Sequential optimization of oil production under uncertainty

Arne B. Huseby
University of Oslo, Norway

Nikita Moratchevski
University of Oslo, Norway

In the present paper we study how to optimize oil production with respect to revenue in a situation where the production rate is uncertain. The oil production in a given period is described in terms of a difference equation, where this equation contains several uncertain parameters. The uncertainty about these parameteres is expressed in terms of a suitable prior distribution. As the production develops, more information about the production parameters is gained. Hence, the uncertainty distributions need to be updated. However, the information comes in the form of inequalities and equalities which makes it very difficult to obtain exact analytical expressions for the posteriors. Still it is possible to estimate the distributions using a combination of rejection sampling and the well-known Metropolis-Hastings algorithm. Armed with these techniques it is possible to solve the optimization problem using stochastic programming. The methods will be demonstrated on a few examples.

1 INTRODUCTION

Optimization is an important element in the management of multiple-field oil and gas assets, since many investment decisions are irreversible and finance is committed for the long term. Optimization of oil and gas recovery in petroleum engineering is a considerable research field. Recent studies of production optimization include (Horne, 2002), (Merabet & Bellah, 2002) and (Neiro & Pinto, 2004).

(Huseby & Haavardsson, 2009) considered the problem of production optimization in an oil or gas field consisting of many reservoirs sharing the same processing facility. In order to satisfy the processing limitations of the facility, the production needs to be choked. Thus, at any given point of time the production from each of the reservoirs are scaled down by suitable *choke factors* chosen so that the total production does not exceed the processing capacity. A *production strategy* is a vector valued function defined for all points of time $t \geq 0$ representing the choke factors applied to the reservoirs at time t. Based on the production profile models introduced in (Arps, 1945), (Haavardsson & Huseby, 2007) developed a general model for oil and gas production using a set of ordinary differential equations. (Huseby & Haavardsson, 2009) used this approach in order to develop a general framework for finding a production strategy which is optimal with respect to various types of objective functions. In (Huseby & Haavardsson, 2010) this work was extended to cases where the production is uncertain.

In the present paper we consider a new variant of this problem where the oil production from a given single reservoir is described relative to a sequence of time periods. In each time period the production is limited by two factors: the potential production volume and the amount of oil that can be processed at the processing facility. Typically, many reservoirs share the same processing facility, so at the start of each period one needs to book a certain processing quota. This quota has a cost which is proportional to the size of the quota. At the same time the production generates an income proportional to the processed volume. If the quota is greater than the potential production volume, one ends up with paying too much for the quota. On the other hand, if the quota is less than the potential production volume, the income from the period is reduced. If the latter situation occurs, this implies that same the oil have to be produced in a later period. From an economical point of view, this reduces the present value of the oil production. Thus, for each period the aim is to find the optimal processing quota, i.e., the quota that maximizes the revenue. If the potential production volume is known, this is trivial since the obvious choice is to choose a quota that is equal to this volume. However, the potential production volume typically depends on a number of uncertain reservoir parameters. Thus, in order to choose the optimal quota, one has to take this uncertainty into account.

The uncertainty about the reservoir parameters is expressed in terms of a suitable prior distribution. As the production develops, more information about the production parameters is gained. Hence, the uncertainty distributions need to be updated. In the present paper we show how this updating can be accomplished. Moreover, we show how to find the optimal quota using Monte Carlo simulation.

2 A GENERAL SAMPLING PROCEDURE

In a situation where decisions are made sequentially, one needs to model how the uncertainty changes as time goes by. In order to take a closer look at this we consider a sequence of n decisions, where the first decision is made at time zero, the second at time 1 etc. We assume that the outcomes of the decisions depend on a vector of uncertain quantities, $\boldsymbol{X} = (X_1, \dots, X_m)$. Moreover, we let $\mathcal{I}_i$ denote the set of information about $\boldsymbol{X}$ available at the point of time when the ith decision is made, and let $\pi(\boldsymbol{x}|\mathcal{I}_i)$ denote the joint conditional density of the X_js given $\mathcal{I}_i$, $i = 0, 1, \dots, n-1$. In the calculations we are about to describe, we need to be able to run Monte Carlo simulations where we sample from these conditional distributions.

A typical example of such a model is a discrete time stochastic process, where the X_js represent the states of the process. In such cases the X_js are observed sequentially. In the litterature there are many such models, including various types of Markov chains. Given such a structure it is usually very easy to derive the sequence of conditional distributions.

When modeling uncertainty related to oil production, however, the uncertain quantities are parameters characterizing properties of the reservoir. Such quantities are typically not directly observable, but as the oil in the reservoir is produced, the uncertainty is gradually reduced. Thus, we let $\boldsymbol{X} = (X_1, \dots, X_m)$ represent the uncertain reservoir parameters. As more and more oil is produced, we get information about these parameters. This information typically limits the variability of $\boldsymbol{X}$ to smaller and smaller sets. Thus, if $\mathcal{X}$ denotes the initial set of possible values for $\boldsymbol{X}$, and $\mathcal{A}_1, \dots, \mathcal{A}_{n-1}$ are subsets of $\mathcal{X}$, for $i = 1, \dots, n-1$ we assume that $\mathcal{I}_i$ is of the following form:

$$\mathcal{I}_i = \{\boldsymbol{X} \in \bigcap_{j=1}^{i} \mathcal{A}_j\}. \tag{1}$$

Moreover, $\pi(\boldsymbol{x}|\mathcal{I}_0)$ is simply the prior density of $\boldsymbol{X}$, which we denote by $\pi(\boldsymbol{x})$.

As long as $P(\boldsymbol{X} \in \cap_{j=1}^{i} \mathcal{A}_j) > 0$, the corresponding conditional density can be derived as:

$$\pi(\boldsymbol{x}|\mathcal{I}_i) = \frac{\pi(\boldsymbol{x})}{P(\boldsymbol{X} \in \cap_{j=1}^{i} \mathcal{A}_j)}, \tag{2}$$

for $i = 1, \dots, n-1$. Given that it is easy to sample from the prior distribution, sampling from the conditional distributions is easily accomplished as well using a standard rejection method. That is, in each iteration of the Monte Carlo simulation we sample repeatedly from $\pi(\boldsymbol{x})$ until we get a value $\boldsymbol{x} \in \cap_{j=1}^{i} \mathcal{A}_j$.

However, if $P(\boldsymbol{X} \in \cap_{j=1}^{i} \mathcal{A}_j) = 0$, rejection sampling will not work. This situation occurs if we observe the value of a function of the uncertain quantities. Assume e.g., that $Y_1 = \Phi_1(\boldsymbol{X})$, and that we at the jth step observe that $Y_1 = y_1$. Moreover, let $\mathcal{A}_j = \{\boldsymbol{X} \in \mathcal{X} : \Phi_1(\boldsymbol{X}) = y_1\}$. Then, assuming that the prior is an absolutely continuous distribution, we typically get that $P(\boldsymbol{X} \in \mathcal{A}_j) = 0$.

Before we explain how to sample from the conditional distribution of $\boldsymbol{X}$ given Y_1, we simplify the problem by assuming that we can find $(m-1)$ functions $\Phi_2, \dots, \Phi_m$ such that $\boldsymbol{Y} = (Y_1, \dots, Y_m) = (\Phi_1(\boldsymbol{X}), \dots, \Phi_m(\boldsymbol{X}))$ forms a one-to-one mapping. Given that this holds true, we can also find an inverse mapping such that $\boldsymbol{X} = (X_1, \dots, X_m) = (\Psi_1(\boldsymbol{Y}), \dots, \Psi_m(\boldsymbol{Y}))$.

By elementary probability theory, the distribution of $\boldsymbol{Y}$ has a prior density, $\nu(\boldsymbol{y})$, given by:

$$\nu(\boldsymbol{y}) = \pi(\Psi_1(\boldsymbol{y}), \dots, \Psi_m(\boldsymbol{y}))|J|, \tag{3}$$

where J is the Jacobian determinant of the mapping. Furthermore, the conditional density of $Y_2, \dots, Y_m$ given $Y_1 = y_1$, denoted $\nu(y_2, \dots, y_m|y_1)$ is given by:

$$\nu(y_2, \dots, y_m|y_1) = \frac{\nu(\boldsymbol{y})}{\int \nu(\boldsymbol{y}) dy_2 \cdots dy_m}. \tag{4}$$

Having derived this, we then get the following sampling algorithm:

Algorithm 2.1 *Assume that we observe that* $Y_1 = \Phi_1(\boldsymbol{X}) = y_1$. *We can then sample* $\boldsymbol{X}$ *as follows:*

STEP 1. *Sample* $Y_2, \dots, Y_m$ *from the conditional distribution* (4). *Let* $\boldsymbol{y} = (y_1, \dots, y_m)$ *denote the resulting value of* $\boldsymbol{Y}$.

STEP 2. *Calculate the resulting value of* $\boldsymbol{X}$ *using the inverse mapping, i.e.,* $\boldsymbol{x} = (\Psi_1(\boldsymbol{y}), \dots, \Psi_m(\boldsymbol{y}))$.

The only remaining problem is how to carry out the first step of Algorithm 2.1. In order to do so, we note that the integral in the denominator of (4) is simply a normalizing constant. Hence, we may rewrite (4) as:

$$\nu(y_2, \dots, y_m|y_1) \propto \nu(\boldsymbol{y}). \tag{5}$$

Thus, $\nu(\boldsymbol{y})$ can be considered to be an *unnormalized* density of the conditional distribution of $Y_2, \ldots, Y_m$ given $Y_1 = y_1$. Sampling from this distribution can easily be done using the well-known Metropolis-Hastings algorithm (Hastings, 1970).

Note that in many situations it is possible to simplify the calculations by letting $\Phi_2, \ldots, \Phi_m$ be identity functions. That is, we let $Y_j = \Phi_j(\boldsymbol{X}) = X_j$, $j = 2, \ldots, m$. In such cases (3) simplifies to the following:

$$\nu(\boldsymbol{y}) = \pi(\Psi_1(\boldsymbol{y}), y_2, \ldots, y_m)|\frac{\partial \Psi_1(\boldsymbol{y})}{\partial y_1}|. \quad (6)$$

More generally, the information available at the ith step, $\mathcal{I}_i$ is a combination of sets $\mathcal{A}_1, \ldots, \mathcal{A}_i$ such that $P(\boldsymbol{X} \in \mathcal{A}_j)$ is positive for some of the js and zero for the others. In such case we use a combination of Metropolis-Hastings algorithm and rejection sampling. We assume that whenever $P(\boldsymbol{X} \in \mathcal{A}_j) = 0$, this corresponds to observing some function of $\boldsymbol{X}$. If this is the case for several js, it is easy to extend Algorithm 2.1 to allow conditioning on more than one variable. Using the Metropolis-Hastings algorithm to sample an $\boldsymbol{x}$ satisfying all the constraints imposed by these functions, we proceed by considering the remaining restrictions on $\boldsymbol{x}$ corresponding to the sets where $P(\boldsymbol{X} \in \mathcal{A}_j) > 0$. If the sampled $\boldsymbol{x}$ does not satisfy all these constraints, it is rejected. Thus, in each iteration the Metropolis-Hastings algorithm is used repeatedly until we end up with an $\boldsymbol{x}$ satisfying *all* constraints.

3 STEPWISE OPTIMIZATION OF PROCESSING QUOTAS

Having established the general sampling procedure, we now apply this to a more specific situation. That is, we consider an oil reservoir, and assume that the production is described in terms of a discrete time process, where q_i denotes the production from the ith period, $i = 1, 2, \ldots$. We also introduce the *cumulative* production from the periods $1, \ldots, i$, denoted Q_i, $i = 1, 2, \ldots$. That is, we let:

$$Q_i = \sum_{j=1}^{i} q_j, \quad i = 1, 2, \ldots. \quad (7)$$

We also let $Q_0 = 0$.

The amount of oil that can be produced within a period is limited by the characteristics of the reservoir determined by a set of uncertain parameters. The main reservoir parameters are the *amount of recoverable oil*, denoted by V, and the so-called *decline rate*, denoted by D. The decline rate represents the fraction of remaining oil that can be produced per unit of time. In general the decline rate may change over time. Still using a production model with a constant decline rate may serve as a satisfactory approximation. The amount of oil that can be produced within the ith period given no other restrictions, can then be expressed as:

$$f(Q_{i-1}) = D(V - Q_{i-1}), \quad i = 1, 2, \ldots. \quad (8)$$

We refer to the function f as the potential production rate function, or PPR function.

Obviously, the amount of recoverable oil is some non-negative number, while the decline rate must be a number between zero and one. Thus, using the notation from Section 2, the set of possible values for the vector (V, D) is given as:

$$\mathcal{X} = \mathbb{R}_+ \times [0, 1]. \quad (9)$$

Moreover, we assume that the distribution of (V, D) has a prior density $\pi(v, d)$ defined over the set $\mathcal{X}$.

In addition to the restrictions imposed by the reservoir itself, the actual production is typically restricted by the available processing capacity. Often the processing facilities are shared between a number of reservoirs. As a result the reservoir manager needs to book a certain processing quota at the start of each period. Thus, we let K_i denote the processing quota available during the ith production period, $i = 1, 2, \ldots$. Hence, the actual production from the ith period is given by:

$$q_i = \min\{f(Q_{i-1}), K_i\}, \quad i = 1, 2, \ldots. \quad (10)$$

By booking the quota K_i at the start of the ith period, the reservoir manager is guaranteed to have this amount of processing capacity available. However, this comes at a price. For simplicity we assume a linear cost model where the quota cost from the ith period is given by $\kappa \cdot K_i$ for some suitable positive constant κ. Similarly, the income from the ith period is given by $\delta \cdot q_i$ for some suitable positive constant δ, where we typically have $\delta > \kappa$. The revenue from the ith period, denoted $R_i(K_i)$ then becomes:

$$R_i(K_i) = \delta \cdot q_i - \kappa \cdot K_i, \quad i = 1, 2, \ldots. \quad (11)$$

We then consider the revenue from the ith period and observe that if $K_i < f(Q_{i-1})$, then $q_i = K_i$. Hence, in this case $R_i(K_i) = (\delta - \kappa) \cdot K_i$. If, on the other hand, $K_i > f(Q_{i-1})$, then $q_i = f(Q_{i-1})$. Hence, in this case $R_i(K_i) = \delta \cdot f(Q_{i-1}) - \kappa \cdot K_i$. Thus, in order to maximize the revenue, one should ideally let $K_i = f(Q_{i-1})$. Since, however, $f(Q_{i-1})$ depends on the uncertain reservoir parameters V and D, the optimal quota becomes uncertain as well. By running a Monte Carlo simulation where V and D are sampled from a suitable joint distribution, we can estimate the expected revenue, $E[R_i(K_i)]$ as a function of the quota K_i, and then choose K_i as the value that maximizes this. Alternatively, we can estimate the derivative of the expected revenue as a function of the quota K_i, and then choose K_i so that this derivative becomes zero. It turns out that the latter approach is both easier and produces more stable results, so this method will be used here.

In order to estimate the derivative of $E[R_i(K_i)]$ with respect to K_i, we argue that:

$$\frac{\partial}{\partial K_i} E[R_i(K_i)] = E[\frac{\partial}{\partial K_i} R_i(K_i)]. \tag{12}$$

This relation can be established using e.g., the well-known Lebesgue's dominated convergence theorem. Furthermore, we observe that:

$$\frac{\partial}{\partial K_i} R_i(K_i) = \begin{cases} (\delta - \kappa) & \text{if } K_i < f(Q_{i-1}) \\ -\kappa & \text{if } K_i > f(Q_{i-1}) \end{cases} \tag{13}$$

If $K_i = f(Q_{i-1})$, the derivative is undefined. However, this event typically occurs with probability zero, so this will not contribute to the expected value of the derivative. As a result we get:

$$E[\frac{\partial}{\partial K_i} R_i(K_i)] = \tag{14}$$

$$(\delta - \kappa) P(f(Q_{i-1}) > K_i) - \kappa P(f(Q_{i-1}) < K_i).$$

From this it follows that the derivative is zero if:

$$P(f(Q_{i-1}) < K_i) = \frac{\delta - \kappa}{\delta}. \tag{15}$$

Hence, for $i = 1, 2, \ldots$, the optimal quota K_i can easily be found by estimating the cumulative distribution of $f(Q_{i-1})$.

For $i = 1$ it follows by (8) that $f(Q_{i-1}) = f(Q_0) = DV$. Hence, the cumulative distribution of $f(Q_0)$ can be estimated by sampling from the prior distribution of (V, D), i.e., $\pi(v, d)$. By (10) the actual production from the first period, q_1, then becomes the minimum of the *true value* of $f(Q_0)$ and the chosen quota K_1.

After the first period is completed, the value of q_1 is observed. If $q_1 = K_1$, this implies that the true value of $f(Q_0)$ is greater than or equal to the observed production q_1. If on the other hand $q_1 < K_1$, this implies that the true value of $f(Q_0)$ is equal to the observed production q_1. Using the notation from Section 2, this means that if $q_1 = K_1$, then:

$$\mathcal{A}_1 = \{(v, d) \in \mathcal{X} : dv \geq q_1\}. \tag{16}$$

Similarly, if $q_1 < K_1$, then:

$$\mathcal{A}_1 = \{(v, d) \in \mathcal{X} : dv = q_1\}. \tag{17}$$

We then proceed to the second period where $f(Q_{i-1}) = f(Q_1) = D(V - Q_1)$, and where of course Q_1 is equal to the observed production in the first period, q_1. In order to estimate the cumulative distribution of $f(Q_1)$ we must sample from the conditional distribution of (V, D) given that $(V, D) \in \mathcal{A}_1$. In either case this can be done using the methods presented in Section 2. As in the previous step the actual production from the second period, q_2, then becomes the minimum of the *true value* of $f(Q_1)$ and the chosen quota K_2.

After the second period is completed, the value of q_2 is observed. If $q_2 = K_2$, the true value of $f(Q_1)$ must be greater than or equal to q_2, while if $q_2 < K_2$, we must have $f(Q_1) = q_2$. Thus, if $q_2 = K_2$,

$$\mathcal{A}_2 = \{(v, d) \in \mathcal{X} : d(v - Q_1) \geq q_2\}, \tag{18}$$

while, if $q_2 < K_2$, then:

$$\mathcal{A}_2 = \{(v, d) \in \mathcal{X} : d(v - Q_1) = q_2\}. \tag{19}$$

Continuing on to the third period where $f(Q_{i-1}) = f(Q_2) = D(V - Q_2)$ and where $Q_2 = q_1 + q_2$, we proceed in a similar fashion. In particular the cumulative distribution of $f(Q_2)$ is estimated by sampling from the conditional distribution of (V, D) given that $(V, D) \in \mathcal{A}_1 \cap \mathcal{A}_2$. Note, however, that if both $q_1 < K_1$ and $q_2 < K_2$, then the set $\mathcal{A}_1 \cap \mathcal{A}_2$ consists of a single point in which case the true values of V and D can be computed. If this happens, the exact optimal quota can be found without the need of a Monte Carlo simulation. If on the other hand $q_1 = K_1$ or $q_2 = K_2$, we have a situation which can be handled by using the sampling procedure from Section 2.

In this way the process continues throughout the periods. At the start of the ith period we determine the optimal quota, K_i by considering the conditional distribution of (V, D) given the observed production values from the previous periods, $q_1, \ldots, q_{i-1}$. By comparing these numbers with their respective quotas $K_1, \ldots, K_{i-1}$, we determine the sets $\mathcal{A}_1, \ldots, \mathcal{A}_{i-1}$. If $\mathcal{A}_1 \cap \cdots \cap \mathcal{A}_{i-1}$ consists of a single point, the true value of V and D can be computed, while if this is not the case, we determine the optimal quota K_i using the sampling procedure from Section 2.

We observe that in cases where $q_j < K_j$, the corresponding $\mathcal{A}_j$ is of the form:

$$\mathcal{A}_j = \{(v, d) \in \mathcal{X} : d(v - Q_{j-1}) = q_j\}. \tag{20}$$

Once again using the notatation from Section 2, we introduce transformed random variables Y_1, Y_2, given by:

$$Y_1 = \Phi_1(V, D) = D(V - Q_{j-1}), \tag{21}$$

$$Y_2 = \Phi_2(V, D) = D. \tag{22}$$

Hence, the inverse transformation is given by:

$$V = \Psi_1(Y_1, Y_2) = Q_{j-1} + Y_1/Y_2, \tag{23}$$

$$D = \Psi_2(Y_1, Y_2) = Y_2. \tag{24}$$

Since Ψ_2 is an identity function the Jacobian in (3) takes the special form given in (6). Thus, we get:

$$|J| = |\frac{\partial \Psi_1(y_1, y_2)}{\partial y_1}| = y_2^{-1}. \tag{25}$$

4 NUMERICAL EXAMPLES

In this section we consider more specific cases. In all these cases V and D are assumed to be independent. The volume, V, is assumed to be lognormally distributed with $E[V] = 12.0$ million barrels of oil and $SD[V] = 1.0$. The decline rate, D is assumed to be uniformly distributed on the interval $[0.1, 0.3]$. The cost and income rates are chosen to be $\kappa = 30$ USD/barrel, and $\delta = 50$ USD/barrel. Finally, we consider the results from 8 production periods.

In all examples the quotas, $K_1, \ldots, K_8$ are found using (15), where the cumulative distribution of $f(Q_{i-1})$ is estimated by running a Monte Carlo simulation. All revenues are presented with a unit of million USD.

In the first case we assume that the *true* values of V and D are 12 and 0.2 respectively. Thus, the true values are equal to the expected values in the respective prior distributions. The results of the simulation are shown in Table 1.

Table 1: Results from simulation where the $V = 12.0$ and $D = 0.2$.

i	Q_{i-1}	K_i	q_i	$f(Q_{i-1})$	R_i
1	0.00	2.14	2.14	2.40	42.82
2	2.14	2.23	1.97	1.97	31.70
3	4.11	1.56	1.56	1.58	31.24
4	5.67	1.28	1.27	1.27	24.82
5	6.94	1.01	1.01	1.01	20.24
6	7.95	0.81	0.81	0.81	16.19
7	8.76	0.65	0.65	0.65	12.95
8	9.41	0.52	0.52	0.52	10.36

The total result, i.e., $R_1 + \cdots + R_8$, is 190.32 million USD, while the total discounted result given a discount rate of 5% per period is 169.15 million USD.

We observe that $q_1 = K_1 = 2.14$. Thus, the set $\mathcal{A}_1$ is given by;

$$\mathcal{A}_1 = \{(v, d) \in \mathcal{X} : dv \geq 2.14\}. \tag{26}$$

Furthermore, $q_2 = 1.97 < K_2 = 2.23$. Thus, since $Q_1 = 2.14$ the set $\mathcal{A}_2$ is given by;

$$\mathcal{A}_2 = \{(v, d) \in \mathcal{X} : d(v - 2.14) = 1.97\}. \tag{27}$$

Continuing in this way, we get that:

$$\mathcal{A}_3 = \{(v, d) \in \mathcal{X} : d(v - 4.11) \geq 1.56\}, \tag{28}$$

and:

$$\mathcal{A}_4 = \{(v, d) \in \mathcal{X} : d(v - 5.67) = 1.27\}, \tag{29}$$

At this stage the intersection $\mathcal{A}_1 \cap \cdots \cap \mathcal{A}_4$ consists of a single point, $(v, d) = (12, 0.2)$. Thus, given all the available information the true value of (V, D) can be computed. As a result we see that the remaining quotas are determined without any uncertainty, i.e., $K_i = f(Q_{i-1})$ for $i = 5, 6, 7, 8$.

In the second case we assume that the *true* values of V and D are 12 and 0.1 respectively. Thus, the true value of the volume is equal to the expected value, while the decline rate is just half of its expectation. The results of the simulation are shown in Table 2.

Table 2: Results from simulation where the $V = 12.0$ and $D = 0.1$.

i	Q_{i-1}	K_i	q_i	$f(Q_{i-1})$	R_i
1	0.00	2.14	1.20	1.20	-4.30
2	1.20	1.07	1.07	1.08	21.41
3	2.27	0.96	0.96	0.97	19.26
4	3.23	0.87	0.87	0.88	17.36
5	4.10	0.78	0.78	0.79	15.66
6	4.88	0.70	0.70	0.71	14.10
7	5.59	0.63	0.63	0.64	12.69
8	6.22	0.57	0.57	0.58	11.47

The total result is reduced to 107.64 million USD, while the total discounted result is reduced to 90.10 million USD. This reduction is not surprising since a lower decline rate implies that one can produce much less oil per period.

In this case we see that the first quota, $K_1 = 2.14$ is way too high. This is of course due to the fact that our prior distribution of D is optimistic compared to the true value of D. As a result we actually lose money during the first period. In the other periods, however, we reduce the quotas according to the information we have and obtain near optimal results. Thus, even though we started out with a far too optimistic prior, the chosen quotas are quickly adjusted in order to minimize loss.

Note that in this case we have:

$$\mathcal{A}_1 = \{(v, d) \in \mathcal{X} : d(v - Q_0) = q_1\}, \tag{30}$$

while:

$$\mathcal{A}_j = \{(v, d) \in \mathcal{X} : d(v - Q_{j-1}) \geq q_j\}, \tag{31}$$

for $j = 2, \ldots, 7$. Thus, the true value of (V, D) is not determined. Still the uncertainty about (V, D) is reduced considerable after the first period of production.

In the third case we assume that the *true* values of V and D are 12 and 0.3 respectively. Thus, the true value of the volume is kept at its expected value, while the decline rate is now increased by 50% with respect to its expectation. The results of the simulation are shown in Table 3.

Table 3: Results from simulation where the $V = 12.0$ and $D = 0.3$.

i	Q_{i-1}	K_i	q_i	$f(Q_{i-1})$	R_i
1	0.00	2.14	2.14	3.60	42.74
2	2.14	2.23	2.23	2.96	44.59
3	4.37	1.95	1.95	2.29	38.97
4	6.32	1.61	1.61	1.71	32.12
5	7.92	1.30	1.22	1.22	22.06
6	9.15	0.88	0.86	0.86	16.30
7	10.00	0.60	0.60	0.60	11.99
8	10.60	0.42	0.42	0.42	8.39

The total result is increased to 217.18 million USD, while the total discounted result is increased to 194.15 million USD. Again this increase is not surprising since a higher decline rate implies that one can produce much more oil per period.

In this case the chosen quotas are too small in the beginning. Thus, the quotas are utilized completely during the first four periods, i.e., $q_j = K_j$ for $j = 1, 2, 3, 4$. As a result we see that the sets $\mathcal{A}_1, \ldots, \mathcal{A}_4$ are given by:

$$\mathcal{A}_j = \{(v, d) \in \mathcal{X} : d(v - Q_{j-1}) \geq q_j\}, \qquad (32)$$

for $j = 1, 2, 3, 4$. Then in periodes 5 and 6 we see that the chosen quotas become too large, i.e., $q_5 < K_5$ and $q_6 < K_5$. Thus, $\mathcal{A}_5$ and $\mathcal{A}_6$ are given by:

$$\mathcal{A}_j = \{(v, d) \in \mathcal{X} : d(v - Q_{j-1}) = q_j\}, \qquad (33)$$

for $j = 5, 6$. At this stage we can compute the true values of V and D. Thus, in the remaining periods we obtain the optimal quotas without any uncertainty.

In the final example we return to the case where the *true* values of V and D are 12 and 0.2 respectively. However, in this case we run the simulations in such a way that only information from the most recent period is used in the optimization of the quotas. The results of the simulation are shown in Table 4.

Compared to the first case we see that total result is reduced from 190.32 million USD to 174.94 million USD, while the total discounted result is reduced from 169.15 million USD to 156.85 million USD.

Table 4: Results from simulation where the $V = 12.0$ and $D = 0.2$, and where only information from the most recent period is used in the optimization.

i	Q_{i-1}	K_i	q_i	$f(Q_{i-1})$	R_i
1	0.00	2.14	2.14	2.40	42.86
2	2.14	2.23	1.97	1.97	31.80
3	4.11	1.56	1.56	1.58	31.21
4	5.67	1.50	1.27	1.27	18.26
5	6.94	0.99	0.99	1.01	19.88
6	7.93	0.97	0.81	0.81	11.53
7	8.75	0.64	0.64	0.65	12.71
8	9.38	0.65	0.52	0.52	6.70

We observe that in the first three periods the results are essentially the same as in the first example. In the fourth period, however, the substantial information gathered in the two first periods is lost. As a result the chosen quota, $K_4 = 1.50$ is too high compared to the potential production rate $f(Q_3) = 1.27$. The same problem occurs in periods 6 and 8 as well. This implies that the total result is significantly worse compared to the case where all information is utilized in the optimization. Note that the effect on the discounted result is less severe since the differences typically occur in the later periods which have less impact on the discounted result. This last example demonstrates the significance of including all available information in the optimization.

5 CONCLUSIONS

In the present paper we have presented a general method for running Monte Carlo simulations in a case where all the distributions are conditioned on a set of equalities and inequalities. This method is then applied to the problem of optimizing production quotas under uncertainty in a discrete time production model. The numerical examples demonstrate how this enables us to fine-tune the quotas as more and more information is gathered. The performance of the method depends on the quality of the prior information. Still even in cases where this true values are significantly different from their respective prior expectations, this is quickly picked up by the updating scheme.

Throughout the paper we have focussed on cases where all the optimization is done based on a short-term perspective. That is, a quota is considered optimal if it is optimal for the upcoming production period. Long-term effects of the quotas, on the other hand, are not considered. It may e.g., happen that by choosing larger quotas in the first periods, more information is gathered, and thus the uncertainty is reduced for the later periods. In order to handle this, one needs to consider a simultaneous optimization of all the quotas. Alternatively, the problem can be analyzed using stochastic dynamic programming. We will return to this issue in an extended study of this problem.

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