate and once in this class we can never leave, so $\mathcal{C}_1$ is (positive) recurrent, once entered we never leave. Finally, $\mathcal{C}_2 = \{4\}$ is an absorbing state, once we enter 4 we will remain in 4 forever.

(b) It is recommended to make a plot to see the situation:



$\mathcal{C}_1$ and $\mathcal{C}_2$ are closed states, so once there we never leave them. Let us consider we start at some state $i \in \mathcal{C}_0$ in the transient class. Denoting $\mu_i = E[T|X_0 = i]$ the expected steps to leave $\mathcal{C}_0$ then, for $i = 0$, we can go either to 0, 1, 2, 3 and 4. If we go to 0 again, the expected time will be $1 + \mu_0$, that is, one step we have already taken and the remaining time is still μ_0. If we go to 1, the expected time to exit $\mathcal{C}_0$ now will be $1 + \mu_1$, that is one step we use to go from 0 to 1 and now we are in state 1 so the remaining expected time is μ_1, while if we go to either 2, 3 or 4 we will have left the class $\mathcal{C}_0$ and hence we just count one step. Thus

$$i = 0, \quad \mu_0 = p_{00}(1 + \mu_0) + p_{01}(1 + \mu_1) + p_{02} + p_{03} + p_{04}$$

which gives

$$i = 0, \quad \mu_0 = \frac{1}{5}(1 + \mu_0) + \frac{1}{5}(1 + \mu_1) + \frac{3}{5}.$$

Using exactly the same reasoning as for $i = 0$ we obtain

$$i = 1, \quad \mu_1 = p_{10}(1 + \mu_0) + p_{11}(1 + \mu_1) + p_{13} + p_{14}.$$

This gives a system of two equations with two unknowns and the solution is: $\mu_0 = \frac{25}{14}$ and $\mu_1 = \frac{15}{7}$.

(c) q_i is the probability that, starting in state $i \in \mathcal{C}_0$ we will be absorbed by state 4.

Again, if $i = 0$, we can go to either 0, 1, 2, 3 or 4. If we go to 0 then the probability of final absorption to 4 is still q_i. If we go to 1 then the probability of final absorption will be now q_1

since we are in 1. If we go to either 2 or 3 then we will never be absorbed by 4 since $\mathcal{C}_1 = \{2,3\}$ is a closed class, so $q_2 = q_3 = 0$. Finally, if we directly go to 4 we are done so $q_4 = 1$. So

$$i = 0, \quad q_0 = p_{00}q_0 + p_{01}q_1 + p_{02}q_2 + p_{03}q_3 + p_{04}q_4$$

which gives

$$i = 0, \quad q_0 = \frac{1}{5}q_0 + \frac{1}{5}q_1 + \frac{1}{5}.$$

Using exactly the same reasoning as for $i = 0$ we obtain

$$i = 1, \quad q_1 = p_{10}q_0 + p_{11}q_1 + p_{14}$$

that is

$$i = 1, \quad q_1 = \frac{2}{5}q_0 + \frac{1}{5}q_1 + \frac{1}{5}$$

which has solution $q_0 = \frac{5}{14}$ and $q_1 = \frac{3}{7}$.

(d) Let $i \in \mathcal{C}_0$. Denoting s_{ij} the expected time of steps until we visit j starting from i. Then for $i = 0$ and $j = 0$

$$s_{00} = 1 + p_{00}s_{00} + p_{01}s_{10}$$

for $i = 0$ and $j = 1$

$$s_{01} = p_{00}s_{01} + p_{01}s_{11}$$

for $i = 1$ and $j = 0$

$$s_{10} = p_{10}s_{00} + p_{11}s_{10}$$

finally for $i = 1$ and $j = 1$

$$s_{11} = 1 + p_{10}s_{01} + p_{11}s_{11}.$$

This can be solved using matrices, see page 243. Let $P_T = (p_{ij})_{i,j \in \mathcal{C}_0}$ be the matrix of only the transient states and $S = (s_{ij})_{i,j \in \mathcal{C}_0}$ be the matrix of expected time spent in the transient states $i \in \mathcal{C}_0$. Then

$$S = (I - P_T)^{-1} = \begin{pmatrix} \frac{10}{7} & \frac{5}{14} \\ \frac{5}{7} & \frac{10}{7} \end{pmatrix} = \begin{pmatrix} 1.43 & 0.36 \\ 0.71 & 1.43 \end{pmatrix}$$

Problem 2

Markov chains are used in numerous situations in real life. In physics, queueing theory, Internet, statistics, economy, social sciences, etc. In this exercise we will try to illustrate how one may use Markov chains in sociology. This exercise is inspired by the work of three economists "D. Acemoglu, G. Egorov, K. Sonin, *Political model of social evolution.* Proceedings of the National Academy of Sciences 108: 21292–21296. (2011)" (You do not need to look at this article).

The authors propose a model using Markov processes for describing the dynamics of political and social changes in a society. We will illustrate in a very simple example how this can be done.

Consider a society or population. Suppose that this society lives in a political regime among the following three: $\Omega = \{\text{Dictatorship, Elections, Democracy}\}$ which we code as $S = \{0, 1, 2\}$

being 0 = Dictatorship, 1 = Elections and 2 = Democracy. This society behaves as follows: While in a dictatorship the probability that a revolution breaks out is α in such a case this society calls for elections to decide. Then democracy is established if there is a majority (more than 50% in favour). The probability that the inhabitants vote NO is β. While in democracy, the probability that a *coup d'état* breaks out will be denoted by γ, being γ a small number hopefully. Once the society has reached democracy it will preferably stay there.

(a) Explain why the political status of this society can be modelled by using a Markov chain. Plot a diagram to help yourself.

(b) Explain why the transition probability matrix of such a process is

$$P = \begin{pmatrix} 1-\alpha & \alpha & 0 \\ \beta & 0 & 1-\beta \\ \gamma & 0 & 1-\gamma \end{pmatrix}$$

and comment on the different states. Is this process ergodic? Remember that in a finite-state Markov chain all recurrent states are positive recurrent.

(c) Given that there is a current dictatorship, what is the probability that there is democracy after three periods?

(d) Assuming $\alpha, \beta, \gamma > 0$, explain why the process is irreducible and aperiodic. Show that the stationary distribution is

$$\pi_0 = \frac{\gamma}{\gamma(\alpha+1)+\alpha(1-\beta)}, \quad \pi_1 = \frac{\alpha\gamma}{\gamma(\alpha+1)+\alpha(1-\beta)}, \quad \text{and} \quad \pi_2 = \frac{\alpha(1-\beta)}{\gamma(\alpha+1)+\alpha(1-\beta)}.$$

Is it unique? Why?

(e) We say that a society is *obedient* if they tend to accept a dictatorial regime, that is, the probability of revolution goes to 0 as time goes by. On the other hand, we say that a society is *revolutionary* if the chances of a revolution breakout get closer to 1 and the probability of obtaining democracy in the elections converges towards 1 as time goes by.

Compute the stationary distributions for both an obedient and revolutionary society and comment on that.

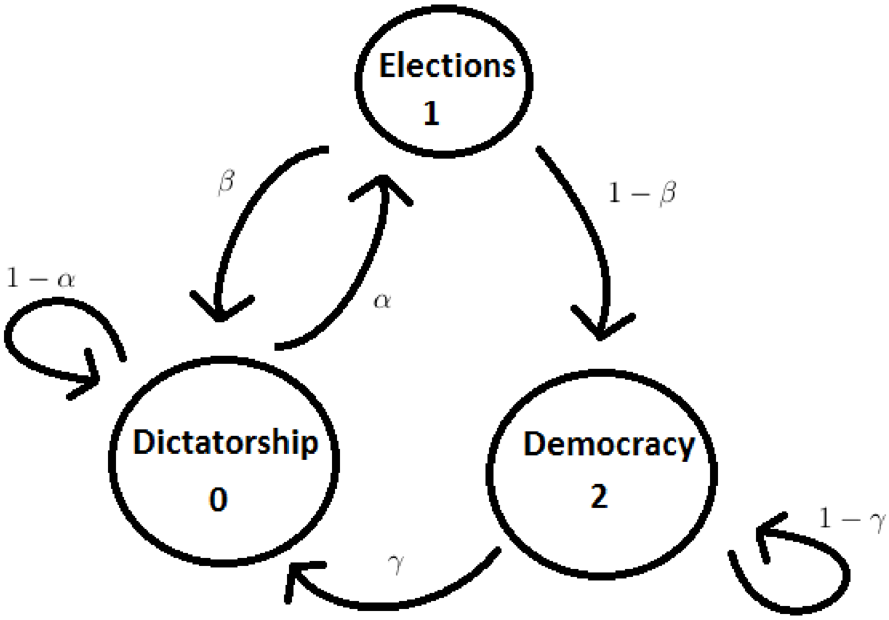
(f) Assume there is democracy. There is a conspiracy in the air. The military is preparing a coup d'état but they still do not know who will take the command. So γ is not clearly specified. Statistical studies have revealed that γ has the following density function $f_\gamma(x) = 3(1-x)^2$ for $0 \leqslant x \leqslant 1$. Compute the probability that the coup d'état will not take place within the next n periods. That is, $P(X_n = 2, X_1 \neq 0, \ldots, X_{n-1} \neq 0 | X_0 = 2)$. What happens when n increases?

Solution:

(a) There are three states in this process (two regimes and a period for elections), we can shift from one to the other according to the possibilities described in the exercise. Let X_n be the political status of the society at period $n \geqslant 0$. Then $X_n \in \{0,1,2\}$ and X_n satisfies indeed the Markov property

$$P(X_{n+1} = j | X_n = i, X_{n-1} = i_{n-1}, \ldots, X_1 = i_1, X_0 = i_0) = P(X_{n+1} = j | X_n = i), \; i, j = 0, 1, 2.$$

The diagram is



(b) Assuming $\alpha, \beta, \gamma > 0$ we see only one class $\{0,1,2\}$ since all states communicate so the chain is irreducible and all states are recurrent. This process is positive recurrent (because it is recurrent and has finitely many states) and aperiodic (see (d) below). Positive recurrent, aperiodic Markov chains are called ergodic.

(c) Given that $X_0 = 0$ we want to find $P(X_3 = 2 | X_0 = 0)$. We just need to find $P^3_{0,2}$. It is not necessary to compute the whole matrix P^3, just

$$P^2 = \begin{pmatrix} (1-\alpha)^2 + \alpha\beta & \alpha(1-\alpha) & \alpha(1-\beta) \\ \beta(1-\alpha) + \gamma(1-\beta) & \alpha\beta & (1-\beta)(1-\gamma) \\ \gamma(1-\alpha) + \gamma(1-\gamma) & \alpha\gamma & (1-\gamma)^2 \end{pmatrix}.$$

Then

$$\begin{aligned} P(X_3 = 2 | X_0 = 0) &= \sum_{k=0}^{2} P(X_3 = 2 | X_2 = k) P(X_2 = k | X_0 = 0) \\ &= 0((1-\alpha)^2 + \alpha\beta) + (1-\beta)(\alpha(1-\alpha)) + (1-\gamma)(\alpha(1-\beta)) \\ &= \alpha(1-\beta)(2-\alpha-\gamma). \end{aligned}$$

(d) There is only one class (all states communicate) so it is irreducible and recurrent. The period of all states is 1 since periodicity is a class property and for instance $p_{00}^{(1)} > 0$. An irreducible ergodic Markov chain has a unique stationary distribution (Theorem 4.1, p215). Let us check that the one given is a solution. First, it is easy to check readily that $\pi_0 + \pi_1 + \pi_2 = 1$. Then we check

$$\pi_j = \sum_{i=0}^{2} P_{ij}\pi_i$$

for $j = 1$ and $j = 2$ (It is only necessary to check two since we only need to check three equations and we already checked one). So for $j = 1$ clearly

$$\sum_{i=0}^{2} P_{i1}\pi_i = \alpha\pi_0 = \alpha\frac{\gamma}{\gamma(\alpha+1)+\alpha(1-\beta)} = \pi_1.$$

and for $j = 2$

$$\sum_{i=0}^{2} P_{i2}\pi_i = (1-\beta)\pi_1+(1-\gamma)\pi_2 = (1-\beta)\frac{\alpha\gamma}{\gamma(\alpha+1)+\alpha(1-\beta)}+(1-\gamma)\frac{\alpha(1-\beta)}{\gamma(\alpha+1)+\alpha(1-\beta)} = \pi_2.$$

(e) The stationary distribution for an obedient society would be

$$\lim_{\alpha\to 0}\pi_0 = \lim_{\alpha\to 0}\frac{\gamma}{\gamma(\alpha+1)+\alpha(1-\beta)} = 1 \quad\text{and}\quad \lim_{\alpha\to 0}\pi_1 = \lim_{\alpha\to 0}\pi_2 = 0.$$

and for a revolutionary society:

$$\lim_{\substack{\alpha\to 1\\ \beta\to 0}}\pi_0 = \lim_{\substack{\alpha\to 1\\ \beta\to 0}}\frac{\gamma}{\gamma(\alpha+1)+\alpha(1-\beta)} = \frac{\gamma}{2\gamma+1},\quad \lim_{\substack{\alpha\to 1\\ \beta\to 0}}\pi_1 = \frac{\gamma}{2\gamma+1}\quad\text{and}\quad \lim_{\substack{\alpha\to 1\\ \beta\to 0}}\pi_2 = \frac{1}{2\gamma+1}.$$

We see that even if a society is revolutionary, the chances of dictatorial regime in the long-run (that is π_0) depend only on how likely a coup d'état is. If $\gamma \to 0$ (as is the case today) the society will get established in a democratic regime sooner or later.

(f) γ is stochastic and it is clear from the diagram that

$$P(X_n = 2, X_1 \neq 0, X_2 \neq 0, \ldots, X_{n-1} \neq 0 | X_0 = 2, \gamma = x) = (1-x)^n.$$

So by the law of total probability (for continuous random variables)

$$P(X_n = 2, X_1 \neq 0, X_2 \neq 0, \ldots, X_{n-1} \neq 0 | X_0 = 2) = \int_0^1 (1-x)^n 3(1-x)^2 dx = \frac{3}{n+3}.$$

As n increases, the probability tends to 0. This will always happen as long as $P(\gamma > 0) > 0$. So sooner or later we will enter a dictatorship (also because of ergodicity).