Interest rate modelling via SPDE's (STK 4530)
Exercises 5, 14.10.2016

Problem 1 (Implementation of the HJM-model) Recall that the risk-neutral dynamics of the forward rate curve $f(t,T)$ in the HJM-model (see Th. 3.3.13) is given by

$$df(t,T) = \sigma(t,T)\sigma^*(t,T)dt + \sigma(t,T)d\widetilde{B}_t,$$

where

$$\sigma^*(t,T) := \int_t^T \sigma(t,u)du$$

and where $\widetilde{B}_t$ is a Brownian motion in the risk-neutral world.

In order to implement the HJM-model let us assume that the volatility process $\sigma(t,T)$ takes the form

$$\sigma(t,T) = s(t) \cdot \widetilde{\sigma}(T-t) \cdot \min\{M, f(t,T)\},$$

where as decribed in the manuscript $s(t)$ and $\widetilde{\sigma}(\tau)$ are nonrandom functions. Further $\min\{M, f(t,T)\}$ is the capped forward rate capped by a constant $M > 0$.

Estimate the volatility function $\widetilde{\sigma}(\tau)$, $0 \leq \tau \leq 1$ (year) by means of principal component analysis from the following data set of Canadian treasury bills quoted at 05/09/07, 12/09/07, 03/10/07 and 10/10/07

$t_1 = 05/09/07$

Time-to-maturity	yield in %
1 month	4.05
3 month	4.08
6 month	4.27
1 year	4.32

$t_1 + \delta = 12/09/07$

Time-to-maturity	yield in %
1 month	3.88
3 month	3.88
6 month	4.20
1 year	4.30

$t_2 = 03/10/07$

Time-to-maturity	yield in %
1 month	3.89
3 month	3.98
6 month	4.21
1 year	4.23

$t_2 + \delta = 10/10/07$

Time-to-maturity	yield in %
1 month	3.93
3 month	3.97
6 month	4.30
1 year	4.44

Here $\delta = \frac{1}{52}$ years. Further, suppose that $M = 0.08$.

Hint: For convenience, use linear interpolation of the bond prices $P(t_j, T)$, $P(t_j + \delta, T)$, $j = 1, 2$ w.r.t. T to determine the historical forward rates $f(t_j, t_j + \tau_k)$, $f(t_j + \delta, t_j + \tau_k)$, $j = 1, 2$, $k = 1, 2, 3, 4$, where the relative maturities τ_k are given by $\tau_k = \frac{k}{5}$, $k = 1, 2, 3, 4$.

Problem 2 Consider the Vasicek model for the overnight rates $r(t)$, that is $r(t)$ is described by

$$dr(t) = a(b - r(t))dt + \sigma dB_t.$$

Find the volatility process $\sigma(t,T)$ of the corresponding forward rates $f(t,T)$ in the HJM-model given by the risk-neutral dynamics

$$df(t,T) = \sigma(t,T)\sigma^*(t,T)dt + \sigma(t,T)d\widetilde{B}_t.$$

Hint: Use the fact that $f(t,T) = -\frac{\partial}{\partial T}\log P(t,T)$. Here $P(t,T)$ is given by an explicit pricing formula. See relation (3.3.9) in the manuscript or the book of Brigo, Mercurio. Then apply Itô's formula to $r(t)$ in the resulting expression for $f(t,T)$ to obtain a SDE dynamics w.r.t. $\widetilde{B}_t$.