Interest rate modelling via SPDE's (STK 4530)
Exercises 3, 30.9.2016

Problem 1 (Estimation of parameters in the Vasicek model) Assume that the dynamics of overnight rates $r(t)$ follows the Vasicek model

$$dr(t) = a(b - r(t)dt + \sigma dB_t,$$

where $a \geq 0$ is the mean reversion, $b \geq 0$ the reversion level and $\sigma \geq 0$ the volatility of the rates. Denote by r_i the overnight rate at the end of the ith interval of constant length δ (in years), $i = 0, 1, ..., n$.

(i) Set $\alpha := e^{-a\delta}$, $\beta := b$ and $V^2 := \frac{\sigma^2}{2a}(1 - e^{-2a\delta})$. Show that the maximum likelihood estimators for α, β and V^2 are given by

$$\widehat{\alpha} = \frac{n \sum_{i=1}^n r_i r_{i-1} - \sum_{i=1}^n r_i \sum_{i=1}^n r_{i-1}}{n \sum_{i=1}^n r_{i-1}^2 - (\sum_{i=1}^n r_{i-1})^2},$$

$$\widehat{\beta} = \frac{\sum_{i=1}^n (r_i - \widehat{\alpha} r_{i-1})}{n(1 - \widehat{\alpha})},$$

$$\widehat{V^2} = \frac{1}{n} \sum_{i=1}^n (r_i - \widehat{\alpha} r_{i-1} - \widehat{\beta}(1 - \widehat{\alpha}))^2.$$

(ii) Consider the following data set of Canadian overnight money market financing rates during 9 consecutive trading days:

Day	r_i %	Day	r_i %
19/09/07	4.4502	27/09/07	4.5421
20/09/07	4.4722	28/09/07	4.5340
21/09/07	4.4555	01/10/07	4.5078
24/09/07	4.4690		
25/09/07	4.4717		
26/09/07	4.4663		

Suppose that $\delta = \frac{1}{252}$ (i.e. 252 trading days per year). Compute the maximum likelihood estimators for a, b and σ.

Hint: (i) Recall that the maximum likelihood estimators of the Vasicek model are obtained by maximizing the joint density of $r_1, ..., r_n$ with respect to the parameters a, b and σ.

(ii) Use the representation

$$r(t) = r(s)e^{-a(t-s)} + b(1 - e^{-a(t-s)}) + \sigma \int_s^t e^{-a(t-u)} dB_u$$

and employ the increments of the Brownian motion B_t to determine the joint density.

Problem 2 Define the process

$$X_t = r(t) - b,$$

where the short rates $r(t)$ are as in Problem 1. Show that

$$Y := \int_0^t X_s ds$$

is a normally distributed random variable on $(\Omega, \mathcal{F}, P)$.

Problem 3 Suppose that the overnight rates $r(t)$ are described by the Vasicek model. Consider a European call option written on a zero-coupon bond with strike price K and maturity $\theta < T$, that is the payoff of this bond option at time θ is given by

$$h = (P(\theta, T) - K)_+.$$

Derive a formula for the present value of the option.