Interest rate modelling via SPDE's (STK 4530)
Exercises 2, 23.9.2016

Problem 1 (Empirical estimation of volatilities) Suppose that the dynamics of a stock price S_t is described by the Black-Scholes model

$$dS_t = \mu \cdot S_t dt + \sigma \cdot S_t dB_t, \; S_0 = x,$$

where μ is the mean return and σ the volatility. Denote by S_i the stock price at the end of the ith interval of constant length τ (in years), $i = 0, 1, ..., n$.

(i) Show that the log-returns u_i given by

$$u_i = \log\left(\frac{S_i}{S_{i-1}}\right), i = 0, 1, ..., n$$

are independent and identically distributed random variables with $Var\left[u_i\right] = \sigma^2 \tau$. So if we replace the variance by the empirical variance in the latter equation we obtain an estimate $\widehat{\sigma}$ of σ given by

$$\widehat{\sigma} = \sqrt{\frac{\frac{1}{n-1}\sum_{i=1}^{n}(u_i - \overline{u})^2}{\tau}},$$

where $\overline{u} = \frac{1}{n}\sum_{i=1}^{n} u_i$ is the emprical mean.

(ii) Consider the following data set of stock prices of Deutsche Börse AG during 11 consecutive trading days:

Day	Closing stock price S_i (Euro)	Day	Closing stock price S_i (Euro)
10/09/07	75.90	18/09/07	83.86
11/09/07	78.81	19/09/07	86.83
12/09/07	78.32	20/09/07	88.53
13/09/07	80.93	21/09/07	90.67
14/09/07	80.18	24/09/07	94.00
17/09/07	82.37		

Assume that $\tau = \frac{1}{252}$ (i.e. 252 trading days per year). Compute the emprical volatility $\widehat{\sigma}$.

Problem 2 Let X be a r.v. with $E\left[|X|\right] < \infty$. Define the process

$$X_t = E\left[X|\, \mathcal{F}_t\right], t \geq 0$$

for a filtration $\mathcal{F}_t$. Verify that X_t is a matingale.

Problem 3 Let $f : [0, T] \longrightarrow \mathbb{R}$ be a deterministic function such that $\int_0^T f^2(s)ds < \infty$. Show that

$$\int_0^T f(s)dB_s$$

is normally distributed with mean zero and variance $\int_0^T f^2(s)ds$.

Hint: Use the following facts:

(i) Let $X, X_n, n \geq 1$ be r.v.'s with existing second moment. Assume that $E[|X_n - X|^2] \underset{n \longrightarrow \infty}{\longrightarrow} 0$. Then there exists a subsequence X_{n_k} of X_n such that

$$X_{n_k} \underset{k \longrightarrow \infty}{\longrightarrow} X \quad \text{a.e.}$$

(ii) For a random vector $Y = (Y_1, ..., Y_n)$ the *characteristic function* $\varphi_Y(\lambda_1, ..., \lambda_n)$ of Y is defined as

$$\varphi_Y(\lambda_1, ..., \lambda_n) = E[\exp(i(\lambda_1 Y_1 + ... + \lambda_n Y_n)], \lambda_1, ..., \lambda_n \in \mathbb{R} \ (i^2 = -1).$$

Then two random vectors $U, Z \in \mathbb{R}^d$ have the same distribution iff

$$\varphi_U(\lambda_1, ..., \lambda_d) = \varphi_Z(\lambda_1, ..., \lambda_d), \lambda_1, ..., \lambda_d \in \mathbb{R}.$$

Further let $X = (X_1, ..., X_m)$ be a random vector. Then $X_1, ..., X_m$ are independent iff

$$\varphi_X(\lambda_1, ..., \lambda_m) = \varphi_{X_1}(\lambda_1) \cdot ... \cdot \varphi_{X_m}(\lambda_m), \lambda_1, ..., \lambda_m \in \mathbb{R}.$$

(iii) $X \sim \mathcal{N}(\mu, \sigma^2)$ iff

$$\varphi_X(\lambda) = \exp(i\mu\lambda - \frac{1}{2}\sigma^2\lambda^2), \lambda \in \mathbb{R}.$$

Problem 4 Consider the probability measure $\widetilde{P}$ given by

$$\widetilde{P}(A) = E_P[1_A \cdot X], A \in \mathcal{F}$$

for a r.v. $X > 0$ on $(\Omega, \mathcal{F}, P)$. Define the process

$$L_t = E_P\left[X \middle| \mathcal{F}_t\right]$$

for a filtration $\mathcal{F}_t$.

Prove that a process M_t is a martingale w.r.t. $\widetilde{P}$ iff $Z_t := M_t L_t$ is a martingale w.r.t. P.

Hint: Derive the *Bayes' rule*

$$E_{\widetilde{P}}\left[Y \middle| \mathcal{F}_t\right] = \frac{E_P\left[Y L_T \middle| \mathcal{F}_t\right]}{L_t}$$

for r.v.'s Y.

Problem 5(General Itô-integral as a local martingale) Let $Y_t, 0 \leq t \leq T$ be a $\mathcal{F}_t-$adapted process on $(\Omega, \mathcal{F}, P)$ such that

$$P(\int_0^T Y_s^2 ds < \infty) = 1.$$

Define the sequence of of bounded random variables

$$R_n = n \wedge \inf\left\{0 \le t \le T : \int_0^t Y_s^2 ds \ge n\right\}, n \ge 1,$$

where $a \wedge b := \min(a, b)$ and $\inf \varnothing := \infty$. Then R_n is a *stopping time* for each n, that is $\{R_n \le t\} \in \mathcal{F}_t$ for all t.This sequence of stopping times is non-decreasing and $\lim_{n \longrightarrow \infty} R_n = \infty$ with probability 1.

Then we may define

$$\int_0^t Y_s dB_s := \int_0^t (1_{\{R_n \ge s\}} \cdot Y_s) dB_s,$$

if $0 \le t \le R_n$. Here the right hand side is defined as the usual Itô-integral.

Show that this stochastic integral is well-defined.

Hint: Let $Z_t, 0 \le t \le T$ be a $\mathcal{F}_t-$adapted process such that $E[\int_0^T Z_s^2 ds] < \infty$ and let τ be a (finite) stopping time. Show that

$$\int_0^\tau Z_s dB_s = \int_0^t (1_{\{\tau \ge s\}} \cdot Z_s) dB_s$$

with probability 1. Verify this equation first for the stopping times

$$\tau_n := \sum_{i=0}^{2^n} \frac{(k+1)T}{2^n} 1_{\left\{\frac{kT}{2^n} \le \tau < \frac{(k+1)T}{2^n}\right\}} \longrightarrow \tau$$

for $n \longrightarrow \infty$ with probability 1.

So we see from the definition that $M_t := \int_0^t Y_s dB_s$ is a *local martingale* in the sense that there exists a sequence of stopping times $\tau_n \nearrow \infty$ with probability 1 such that $M_{t \wedge \tau_n}$ is a martingale.