Home exam in STK4011/9011 autumn 2015

The exam in STK4011/9011 consists of this home exam and a written exam.

The deadline for turning in your written solution of the home exam is Thursday 19 November at 2 pm. Two copies marked with your candidate number should be submitted to the reception office at the Department of Mathematics on the seventh floor of N. H. Abel's house. You may deliver a handwritten or Latex/Word-processed solution to the problems in English, Norwegian or any other Scandinavian language.

Note that you are *not allowed* to collaborate with other students or anyone else on the home exam. Moreover you should submit a special extra page with your written solution. This page is the self-declaration form, which is available at the course webpage.

The written exam in STK4011/9011 takes place Tuesday 8 December at 9 am. Details are posted on the course web-page.

Problem 1

Let (X_1, X_2, X_3) have joint probability density function (pdf)

$$f(x_1, x_2, x_3) = \begin{cases} \lambda^3 e^{-\lambda x_3} & \text{for } 0 < x_1 < x_2 < x_3 \\ 0 & \text{otherwise} \end{cases}$$

Consider the transformation $U_1 = X_1/X_2$, $U_2 = X_2/X_3$ and $U_3 = X_3$.

a) Find the joint pdf of (U_1, U_2, U_3).

b) Find the marginal pdfs of U_1, U_2 and U_3. Are the U_i's independent?

Problem 2

Let X be the income of a randomly chosen employee from a given group of the population. It is common to assume that X has a Pareto distribution, i.e., X has pdf

$$f(x|\theta) = \begin{cases} \theta\kappa^\theta \left(\frac{1}{x}\right)^{\theta+1} & \text{if} \quad x > \kappa \\ 0 & \text{if} \quad x \leq \kappa \end{cases} \tag{1}$$

Here $\kappa > 0$ is the lowest possible income in the given population group, while $\theta > 1$ is a parameter that depends on the differences in income within the group. We will assume throughout that the minimum income κ is known.

a) Find an expression for $\mathrm{E}_\theta(X^k)$. For which values of θ is the formula valid?

b) Show that $Y = 2\theta(\log X - \log \kappa)$ is chi squared distributed with 2 degrees of freedom.

Let $X_1, \dots, X_n$, with $n \geq 3$, be a random sample from the Pareto distribution (1).

c) Determine the method of moments estimator θ^* of θ.

d) Determine the maximum likelihood estimator $\hat{\theta}$ of θ.

e) Derive expressions for $\mathrm{E}_\theta \hat{\theta}$ and $\mathrm{Var}_\theta \hat{\theta}$. (*Hint*: Use the result in question b together with exercise 3.17 in the book by Casella and Berger. Note that the formula in exercise 3.17 is valid for $\nu > -\alpha$.)

f) Show that $\sum_{i=1}^n \log X_i$ is a complete sufficient statistic for θ.

g) Find the best unbiased estimator of θ. Does the estimator achieve the Cramér-Rao lower bound?

Problem 3

Assume that $X_1, X_2, \dots$ are independent and identically distributed with pdf given by (1) in the previous problem, and assume that $\theta > 2$. Let θ_n^* and $\hat{\theta}_n$ be the method of moments and maximum likelihood estimators, respectively, based on $X_1, \dots, X_n$ (cf. questions c and d in the previous problem).

a) Show that θ_n^* is consistent.

b) Show that $\sqrt{n}(\theta_n^* - \theta) \to n(0, \sigma^2_{\text{mom}})$ in distribution, and find an expression for the asymptotic variance σ^2_{mom}.

c) Show that $\sqrt{n}(\hat{\theta}_n - \theta) \to n(0, \sigma^2_{\text{max}})$ in distribution, and find an expression for the asymptotic variance σ^2_{max}.

d) Find the asymptotic relative efficiency of the maximum likelihood estimator with respect to the method of moments estimator. Describe how the asymptotic relative efficiency depends on θ and discuss what it tells you about the properties of the two estimators.

END