MAT4300 - FALL 2009 - MANDATORY ASSIGNMENT

Solution (the easy parts 1b) and 1c) are only sketched)

Problem 1

a) Let $\rho, f \in \bar{\mathcal{M}}^+$ and $A \in \mathcal{A}$. Then

$$(*) \qquad \int_A f\, d\mu_\rho = \int_A f\, \rho\, d\mu\,.$$

We consider first $f = \mathbf{1}_E$, $E \in \mathcal{A}$. Then

$$\int_A \mathbf{1}_E\, d\mu_\rho = \int \mathbf{1}_{A\cap E}\ d\mu_\rho = \mu_\rho(A\cap E)$$

$$= \int_{A\cap E} \rho\, d\mu = \int_A \mathbf{1}_E\, \rho\, d\mu\,,$$

so $(*)$ holds in this case.

Next, assume that $f \in \mathcal{E}(\mathcal{A})^+ = \{\mathcal{E}(\mathcal{A}) \mid f \geq 0\}$. Write $f = \sum_{j=1}^n y_j \mathbf{1}_{E_j}$ for some $y_j \in \mathbb{R}^+$, $E_j \in \mathcal{A}$. Linearity of the integral clearly passes over to linearity of the integral over A. Using this and the first step, we get

$$\int_A f\, d\mu_\rho = \int_A (\sum_{j=1}^n y_j\, \mathbf{1}_{E_j})\ d\mu_\rho = \sum_{j=1}^n y_j\, (\int_A \mathbf{1}_{E_j}\, d\mu_\rho)$$

$$= \sum_{j=1}^n y_j\, (\int_A \mathbf{1}_{E_j}\, \rho\, d\mu) = \int_A (\sum_{j=1}^n y_j\, \mathbf{1}_{E_j})\, \rho\, d\mu = \int_A f\, \rho\, d\mu\,,$$

so $(*)$ also holds in this case.

Finally, we consider $f \in \bar{\mathcal{M}}^+$. Using Thm 8.8, we pick an increasing sequence f_j in $\mathcal{E}(\mathcal{A})^+$ converging pointwise to f, i.e. $f_j \uparrow f$. Then $\mathbf{1}_A\, f_j$ and $\mathbf{1}_A\, f_j\, \rho$ are sequences in $\bar{\mathcal{M}}(\mathcal{A})^+$ such that $\mathbf{1}_A\, f_j \uparrow f$, $\mathbf{1}_A\, f_j\, \rho \uparrow f\, \rho$. Hence, using B. Lévi's MCT (twice) and the second step, we get

$$\int_A f\, d\mu_\rho = \int \mathbf{1}_A\, f\, d\mu_\rho = \lim_{j\to\infty} \int \mathbf{1}_A\, f_j\, d\mu_\rho$$

$$= \lim_{j\to\infty} \int \mathbf{1}_A\, f_j\, \rho\, d\mu = \int \lim_{j\to\infty} \mathbf{1}_A\, f_j\, \rho\, d\mu = \int_A f\, \rho\, d\mu\,,$$

which shows that $(*)$ holds.

b) Let $\rho \in \bar{\mathcal{M}}^+$ *and* $f \in \mathcal{M}$. *Then*

$f \in \mathcal{L}^1(\mu_\rho)$ *if and only if* $f\,\rho \in \bar{\mathcal{L}}^1(\mu)$, *in which case we have*

$$\int_A f\, d\mu_\rho = \int_A f\,\rho\, d\mu \quad \textit{for all } A \in \mathcal{A}\,.$$

Indeed, this follows easily by noticing that $(f\,\rho)^\pm = f^\pm\,\rho$ (since $\rho \geq 0$) and using a) on $f^\pm$, together with the definitions of integrability and of the integral. We skip the details.

c) Assume $\rho = \sum_{j=1}^\infty c_j\,\rho_j$ *for some sequences* $\{c_j\}_{j\in\mathbb{N}} \subset (0,\infty]$ *and* $\{\rho_j\}_{j\in\mathbb{N}} \subset \bar{\mathcal{M}}^+$. *Then*

$$\mu_\rho(A) = \sum_{j=1}^\infty c_j\,\mu_{\rho_j}(A)\,, \ \textit{for all } A \in \mathcal{A}\,.$$

This is a simple consequence of Cor. 9.9. We skip the details.

d) Let $\rho \in \bar{\mathcal{M}}^+$. *Assume* $f, g \in \bar{\mathcal{M}}$. *Then*

$$f = g \ \ \mu\text{-a.e.} \ \Rightarrow \ f = g \ \ \mu_\rho\text{-a.e.}$$

Give also an example showing that the converse implication is not necessarily true in general.

Assume $f = g$ μ-a.e.; that is, $\mu(N) = 0$, where $N = \{f \neq g\}$.
Then $\mu_\rho(N) = \int_N \rho\, d\mu = \int \mathbf{1}_N\,\rho\, d\mu = 0$ (see Thm 10.9 (ii) and its proof).
Hence, $f = g$ μ_ρ-a.e..

To see that the converse is not necessarily true, one may of course consider the "trivial" case where $\rho = 0$, so μ_ρ is the zero measure. But the converse can fail even if $\rho \neq 0$: consider f.ex.

$$X = \mathbb{R}\,,\ \mathcal{A} = \mathcal{B}(\mathbb{R})\,,\ \mu = \lambda \text{ and } \rho = \mathbf{1}_{[0,1]}\,,\ f = \mathbf{1}_{[-1,1]}\,,\ g = \mathbf{1}_{[0,2]}.$$

Then $N = \{f \neq g\} = [-1,0) \cup (1,2]$, so $\mu_\rho(N) = \mu(N \cap [0,1]) = \mu(\emptyset) = 0$, while $\mu(N) = 2 \neq 0$.
Hence $f = g$ μ_ρ-a.e., while it is not true that $f = g$ μ-a.e..

e) Assume $(X, \mathcal{A}, \mu) = (\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$ *and* $\rho : \mathbb{R} \to [0,\infty)$ *is continuous. Let* $F(x) = \int_0^x \rho(t)dt$, $x \in \mathbb{R}$, *and let* ν_F *be the associated Stieljes measure on* $\mathcal{B}(\mathbb{R})$.

Then $\nu_F = \rho \cdot \lambda$ *and* $\int f \, d\nu_F = \int f \rho \; d\lambda$ *for all* $f \in \bar{\mathcal{M}}^+$.

For all $a, b \in \mathbb{R}$, $a \leq b$, we have

$$\nu_F([a,b)) = F(b) - F(a) = \int_0^b \rho(t)dt - \int_0^a \rho(t)dt = \int_a^b \rho(t)dt = \int_{[a,b)} \rho \, d\lambda = \lambda_\rho([a,b))$$

Since the family $\mathcal{J} = \{[a,b) \,|\, a,b \in \mathbb{R}\,,\, a \leq b\}$ generates $\mathcal{B}(\mathbb{R})$, is stable under finite intersections, and the exhausting sequence $[-k,k) \uparrow \mathbb{R}$ in $\mathcal{J}$ satisfies $\nu_F([-k,k)) = F(k) - F(-k) < \infty$ for all $k \in \mathbb{N}$, Thm 5.7 (or 6.1) applies and gives that $\nu_F = \lambda_\rho = \rho \cdot \lambda$. The final assertion follows then from 1a).

f) Assume $\mathcal{A} = \mathcal{P}(X)$. *Let* μ *be the counting measure on* $\mathcal{A}$, $\rho \in \bar{\mathcal{M}}^+$ *and* $A \in \mathcal{A}$, *i.e.* $A \subset X$. *Then*

$$\mu_\rho(A) = \sum_{x \in A} \rho(x)\,.$$

Since $\mu_\rho(A) = \int_A d\mu_\rho = \int \mathbf{1}_A \, \rho \, d\mu$, we have to show that

$$(**) \quad \int \mathbf{1}_A \, \rho \, d\mu = \sum_{x \in A} \rho(x)\,.$$

Assume first there exists some $x_0 \in A$ such that $\rho(x_0) = \infty$.
Then, by definition, we have $\sum_{x \in A} \rho(x) = \infty$ (cf. Extra-Exercise 1).
On the other hand, let $h_n \in \mathcal{E}^+(\mathcal{A})$ be given by $h_n = n\, \mathbf{1}_{\{x_0\}}$, $n \in \mathbb{N}$. Then for each $n \in \mathbb{N}$ we have $h_n \leq \mathbf{1}_A \, \rho$; therefore

$$n = n \int_{\{x_0\}} d\mu = \int h_n \, d\mu \leq \sup\{\int h \, d\mu \,|\, h \in \mathcal{E}^+(\mathcal{A}),\, h \leq \mathbf{1}_A \, \rho\} = \int \mathbf{1}_A \, \rho \, d\mu\,.$$

This implies that $\int \mathbf{1}_A \, \rho \, d\mu = \infty$. Hence we see that $(**)$ holds in this case.

Next, we assume that $\rho(x) < \infty$ for all $x \in A$.

Let $B = \{x_1, \cdots, x_k\}$ be a finite subset of A.

Set $g = \sum_{j=1}^k \rho(x_j)\, \mathbf{1}_{\{x_j\}} \in \mathcal{E}^+(\mathcal{A})$. Then $g \leq \mathbf{1}_A \, \rho$, so

$$\sum_{x \in B} \rho(x) = \sum_{j=1}^k \rho(x_j) = \int g \, d\mu \leq \int \mathbf{1}_A \, \rho \, d\mu\,.$$

Taking the supremum over all such B's, we get

$$\sum_{x\in A}\rho(x) \leq \int \mathbf{1}_A\,\rho\,d\mu\,.$$

Especially, if $\sum_{x\in A}\rho(x) = \infty$, then this implies that

$$\sum_{x\in A}\rho(x) = \int \mathbf{1}_A\,\rho\,d\mu\; = \infty$$

and $(**)$ holds.

Thus to show that $(**)$ always holds, it remains only to check that

$$(***)\quad \int \mathbf{1}_A\,\rho\,d\mu\; \leq \sum_{x\in A}\rho(x) \text{ whenever } \sum_{x\in A}\rho(x) < \infty\,.$$

So assume now that $\sum_{x\in A}\rho(x) < \infty\,.$

Set $A_0 = \{x\in A\,|\,\rho(x) > 0\}\,,\ A_k = \{x\in A\,|\,\rho(x) > 1/k\}\,,\ k\in\mathbb{N}$.

Then $A_0 = \cup_{k\in\mathbb{N}} A_k$ is countable, as each A_k is finite: indeed, if A_k is infinite for some $k\in\mathbb{N}$, then

$$\sum_{x\in A}\rho(x) \geq \sum_{x\in A_k}\rho(x) \geq \frac{1}{k}\sum_{x\in A_k} 1 = \infty\,,$$

contradicting our assumption.

Note also that ρ is zero on $A\setminus A_0$, by definition of A_0.

If A_0 is finite, then trivially

$$\int \mathbf{1}_A\,\rho\,d\mu = \int \mathbf{1}_{A_0}\,\rho\,d\mu = \sum_{x\in A_0}\rho(x) = \sum_{x\in A}\rho(x)$$

and $(**)$ holds.

If A_0 is countably infinite, let $\{a_1, a_2, \cdots\}$ be an enumeration of the elements of A_0 and set $\rho_n = \sum_{j=1}^n \rho(a_j)\,\mathbf{1}_{\{a_j\}}\,,\ n\in\mathbb{N}$. Then $\{\rho_n\}\uparrow \mathbf{1}_A\,\rho$, and B. Lévi's MCT gives

$$\int \mathbf{1}_A\,\rho\,d\mu = \lim_{n\to\infty}\int \rho_n\,d\mu = \lim_{n\to\infty}\sum_{j=1}^n \rho(a_j) \leq \sum_{x\in A}\rho(x)\,.$$

Hence $(***)$ holds (actually one can easily show that we have equality above), and this finishes the proof.

Problem 2.

a) Let $T : X \to Y$*,* $B \subset Y$ *and set* $A = T^{-1}(B) \subset X$*.*
Then $T(A) = B \cap T(X)$*. If* T *is injective and* X *is finite, then* $\#(T^{-1}(B)) = \#(B \cap T(X))$*.*

Since $A = \{x \in X \,|\, T(x) \in B\}$ we have

$$T(A) = \{T(x) \,|\, x \in X\,,\, T(x) \in B\} = \{\, y \in T(X) \,|\, y \in B\} = T(X) \cap B\,.$$

If T is injective, then $x \to T(x)$ gives a bijection from A onto $T(A)$, so A and $T(A)$ have the same cardinality. Now cardinality of a finite set may be interpreted as the number of elements in the set. Hence, if T is injective and X is finite, we get $\#(T^{-1}(B)) = \#(A) = \#(T(A)) = \#(B \cap T(X))$.

For each $n \in \mathbb{N}$*, set* $X_n = \{0,1\}^n$ *and* $\mu_n(A) = \frac{\#(A)}{2^n}$*,* $A \subset X_n$*.*
Further, let $T_n(a_1, a_2, \cdots, a_n) = a_1 \cdot \frac{1}{2} + a_2 \cdot \frac{1}{2^2} + \cdots + a_n \cdot \frac{1}{2^n}$ *when* $(a_1, a_2, \cdots, a_n) \in X_n$*, and set* $\lambda_n = T_n(\mu_n)$ $(= \mu_n \circ T_n^{-1})$*.*

b) Let $B \subset [0,1]$*,* $n \in \mathbb{N}$*. Then* $\lambda_n(B) = \frac{1}{2^n} \cdot \#\big(B \cap D_n\big)$*, where* $D_n = \big\{0, \frac{1}{2^n}, \frac{2}{2^n}, \frac{3}{2^n}, \cdots, \frac{2^n-1}{2^n}\big\}$*.*

Note that

$$T_n(X_n) = \{a_1 \cdot \frac{1}{2} + a_2 \cdot \frac{1}{2^2} + \cdots + a_n \cdot \frac{1}{2^n} \mid (a_1, a_2, \cdots, a_n) \in X_n\}$$

$$= \{\frac{1}{2^n}\big[a_1 \cdot 2^{n-1} + a_2 \cdot 2^{n-2} + \cdots + a_{n-1} \cdot 2^1 + a_n \cdot 2^0\big] \mid (a_1, a_2, \cdots, a_n) \in X_n\}$$

$$= \{\frac{1}{2^n} m \,|\, m \in \mathbb{Z}\,,\, 0 \le m \le 2^n - 1\} = D_n$$

since every every integer m from 0 to $2^n - 1$ may be (uniquely) written as $a_1 \cdot 2^{n-1} + a_2 \cdot 2^{n-2} + \cdots + a_{n-1} \cdot 2^1 + a_n \cdot 2^0$ for some $(a_1, a_2, \cdots, a_n) \in X_n$; the string $a_1 a_2 \cdots a_{n-1} a_n$ is then the base 2 representation of m.

Hence, using a), we get

$$\lambda_n(B) = \mu_n(T_n^{-1}(B)) = \mu_n(B \cap T_n(X_n)) = \frac{1}{2^n} \cdot \#\big(B \cap D_n\big)\,.$$

c) Let μ_c *denote the counting measure on* $\mathcal{P}([0,1])$*. Let* $n \in \mathbb{N}$ *and* $f : [0,1] \to \mathbb{R}$*.*

Then $\lambda_n = \frac{1}{2^n}\,\mathbf{1}_{D_n}\cdot\mu_c$, $f \in \mathcal{L}^1(\lambda_n)$ *and*

$$\int f \; d\lambda_n = \frac{1}{2^n}\cdot\sum_{x\in D_n} f(x) = \frac{1}{2^n}\cdot\sum_{j=0}^{2^n-1} f\big(\frac{j}{2^n}\big)\,.$$

Let $B \subset [0,1]$. Using b) we get

$$\lambda_n(B) = \frac{1}{2^n}\cdot\#\big(\,B\cap D_n\,\big) = \frac{1}{2^n}\int_{B\cap D_n} d\mu_c = \int_B \frac{1}{2^n}\mathbf{1}_{D_n}\,d\mu_c\,.$$

This shows that $\lambda_n = \frac{1}{2^n}\,\mathbf{1}_{D_n}\cdot\mu_c$.

Trivially, f is measurable. Further, using 1a) and 1f), we get

$$\int f^{\pm}\,d\mu_n = \int f^{\pm}\,\frac{1}{2^n}\,\mathbf{1}_{D_n}\,d\mu_c = \frac{1}{2^n}\int_{D_n} f^{\pm}\,d\mu_c = \frac{1}{2^n}\sum_{x\in D_n} f^{\pm}(x) < \infty$$

since D_n is finite. This shows that $f \in \mathcal{L}^1(\lambda_n)$. Further we then get

$$\int f\,d\mu_n = \int f^{+}\,d\mu_c - \int f^{-}\,d\mu_c = \frac{1}{2^n}\sum_{x\in D_n} f^{+}(x)\; - \;\frac{1}{2^n}\sum_{x\in D_n} f^{-}(x)$$

$$= \frac{1}{2^n}\sum_{x\in D_n} f(x) = \frac{1}{2^n}\sum_{j=0}^{n-1} f(j/2^n)\,,$$

as desired.

d) Let f be a bounded, Riemann integrable function on $[0,1]$. *Then*

$$\lim_{n\to\infty}\int f\;d\lambda_n = \int_0^1 f(x)\,dx\,.$$

For each n we consider the regular partition

$P_n = \{0 < \frac{1}{2^n} < \frac{2}{2^n} < \frac{3}{2^n} < \cdots < \frac{2^n-1}{2^n} < 1\}$ of $[0,1]$.

The Riemann sum associated to f by choosing as intermediate points the left endpoint of each of the subintervals $[\frac{j}{2^n}\,,\,\frac{j+1}{2^n}]\,,\ j = 0,1,\cdots,n-1\,,$ is then given by

$$R_n(f) = \sum_{j=0}^{n-1} f(j/2^n)\frac{1}{2^n} = \int f\,d\mu_n\,.$$

As the mesh of P_n, which is equal to 2^{-n}, goes to zero as $n \to \infty$, the Riemann integrability of f implies that $\lim_{n\to\infty} R_n(f) = \int_{[0,1]} f(x)\,dx$ (cf. Appendix E. 6). The assertion clearly follows.

e) Let λ denote the Lebesgue measure on $[0,1]$ and $0 \le a \le b \le 1$. Then

$$\lim_{n\to\infty} \lambda_n([a,b]) = \lambda([a,b])\,.$$

As the function $\mathbf{1}_{[a,b]}$ is Riemann integrable, d) gives:

$$\lim_{n\to\infty} \lambda_n([a,b]) = \lim_{n\to\infty} \int \mathbf{1}_{[a,b]}\,d\lambda_n = \int_0^1 \mathbf{1}_{[a,b]}\,dx = \int_{[0,1]} \mathbf{1}_{[a,b]}\,d\lambda = \lambda([a,b])\,.$$

f) The formula

$$\lim_{n\to\infty} \int f\ d\lambda_n = \int f\,d\lambda$$

does not hold for all $f \in \mathcal{L}^1(\lambda)$.

Indeed, let f be the restriction of $\mathbf{1}_{\mathbb{Q}}$ to $[0,1]$.
Then $f\left(\frac{j}{2^n}\right) = 1$ for all $j = 0, 1, \cdots, n-1$, and all $n \in \mathbb{N}$.
Using c) we get

$$\int f\ d\lambda_n = \frac{1}{2^n} \cdot \sum_{j=0}^{2^n-1} 1 = \frac{2^n}{2^n} = 1$$

for all $n \in \mathbb{N}$. Hence $\lim_{n\to\infty} \int f\ d\lambda_n = 1$.

On the other hand, $f = 0$ λ-a.e. (since $\mathbb{Q} \cap [0,1]$ is countable).

So $f \in \mathcal{L}^1(\lambda)$ and $\int f\,d\lambda = 0 \neq \lim_{n\to\infty} \int f\ d\lambda_n$.