MAT3300/4300 – Exam – Fall 09 – Solution.

Exercise 1

a) Since $\emptyset \in \mathcal{A}'$ and $\emptyset \in \mathcal{A}''$ we have $\emptyset = \emptyset \cup \emptyset \in \mathcal{A}$.

Next, let $A \in \mathcal{A}$, so $A = A' \cup A''$, where $A' \in \mathcal{A}'$, $A'' \in \mathcal{A}''$. Then the complement of A in X, $X \setminus A$, is also in $\mathcal{A}$. (To avoid confusion, we don't use the usual notation for complement). Indeed, we have:

$$X\setminus A = (X' \cup X'')\setminus(A' \cup A'') = (X'\setminus(A' \cup A'')) \cup (X''\setminus(A' \cup A'')) = (X'\setminus A') \cup (X''\setminus A')$$

As $(X' \setminus A') \in \mathcal{A}'$, $(X'' \setminus A'') \in \mathcal{A}''$, we have $(X \setminus A) \in \mathcal{A}$.

Assume now $A_j \in \mathcal{A}$ for each $j \in \mathbb{N}$ and write $A_j = A'_j \cup A''_j$, where $A'_j \in \mathcal{A}'$, $A''_j \in \mathcal{A}''$. Then

$$\cup_{j=1}^{\infty} A_j = \cup_{j=1}^{\infty} (A'_j \cup A''_j) = (\cup_{j=1}^{\infty} A'_j) \cup (\cup_{j=1}^{\infty} A''_j) \in \mathcal{A}$$

since $\cup_{j\in\mathbb{N}} A'_j \in \mathcal{A}'$, $\cup_{j\in\mathbb{N}} A''_j \in \mathcal{A}''$.

Thus we have shown that $\mathcal{A}$ is a σ-algebra in X, as desired.

Further, if $A' \in \mathcal{A}'$ then $A' = A' \cup \emptyset \in \mathcal{A}$. So $\mathcal{A}' \subset \mathcal{A}$. Similarly, $\mathcal{A}'' \subset \mathcal{A}''$.

Let now $\mathcal{B}$ be any σ-algebra in X which contains $\mathcal{A}'$ and $\mathcal{A}''$. Let $A \in \mathcal{A}$, so $A = A' \cup A''$, where $A' \in \mathcal{A}'$, $A'' \in \mathcal{A}''$. Then $A' \in \mathcal{B}$ and $A'' \in \mathcal{B}$. So $A = A' \cup A'' \in \mathcal{B}$. This shows that $\mathcal{A} \subset \mathcal{B}$.
Hence $\mathcal{A}$ is the smallest σ-algebra in X which contains $\mathcal{A}'$ and $\mathcal{A}''$, that is, $\mathcal{A} = \sigma(\mathcal{A}' \cup \mathcal{A}'')$, as desired.

b) We have $\mu(\emptyset) = \mu(\emptyset \cup \emptyset) = \mu'(\emptyset) + \mu''(\emptyset) = 0 + 0 = 0$.

Further, let $\{\mathcal{A}_j\}_{j\in\mathbb{N}}$ be a sequence in $\mathcal{A}$ of pairwise disjoint subsets of X.

For each $j \in \mathbb{N}$, write $A_j = A'_j \cup A''_j$, where $A'_j \in \mathcal{A}'$, $A''_j \in \mathcal{A}''$.

Then, when $j \neq k$, we have $A'_j \cap A'_k = \emptyset$ and $A''_j \cap A''_k = \emptyset$ (otherwise $A_j \cap A_k$ would be nonempty). Hence, using the computation in a), we get

$$\mu(\cup_{j=1}^{\infty} A_j) = \mu\big((\cup_{j=1}^{\infty} A'_j) \cup (\cup_{j=1}^{\infty} A''_j)\big) = \mu'(\cup_{j=1}^{\infty} A'_j) + \mu''(\cup_{j=1}^{\infty} A''_j)$$

$$= \sum_{j=1}^{\infty} \mu'(A'_j) + \sum_{j=1}^{\infty} \mu''(A''_j) = \sum_{j=1}^{\infty} (\mu'(A'_j) + \mu''(A''_j)) = \sum_{j=1}^{\infty} \mu(A_j) .$$

Hence, μ is a measure on $\mathcal{A}$. If $A' \in \mathcal{A}'$, then

$$\mu(A') = \mu(A' \cup \emptyset) = \mu'(A') + \mu''(\emptyset) = \mu'(A') + 0 = \mu'(A') .$$

So μ agrees with μ' on $\mathcal{A}'$. Similarly, μ agrees with μ'' on $\mathcal{A}''$.

Finally, assume both μ' and μ'' are σ-finite.

Let $\{A'_j\}_{j\in\mathbb{N}} \subset \mathcal{A}'$ be such that $A'_j \uparrow X'$ and $\mu'(A'_j) < \infty$ for alle j.

Let also $\{A''_j\}_{j\in\mathbb{N}} \subset \mathcal{A}''$ such that $A''_j \uparrow X''$ and $\mu''(A''_j) < \infty$ for alle j.

For each j, set $A_j = A'_j \cup A''_j \in \mathcal{A}$.

Then, clearly we have $A_j \uparrow (X' \cup X'') = X$ and

$$\mu(A_j) = \mu'(A'_j) + \mu''(A''_j) < \infty \text{ for all } \ j \in \mathbb{N}\,.$$

Hence, μ is σ-finite, as desired.

c) Note that if $A \in \mathcal{A}$, so $A = A' \cup A''$ with $A' \in \mathcal{A}'$, $A'' \in \mathcal{A}''$, then $A \cap X' = A' \in \mathcal{A}'$.

Now, for each $B \in \mathcal{B}(\mathbb{R})$, measurability of f means that $f^{-1}(B) \in \mathcal{A}$; hence, we then get $(f')^{-1}(B) = f^{-1}(B) \cap X' \in \mathcal{A}'$. Thus, $f' \in \mathcal{M}^+(\mathcal{A}')$.

Again, consider $A = A' \cup A''$ with $A' \in \mathcal{A}'$, $A'' \in \mathcal{A}''$.

Then, clearly, $(\mathbf{1}_A^X)' = \mathbf{1}_{A'}^{X'}$ and $(\mathbf{1}_A^X)'' = \mathbf{1}_{A''}^{X''}$. Therefore we get

$$\int \mathbf{1}_A^X \, d\mu = \mu(A) = \mu'(A') + \mu''(\mathcal{A}'')$$

$$= \int \mathbf{1}_{A'}^{X'} \, d\mu' + \int \mathbf{1}_{A''}^{X''} \, d\mu'' = \int (\mathbf{1}_A^X)' \, d\mu' + \int (\mathbf{1}_A^X)'' \, d\mu''\,,$$

i.e. the desired formula holds for $\mathbf{1}_A^X$. By linearity (of integrals), it also holds for simple functions in $\mathcal{M}^+(\mathcal{A})$, i.e. when $f \in \mathcal{E}^+(\mathcal{A})$.

Now, (using Thm 8.8), we pick a sequence f_n in $\mathcal{E}^+(\mathcal{A})$ converging pointwise on X to the given $f \in \mathcal{M}^+(\mathcal{A})$, i.e. $f_n \uparrow f$. Then $(f_n)'$ is a sequence in $\mathcal{E}^+(\mathcal{A}')$ such that $(f_n)' \uparrow f'$, and similarly for $(f_n)''$.

Using B. Lévi's MCT (three times), we get

$$\int f \, d\mu = \lim_{n\to\infty} \int f_n \, d\mu = \lim_{n\to\infty} (\int (f_n)' d\mu' + \int (f_n)'' d\mu'')$$

$$= \lim_{n\to\infty} \int (f_n)' d\mu' + \lim_{n\to\infty} \int (f_n)'' d\mu'' = \int f' d\mu' + \int f'' d\mu''\,,$$

as desired.

Exercise 2.

a) Set $\mathcal{B} = \{B \in \mathcal{B}(\mathbb{R}) \,|\, B \subset [0,1]\}$. Being continuous, f is $\mathcal{B}/\mathcal{B}(\mathbb{R})$-measurable . Since g is $\mathcal{A}/\mathcal{B}(\mathbb{R})$-measurable and takes values in $[0,1]$, g is $\mathcal{A}/\mathcal{B}$-measurable. Being a product of $\mathcal{A}/\mathcal{B}$-measurable functions, each function $g_n : x \to g(x)^n$ is then $\mathcal{A}/\mathcal{B}$-measurable. Hence, each $h_n = f \circ g_n$ is $\mathcal{A}/\mathcal{B}(\mathbb{R})$-measurable, that is, each $h_n \in \mathcal{M}(\mathcal{A})$.

Further, as f is continuous on the closed intervall $[0,1]$, f is bounded; that is, $|f(t)| \leq M$ for all $t \in [0,1]$ for some constant $M > 0$. Then we clearly have $|h_n(x)| \leq M$ for all $x \in X$ and all $n \in \mathbb{N}$.

Setting $H(x) = M$, $x \in X$, we have $H \in \mathcal{L}^1(\mu)$ (since μ is finite). Since $|h_n| \leq H$, it follows that each $h_n \in \mathcal{L}^1(\mu)$.

Now, set $E = \{x \in X \,|\, g(x) = 1\} \in \mathcal{A}$. Then $\lim_{n\to\infty} h_n(x) = f(1)$ for all $x \in E$. Further, when $x \in E^c = [0,1] \setminus E$, we have $\lim_{n\to\infty} h_n(x) = f(0)$.

Hence, $\lim_{n\to\infty} h_n(x) = f(1)\mathbf{1}_E + f(0)\mathbf{1}_{E^c}$.

We can now apply Lebesgue's DCT.

We get that $f(1)\mathbf{1}_E + f(0)\mathbf{1}_{E^c} \in \mathcal{L}^1(\mu)$ (which is obvious as μ is finite) and

$$\lim_{n\to\infty} \int h_n \, d\mu = \int (f(1)\mathbf{1}_E + f(0)\mathbf{1}_{E^c}) \, d\mu = f(1) \cdot \mu(E) + f(0) \cdot \mu(E^c) \in \mathbb{R} \, .$$

b) In this case, using the notation introduced in a), we see that $E = \{1, 2\}$, so $\mu(E) = 2$ and $\mu(E^c) = \mu(X) - \mu(E) = 10 - 2 = 8$.

From the formula above, we get $\lim_{n\to\infty} \int h_n \, d\mu = -4 \cdot 2 + 1 \cdot 8 = 0$.

Exercise 3

We will first show that $f - f_N \in \mathcal{L}^1(\mu)$. As f and f_N are measurable, $f - f_N$ is measurable. So it suffices to show that $\int |f - f_N| \, d\mu < \infty$.

For each $n \in \mathbb{N}$, we let $g_n \in \mathcal{M}^+(\mathcal{A})$ be given by $g_n = |f_n - f_N|$. We also set $A = \{x \in X |\, \lim_{n\to\infty} f_n(x) = f(x)\}$. Since $\lim_{n\to\infty} f_n = f$ μ-a.e., we have $A \in \mathcal{A}$ and $\mu(A^c) = 0$. This implies that

$\int h \, d\mu = \int_A h \, d\mu + \int_{A^c} h \, d\mu = \int_A h \, d\mu$ for all $h \in \mathcal{M}^+(\mathcal{A})$.

Using the the second assumption, we then get

$\int_A |f_n - f_N| \, d\mu = \int |f_n - f_N| \, d\mu = \|f_n - f_N\|_1 \leq C$ for all $n \in \mathbb{N}$.

Hence $\inf_{n\geq k}\{\int_A |f_n - f_N|\,d\mu\} \leq C$ for all $k \in \mathbb{N}$. So

$$\liminf_{n\to\infty} \int_A |f_n - f_N|\,d\mu = \sup_{k\in\mathbb{N}} \inf_{n\geq k} \{\int_A |f_n - f_N|\,d\mu\} \leq C\,.$$

Using Fatou's lemma, we get

$$\int |f-f_N|\,d\mu = \int_A |f-f_N|\,d\mu = \int \lim_{n\to\infty} \mathbf{1}_A\,|f_n-f_N|\,d\mu = \int \liminf_{n\to\infty} \mathbf{1}_A\,|f_n-f_N|\,d\mu$$

$$\leq \liminf_{n\to\infty} \int \mathbf{1}_A\,|f_n - f_N|\,d\mu = \liminf_{n\to\infty} \int_A |f_n - f_N|\,d\mu \leq C < \infty\,,$$

as desired.

Now we know that both f_N and $f - f_N$ are in $\mathcal{L}^1(\mu)$. As $\mathcal{L}^1(\mu)$ is vector space, we get $f = (f - f_N) + f_N \in \mathcal{L}^1(\mu)$, as was to be shown.

Exercise 4.

a) We have

$$\int |\varphi|^p d\nu = \int |\varphi|^p\,\rho\,d\mu = \int \frac{1}{x^{p/2}}\,\frac{x}{2}\,d\mu(x) = \frac{1}{2}\int \frac{1}{x^{\frac{p}{2}-1}}\,d\mu\,.$$

As we know that the integral on the right is finite if and and only if $\frac{p}{2} - 1 < 1$, we get that $\int |\varphi|^p\,d\nu < \infty \Leftrightarrow p < 4$, as desired.

b) Using Tonelli's theorem, we get

$$\int |F|\,d(\mu\times\mu) = \int\int \frac{x\,|f(y)|}{y}\,\mathbf{1}_D(x,y)\,d\mu(x)\,d\mu(y)$$

$$= \int \frac{|f(y)|}{y} \int_{(0,y]} x\,d\mu(x)\,d\mu(y) = \int \frac{|f(y)|}{y}\,\frac{y^2}{2}\,d\mu(y)$$

$$= \int |f(y)|\,\frac{y}{2}\,d\mu(y) = \int |f(y)|\,d\nu(y) = \int |f|\,d\nu < \infty\,.$$

c) Let $x \in (0,1]$. Then $\frac{1}{y^2} \leq \frac{1}{x^2}$ for all $y \in (x,1]$. Hence

$$\int_{(x,1]} \frac{|f(y)|}{y}\,d\mu(y) = \int_{(x,1]} 2\,\frac{|f(y)|}{y^2}\,\frac{1}{2}\,y\,d\mu(y)$$

$$\leq \frac{2}{x^2}\int_{(x,1]} |f(y)|\,\frac{1}{2}\,y\,d\mu(y) \leq \frac{2}{x^2}\int |f(y)|\,d\nu(y) < \infty\,.$$

Consider now the (measurable) function given by

$$h_x(y) = 2 \cdot \mathbf{1}_{(x,1]}(y)\, \frac{f(y)}{y}\,,\ y \in (0,1]\,.$$

We then have

$$\int |h_x|\, d\mu(y) = 2 \int_{(x,1]} \frac{|f(y)|}{y}\, d\mu(y) < \infty\,,$$

so $h_x \in \mathcal{L}^1(\mu)$. Thus, the integral

$$\int h_x\, d\mu = 2 \int \mathbf{1}_{(x,1]}(y)\, \frac{f(y)}{y}\, d\mu(y)$$

makes sense and we may call it $g(x)$, as wanted.

d) Being clearly continuous on $(0,1]$, g is $\mathcal{A}/\mathcal{B}(\mathbb{R})$-measurable. Further, using Tonelli's theorem and b), we get

$$\int |g|\, d\nu = \int \Big|\, 2 \int \mathbf{1}_{(x,1]}(y)\, \frac{f(y)}{y}\, d\mu(y)\,\Big|\, \rho(x)\, d\mu(x)$$
$$\leq \int 2 \int_{(x,1]} \frac{|f(y)|}{y}\, d\mu(y)\, \frac{x}{2}\, d\mu(x) = \int \int_{(x,1]} x\, \frac{|f(y)|}{y}\, d\mu(y)\, d\mu(x)$$
$$= \int \int \frac{x\,|f(y)|}{y}\, \mathbf{1}_D(x,y)\, d\mu(y)\, d\mu(x) = \int |F|\, d(\mu \times \mu) = \int |f|\, d\nu < \infty\,.$$

Hence $g \in \mathcal{L}^1(\nu)$.

Further, as $\int |F|\, d(\mu \times \mu)$ is finite, we may apply Fubini's theorem. In the same way, we get

$$\int F\, d(\mu \times \mu) = \int \frac{f(y)}{y} \int_{(0,y]} x\, d\mu(x)\, d\mu(y) = \int \frac{f(y)}{y}\, \frac{y^2}{2}\, d\mu(y)$$
$$= \int f(y)\, \frac{y}{2}\, d\mu(y) = \int f(y)\, d\nu(y) = \int f\, d\nu$$

and

$$\int F\, d(\mu \times \mu) = \int x \int \mathbf{1}_{(x,1]}(y)\, \frac{f(y)}{y}\, d\mu(y)\, d\mu(x)$$
$$= \int \frac{x}{2}\, g(x)\, d\mu(x) = \int g\, d\nu\,,$$

and the last assertion to be proved clearly follows.