Interest rate modelling via SPDE's (STK 4530)
Exercises 1, 16.2.2016

Problem 1 (Bootstrap method) A coupon bearing bond provides coupon payments $C_1, ..., C_m$ at future times $T_1 < ... < T_m$ and a principal at maturity T_m. Then using continuous compounding of interest rates the present value $B(t)$ of such a bond is given by the formula

$$B(t) = \sum_{i=1}^{m} C_i e^{-(T_i - t)\cdot R(t,T_i)} + X e^{-(T_m - t)\cdot R(t,T_m)}.$$

Consider coupon bonds given by the following 5 Treasury bonds:

Bond Principal	Time to maturity	Annual coupon	Bond price
(\$)	(years)	(\$)	(\$)
100	0.25	0	98.7
100	0.50	0	97.3
100	1.00	0	94.6
100	1.50	8	102.4
100	2.00	12	110.1

Suppose that the above coupons are semiannually paid .

Bootstrap the Treasury zero rates up to time-to-maturity of 2 years, that is find the initial term structure of interest rates by computing the rates $R(t, t + 0.25), ..., R(t, t + 2)$ based on the above data.

Problem 2

(i) Construct the smallest $\sigma-$algebra $\mathcal{F}$ containing the events

$$F_1 = \text{"stock price } S = 1 \text{ \$" and } F_2 = \text{"stock price } S = 2 \text{ \$"}.$$

(ii) Assume that the entirety of market information at terminal time T is modelled by the $\sigma-$algebra $\mathcal{F}_T$. Consider the market event $A \in \mathcal{F}_T$ given by

$$A = \{S_T > 100 \text{ \$}\}$$

for a stock with price S_T at maturity T. Further let h be a terminal (random) payment based on the market information $\mathcal{F}_T$, that is h is a random variable on the probability space $(\Omega, \mathcal{F}_T, P)$. Suppose that the terminal payoff of a claim is given by

$$C = \begin{cases} h & \text{if } A \text{ occurs} \\ 0 & \text{else} \end{cases}$$

Verify that C is a random variable.

(iii) Assume that $S_t^{(i)}, i = 1, 2$ are $\mathcal{F}_t-$adapted stock price processes. Show that the maximum of these prices, that is $\max(S_t^{(1)}, S_t^{(2)})$ is adapted, too.

Problem 3 Let S_t be the price of a stock at time t with initial value $S_0 = x \geq 0$. Assume that the dynamics of S_t is described by the Black-Scholes model:

$$S_t = x \exp\left(\mu t - \frac{\sigma^2}{2} t + \sigma B_t\right),$$

where μ, σ are constants and B_t the Brownian motion. Employ the Itô formula to verify that S_t (uniquely) satisfies the SDE

$$dS_t = \mu \cdot S_t dt + \sigma \cdot S_t dB_t, \; S_0 = x.$$

Problem 4 Determine in Problem 4 the Girsanov transformation $\widetilde{P}$ of P which makes S_t a martingale. Assume that $\sigma \neq 0$.

Problem 5 Let Z_t be a process defined by

$$Z_t = \exp\left(\int_0^t X_s dB_s - \frac{1}{2}\int_0^t X_s^2 ds\right), 0 \leq t \leq T,$$

where X_t is a bounded process. Prove that the Girsanov transformation $\widetilde{P}$ given by

$$\widetilde{P}(A) := E\left[1_A \cdot Z_T\right], \; A \in \mathcal{F}$$

is a probability measure.

Hint: Use Itô's formula applied to $Y_t := \int_0^t X_s dB_s - \frac{1}{2}\int_0^t X_s^2 ds$.