EXAM PROBLEMS

IN

ECON4120 MATHEMATICS 2

MATHEMATICAL ANALYSIS

AND

LINEAR ALGEBRA

Department of Economics

2003

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Preface

This problem collection is meant for students who prepare for the exam in ECON4120 Mathematics 2. The problems are taken from previous exams at roughly the same level.

In the back (mostly brief) answers to the problems are supplied. We thank Li Cen for excellent help with this booklet.

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Problem 1

Let $f(x) = x(x-1)(x-2)$.

(a) Decide where $f(x) > 0$.

(b) Compute $f'(x)$. Decide where the function increases/decreases. Find possible local extreme points and values. Where is $f(x)$ strictly convex?

(c) Sketch the graph. Compute $\int_0^1 f(x)\,dx$.

Problem 2

(a) The function g is defined by $g(x,y) = 3 + x^3 - x^2 - y^2$, and its domain, D, is given by $x^2 + y^2 \leq 1$ and $x \geq 0$. Sketch the domain D in the xy-plane.

(b) Find the stationary points of the function g, and classify them.

(c) Find the (global) extreme points and extreme values of g in D.

Problem 3

Let $D = \begin{vmatrix} 0 & x & y \\ x & 1 & a \\ y & b & ab \end{vmatrix}$, where a and b are positive constants, with $a > b$. Along what straight lines in the xy-plane is $D = 0$? In what part(s) of the xy-plane is $D > 0$?

Problem 4

The daily production of a firm is given by $F(L,K) = L^{1/2}K^{1/2}$, where L is the number of workers and K is invested capital.

(a) Show that $F(tL,tK) = tF(L,K)$ for all $t \geq 0$, and that

$$L\frac{\partial F}{\partial L} + K\frac{\partial F}{\partial K} = F(L,K)$$

Each worker has a year salary of 50 000 kr., and interest on capital investment is paid at 8% p.a. The firm has 1 million kr. to spend for salary expenses and interest payment each year.

(b) Find the values of L and K that maximize production capacity subject to the budget constraint.

Problem 5

(a) Let $\mathbf{A} = \begin{pmatrix} a & b & 0 \\ -b & a & b \\ 0 & -b & a \end{pmatrix}$, where a and b are arbitrary constants. Find the determinant of $\mathbf{A}$, and compute $\mathbf{A} \cdot \mathbf{A} = \mathbf{A}^2$.

(b) A square matrix $\mathbf{B}$ is called *skew-symmetric* if $\mathbf{B} = -\mathbf{B}'$, where $\mathbf{B}'$ is the transpose of $\mathbf{B}$. Show that if $\mathbf{C}$ is an arbitrary matrix such that $\mathbf{C}'\mathbf{B}\mathbf{C}$ is defined, then $\mathbf{C}'\mathbf{B}\mathbf{C}$ is skew-symmetric if $\mathbf{B}$ is.

(c) When is the matrix $\mathbf{A}$ defined in part (a) above skew-symmetric?

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Problem 6

A function f is given by the formula $f(x) = (1 + 2/x)\sqrt{x+6}$.

(a) Determine the domain of f.

(b) Determine the zeros of f, and the intervals where $f(x)$ is positive.

(c) Find the local extreme points of the function, if any.

(d) Determine the limits of $f(x)$ as $x \to 0^-$, $x \to 0^+$, and $x \to \infty$. Determine the limit of $f'(x)$ as $x \to \infty$. Sketch the graph of f.

Problem 7

A firm produces x units of one commodity and y units of another. The selling prices per unit are determined by the demand relations

$$p = a - 2x^2, \qquad q = by^{-1/2}$$

The cost function is $\pi(x, y) = cx + dy + e$. The constants a, b, c, d, e are positive.

(a) Determine the values of x and y that maximize the firm's net profits, N.

(b) Find the elasticity of N w.r.t. y. What is this elasticity at the maximum net profit?

Problem 8

The function f is defined by

$$f(x, y) = 5xy - x^a y^a - 4 \quad \text{for all } x > 0,\ y > 0 \qquad (a > 1)$$

(a) Compute the partial derivatives of f of the first and second order.

(b) Find all stationary points of f and classify them, if possible.

(c) Show that the hyperbolas $xy = k$ (k constant > 0) are level curves for f. f attains its maximum on one of these level curves. Which one?

(d) Suppose $c > 0$. Find conditions on the constants a and c for $h(z) = 5z - z^a - c = 0$ to have no, one, or two solutions, respectively, in $(0, \infty)$.

(e) Let p and q be positive constants and solve the problem

$$\text{minimize} \quad px + qy \quad \text{subject to} \quad 5xy - x^2y^2 = 4, \quad x > 0, \quad y > 0$$

(Take it for granted that the problem has a solution.)

Problem 9

Suppose that the equation

$$\ln x + 2(\ln x)^2 = \frac{1}{2}\ln K + \frac{1}{3}\ln L$$

defines x as a differentiable function of K and L.

(a) Find expressions for $\dfrac{\partial x}{\partial K}$, $\dfrac{\partial x}{\partial L}$, and $\dfrac{\partial^2 x}{\partial K \partial L}$.

(b) Show that $\text{El}_K x + \text{El}_L x = \dfrac{5}{6}\left(\dfrac{1}{1 + 4\ln x}\right)$.

Problem 10

Let the utility function U be defined by the formula

$$U(x, y) = A\ln(x - a) + B\ln(y - b)$$

where a, b, A and B are positive constants, $A + B = 1$.

(a) For what values of x and y is U defined?

(b) Let p, q and R be positive constants. Use Lagrange's method to show that if $x = x^*$, $y = y^*$ solve the problem

$$\max_{x,\,y} \big[A\ln(x - a) + B\ln(y - b)\big] \quad \text{subject to} \quad px + qy = R \tag{$*$}$$

then

$$x^* = a + \frac{A\big(R - (pa + qb)\big)}{p}, \qquad y^* = b + \frac{B\big(R - (pa + qb)\big)}{q} \tag{$**$}$$

(c) What conditions must the constants satisfy for x^* and y^* given in $(**)$ really to solve problem $(*)$? Draw a diagram that shows the domain of U and the budget line $px + qy = R$.

(d) Let $U^*(p, q, R) = U(x^*, y^*)$ where x^*, y^* are given in $(**)$. Show that $\partial U^*/\partial R > 0$.

Problem 11

In a model from economic growth theory one encounters the function f defined by

$$f(x) = \frac{1}{x} - \frac{1}{e^x - 1} \qquad \text{for alle } x > 0$$

(a) Compute $\lim\limits_{x\to 0} f(x)$ and $\lim\limits_{x\to\infty} f(x)$, and find $f'(x)$.

(b) Let $g(x) = x^2e^x - (e^x - 1)^2$. Show that $g'(x) < 0$ for all $x > 0$. (Taylor's formula for e^x may be useful.) Prove that $g(x) < 0$ for all $x > 0$, and use this result to show that f is strictly decreasing for $x > 0$.

(c) Sketch the graph of f.

Problem 12

Consider the matrices $\mathbf{T} = \begin{pmatrix} p & q & 0 \\ \frac{1}{2}p & \frac{1}{2} & \frac{1}{2}q \\ 0 & p & q \end{pmatrix}$ and $\mathbf{S} = \begin{pmatrix} p^2 & 2pq & q^2 \\ p^2 & 2pq & q^2 \\ p^2 & 2pq & q^2 \end{pmatrix}$.

(a) Compute $|\mathbf{T}|$. Suppose that $p \cdot q \neq 0$. Find a necessary and sufficient condition for $\mathbf{T}^{-1}$ to exist. Does $\mathbf{S}$ have an inverse?

Suppose from now on that $p + q = 1$.

(b) Let $\mathbf{T}$ and $\mathbf{S}$ be as given above. Show that $\mathbf{T} \cdot \mathbf{S} = \mathbf{S}$. It is easy to show (but you are not supposed to do it) that $\mathbf{T}^2 = \frac{1}{2}\mathbf{T} + \frac{1}{2}\mathbf{S}$. Show that it follows that $\mathbf{T}^3 = \frac{1}{4}\mathbf{T} + \frac{3}{4}\mathbf{S}$.

(c) Use the results in (b) to find a formula expressing $\mathbf{T}^n$ $(n = 2, 3, \dots)$ as a linear function of $\mathbf{T}$ and $\mathbf{S}$. Prove the formula by induction. Find the limit of $\mathbf{T}^n$ as $n \to \infty$.

Problem 13

Let $U(x, y)$ denote the utility of a person from x hours of leisure per day (24 hours) and y units of other commodities. The person gets an hourly wage of w and pays an average of p per unit for the other goods, so that

$$py = w(24 - x), \tag{1}$$

where we assume that the person spends all he earns.

(a) Show that Lagrange's method applied to the problem of maximizing $U(x, y)$ subject to the constraint (1), leads to the equation

$$pU_1'(x, y) = wU_2'(x, y). \tag{2}$$

(b) Suppose that equations (1) and (2) define x and y as differentiable functions of p and w. Show that with appropriate conditions on $U(x, y)$, we have

$$\frac{\partial x}{\partial w} = \frac{(24 - x)(wU_{22}'' - pU_{12}'') + pU_2'}{p^2U_{11}'' - 2pwU_{12}'' + w^2U_{22}''}.$$

(c) Find $\dfrac{\partial x}{\partial w}$ when $U(x, y) = \ln x \cdot \ln(8 + y)$, $x = 16$, $y = 8$, and $p = w = 1$.

Problem 14

Find the general solution of the differential equation

$$t\dot{x} + (2 - t)x = e^{2t}, \quad t > 0$$

Determine the particular solution with $x(1) = 0$.

Problem 15

In a problem about optimal harvesting of a fish population one needs to study the function f defined by

$$f(q) = \frac{2q\hat{z}}{2q - (p - q)^2}$$

where p and $\hat{z}$ are positive constants.

(a) For what values of q is f defined? Find $\lim\limits_{q\to\infty} f(q)$ and $\lim\limits_{q\to-\infty} f(q)$.

(b) Compute $f'(q)$ and show that f has two stationary points. Decide the character of these stationary points by studying the expression for $f'(q)$.

(c) Sketch the graph of f.

Problem 16

Let f be defined by $f(x, y) = (x + y - 2)^2 + (x^2 + y - 2)^2 - 8$ for all (x, y).

(a) Compute the first- and second-order partial derivatives of f.

(b) Find the three stationary points of f, and classify them. Prove that f has a global minimum at two of the stationary points.

(c) Let p and q be given real numbers, not both equal to 0, and let $g(t) = f(pt, qt)$. Compute $g'(t)$, and show that $g'(t) \to \infty$ as $t \to \infty$.

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Problem 17

Consider the matrices $\mathbf{C} = \begin{pmatrix} 1 & 3 & -7 \\ 2 & 5 & 1 \\ 1 & 2 & 7 \end{pmatrix}$ and $\mathbf{D} = \begin{pmatrix} a & b & c \\ -13 & 14 & -15 \\ -1 & 1 & -1 \end{pmatrix}$.

(a) Compute the determinant $|\mathbf{D}|$. Compute the matrix product $\mathbf{C} \cdot \mathbf{D}$, and show that for appropriate choices of a, b, and c, one has $\mathbf{D} = \mathbf{C}^{-1}$.

(b) Let $\mathbf{A} = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & -2 \\ 0 & 0 & -1 \end{pmatrix}$ and put $\mathbf{B} = \mathbf{C}^{-1} \cdot \mathbf{A} \cdot \mathbf{C}$. Let $\mathbf{X} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ and $\mathbf{H} = \begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix}$.
Show that there exists exactly one 3×1 matrix $\mathbf{Y}$ such that $\mathbf{A} \cdot \mathbf{Y} = \mathbf{C} \cdot \mathbf{H}$. (You need not find $\mathbf{Y}$.) Show next that $\mathbf{X} = \mathbf{C}^{-1} \cdot \mathbf{Y}$ is the solution of the equation $\mathbf{B} \cdot \mathbf{X} = \mathbf{H}$.

Problem 18

Find the limit $\lim\limits_{x \to 0} \dfrac{e^x - 1 - x}{x\sqrt{1+x} - x}$.

Problem 19

Let the function f be given by $f(x, y) = \ln(2x + y + 2) - 2x - y$.

(a) Find the first- and second-order partial derivatives of f.

(b) Determine all the stationary points of f.

(c) Sketch the set $S = \{ (x, y) : x^2 + y^2 \le 1,\ x + y \ge 0 \}$ in the xy-plane and find the maximum of the function f over this set.

Problem 20

Let f be defined by $f(x, y) = \frac{1}{2}e^{-x-y} - e^{-x} - e^{-y}$ for all $x > 0$, $y > 0$. Compute the Hessian matrix of f.

Problem 21

Evaluate the following integrals:

(a) $\displaystyle\int \left((2x-1)^2 + e^{2x-2}\right) dx$ (b) $\displaystyle\int \frac{x^2 - 2x}{x - 1}\, dx$ (c) $\displaystyle\int_0^1 \left(\int_1^2 \frac{1}{(x+y)^2}\, dx \right) dy$

Problem 22

In several economic models one studies the function U defined by

$$U(x) = -Ae^{-ax} - Be^{bx}$$

where A, B, a, and b are positive constants.

(a) Compute $U'(x)$ and show that U has a (global) maximum at $x^* = \dfrac{1}{a+b} \ln\left(\dfrac{aA}{bB}\right)$.

(b) Where is U convex/concave? Sketch the graph of U when $aA > bB$,

(c) Show that

$$U(x) = -Ae^{-ax^*}e^{-a(x-x^*)} - Be^{bx^*}e^{b(x-x^*)} = -\frac{C}{a}e^{-a(x-x^*)} - \frac{C}{b}e^{b(x-x^*)}$$

for an appropriate choice of C (x^* as given in (a)). Use this to show that the graph of U is symmetric about the line $x = x^*$ if $b = a$.

(d) Show that the quadratic approximation to $U(x)$ around x^* is

$$U(x) \approx -C\Big(\frac{1}{a} + \frac{1}{b}\Big) - \frac{1}{2}C(a+b)(x-x^*)^2.$$

Problem 23

Find the elasticity of y w.r.t. x when y is given as a function of x by

$$\ln y - a\ln x - b(\ln x)^2 - c\ln(\ln x) = 0$$

where a, b and c are constants. For what values of x is this function defined?

Problem 24

Consider the function f defined by $f(x,y) = e^{-2x-x^2-2y^2}$ for all (x,y).

(a) Find any stationary points for f and classify them.

(b) Sketch the set $S = \{\,(x,y) : x \geq 0,\ \ y \geq \dfrac{1}{1+x}\,\}$ in the xy-plane.

(c) Suppose that the problem

$$\text{maximize} \quad f(x,y) \quad \text{subject to} \quad (x,y) \in S$$

has a solution. Find the solution.

(d) Try to prove that the problem in (c) has a solution. Does the problem of minimizing $f(x,y)$ subject to $(x,y) \in S$ have a solution?

Problem 25

Let $\mathbf{A}_t = \begin{pmatrix} 1 & 0 & t \\ 2 & 1 & t \\ 0 & 1 & 1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$.

(a) For what values of t does $\mathbf{A}_t$ have an inverse? Does $\mathbf{I} - \mathbf{B}\mathbf{A}_t$ have an inverse for any value of t? ($\mathbf{I}$ is the identity matrix of order 3.)

(b) Find a matrix $\mathbf{X}$ such that $\mathbf{B} + \mathbf{X}\mathbf{A}_1^{-1} = \mathbf{A}_1^{-1}$. ($\mathbf{A}_1$ is the matrix we obtain from $\mathbf{A}_t$ when $t = 1$.)

Problem 26

Consider the function f defined by the formula $f(x) = \dfrac{1}{3}x^3\sqrt{4-x^2}$.

(a) Determine the domain of f. Compute $f(x) + f(-x)$ and give a geometric interpretation of the result.

(b) Compute $f'(x)$ and determine where f is increasing and where it is decreasing.

(c) Sketch the graph of f.

(d) Explain why the function f restricted to $[0, \sqrt{3}\,]$ has an inverse function g. Compute $g'(\frac{1}{3}\sqrt{3}\,)$. (*Hint:* $f(1) = \frac{1}{3}\sqrt{3}$.)

Problem 27

Compute $\lim_{x\to 7} \dfrac{\sqrt[3]{x+1}-\sqrt{x-3}}{x-7}$.

Problem 28

The function g is given by $g(x) = 2x - ae^{-x}(1+x^2)$, where a is a positive constant.

(a) Determine where the function g is convex.

(b) Find $\lim_{x\to\infty} g(x)$. Show that $g(x) = 0$ has exactly one solution, x_0, and that $x_0 > 0$.

(c) Show that $x_0 < a/2$. (*Hint:* Show that $g'(x) > 2$ for $x \neq 1$.)

(d) Define the function f by $f(x) = ae^{-x} + \ln(1+x^2)$. Show that the point x_0 that you found in (b) is a global minimum point of f.

(e) The point x_0 defined by the equation $g(x_0) = 0$ depends on a. Find an expression for dx_0/da.

(f) Compute $\lim_{a\to 0^+} \dfrac{x_0}{a}$.

Problem 29

Suppose that the demand for a certain commodity from a representative family depends on the good's price p and the family's income r, according to the function

$$E(p,r) = Ap^{-a}r^b \qquad (A,\ a \text{ and } b \text{ are positive constants}) \qquad (*)$$

(a) Find a constant k such that $p\dfrac{\partial E(p,r)}{\partial p} + r\dfrac{\partial E(p,r)}{\partial r} = kE(p,r)$.

In a study on the demand for milk in Norway (1925–1935), Frisch and Haavelmo found that demand could be represented by $(*)$ with $a = 1.5$ and $b = 2.08$. Verify that in this case $k = 0.58$.

(b) Show that for the function E in $(*)$ one has

$$p^2\frac{\partial^2 E(p,r)}{\partial p^2} + 2pr\frac{\partial^2 E(p,r)}{\partial p\partial r} + r^2\frac{\partial^2 E(p,r)}{\partial r^2} = (a-b)(a-b+1)E(p,r)$$

(c) Suppose that p and r are both differentiable functions of time t. Then E given in $(*)$ is a function of t alone. Find an expression for dE/dt.

Put $p(t) = p_0(1.06)^t$ and $r(t) = r_0(1.08)^t$, where p_0 is the price and r_0 is the incom at time $t = 0$. Show that in this case $dE/dt = E(p_0, r_0)\,Q^t \ln Q$, where $Q = (1.08)^b/(1.06)^a$.

(d) Find a condition on a and b that ensures that E increases as t increases.

Problem 30

Consider the problem

$$\text{maximize (minimize)} \quad x^2 + y^2 - 2x + 1 \quad \text{s.t} \quad \tfrac{1}{4}x^2 + y^2 = b \qquad (*)$$

where b is a constant $> 4/9$. (The constraint defines a closed and bounded set in the xy-plane, an ellipse.)

(a) Solve the problem by using Lagrange's method.

(b) The maximum value of $x^2 + y^2 - 2x + 1$ in problem $(*)$ will be a function $f^*(b)$ of b. Show that $df^*(b)/db = \lambda$, where λ is the Lagrange multiplier.

Problem 31

(a) Find the degree of homogeneity (if the function is homogeneous) for

(i) $f(x_1, x_2) = 5x_1^4 + 6x_1x_2^3$

(ii) $F(x_1, x_2, x_3) = e^{x_1+x_2+x_3}$

(iii) $G(K, L, M, N) = K^{a-b} \cdot L^{b-c} \cdot M^{c-d} \cdot N^{d-a}$

(b) Test Euler's theorem on the function in (i).

Problem 32

Let f be defined by the formula $f(x) = \dfrac{xe^{2x}}{x+1}, \quad x \neq -1.$

(a) Compute $f'(x)$. Does f have any local extreme points?

(b) Examine $f(x)$ as $x \to (-1)^+$, $x \to (-1)^-$, $x \to -\infty$ and $x \to \infty$.

(c) Show that f has only one inflection point, x_0, and that x_0 lies in $(-1/2, 0)$.

(d) Where is f concave? Sketch the graph of f.

Problem 33

Let the function f be defined by $f(x, y) = -\frac{1}{3}y^3 + 4y^2 - 15y + x^2 - 8x$.

(a) Sketch the set A in the xy-plane consisting of all (x, y) where $x \geq 0$, $10 \geq y \geq 0$, $x + y \geq 8$.

(b) Find the minimum value of $f(x, y)$ in the set A, taking it for granted that tere is a minimum.

Problem 34

Suppose that $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$, and $\mathbf{E}$ are $n \times n$ matrices such that $\mathbf{D}$ and $\mathbf{B} - \mathbf{C}$ have inverses. Solve the matrix equation $\mathbf{A} + \mathbf{BXD} - \mathbf{CXD} = \mathbf{E}$ for the $n \times n$ matrix $\mathbf{X}$.

Problem 35

A standard macro model leads to the equation system

$$\begin{aligned} M &= lPy + L(r) \\ S(y, r, g) &= I(y, r) \end{aligned} \qquad (*)$$

Here M, l and P are constants and L, S and I are differentiable functions.

(a) Explain why it is reasonable to assume that the system $(*)$ defines y and r as differentiable functions of g.

(b) Differentiate the system $(*)$. Find expressions for dy/dg and dr/dg.

Problem 36

In a study of a country's population one studies the function f defined by

$$f(x) = x - (\alpha + \beta)e^{-x} + \alpha e^{-2x} + \beta \quad \text{for all } x$$

where α and β are positive constants and $\alpha > \beta$.

(a) Compute $f'(x)$ and $f''(x)$.

(b) Show that f has exactly one inflection point, $\bar{x}$, and that $\bar{x} > 0$.

(c) Show that the equation $2\alpha z^2 - (\alpha + \beta)z - 1 = 0$ has exactly one positive solution. (Here z is the unknown.)

(d) Show that f has exactly one stationary point, x_0. (Use $z = e^{-x}$ as a new variable.) Show that x_0 is a global minimum point of f.

(e) Find a necessary and sufficient condition on α and β for x_0 to be positive.

Problem 37

Let $\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$.

(a) Compute $|\mathbf{A}|$, $\mathbf{A}^2$ and $\mathbf{A}^3$. Show that $\mathbf{A}^3 - 2\mathbf{A}^2 + \mathbf{A} - \mathbf{I} = \mathbf{0}$, where $\mathbf{I}$ is the identity matrix of order 3.

(b) Show that $\mathbf{A}$ has an inverse $\mathbf{A}^{-1} = (\mathbf{A} - \mathbf{I})^2$.

Problem 38

Consider the system

$$\begin{aligned} u^2 v - u &= x^3 + 2y^3 \\ e^{ux} &= vy \end{aligned}$$

(a) The system defines u and v as differentiable functions of x and y about the point $P\colon (x, y, u, v) = (0, 1, 2, 1)$. Find the differentials of u and v expressed in terms of the differentials of x and y at that point. What are $\partial u/\partial y$ and $\partial v/\partial x$ at P?

(b) If x increases by 0.1 and y decreases by -0.2 from their values at P, what are the approximate changes in u and v?

Problem 39

The function f is defined by $f(x, y) = \ln(x + y) - x^2 - y^2 + x$ for all $x > 0$, $y > 0$.

(a) Find the stationary points for f, if any.

(b) Find the (global) maximum and minimum points of f, if any.

Problem 40

Compute the following integrals:

(a) $\displaystyle\int (1 - x^2)^2 \, dx$ (b) $\displaystyle\int_{P_N}^{P_L} \left(a - bP^{1-\alpha}\right) dP \quad (\alpha \neq 2)$

Problem 41

Define $f(x,y)$ for all (x,y) by $f(x,y) = e^{x+y} + e^{x-y} - \frac{3}{2}x - \frac{1}{2}y$.

(a) Compute the partial derivatives of f of the first and second order.

(b) Show that f has a (global) minimum point.

Problem 42

Consider the matrix $\mathbf{A}_t = \begin{pmatrix} t & 1 & 1 \\ t & 2 & 1 \\ 4 & t & 2 \end{pmatrix}$.

(a) Compute $|\mathbf{A}_t|$ and determine for what values of t the matrix $\mathbf{A}_t$ has an inverse.

(b) Show that for $t = 1$, the inverse of $\mathbf{A}_t$ is $\mathbf{A}_1^{-1} = \frac{1}{2}\begin{pmatrix} -3 & 1 & 1 \\ -2 & 2 & 0 \\ 7 & -3 & -1 \end{pmatrix}$.

(c) Write the equation system

$$\begin{aligned} x + y + z &= 2 \\ x + 2y + z &= 1 \\ 4x + y + 2z &= 0 \end{aligned}$$

as a matrix equation. Use the result in (b) to solve the system.

Problem 43

Assume that production, X, depends on the number N of workers by $X = Ng\left(\frac{\varphi(N)}{N}\right)$, where g and φ are given, differentiable functions. Find expressions for $\frac{dX}{dN}$ and $\frac{d^2X}{dN^2}$.

Problem 44

Consider the function f defined by $f(x,y) = xye^{-x/y}$ for $x > 0$, $y > 0$.

(a) Compute the first-order partial derivatives of f.

(b) Compute $\mathrm{El}_x f(x,y)$ and $\mathrm{El}_y f(x,y)$ by using the rules for elasticities. (Check by using the results in (a).)

(c) Argue why f does not attain a maximum value over its domain.

(d) Find the values of x and y that maximize $f(x,y)$ subject to $x + y = c$, $x > 0$, $y > 0$, where c is a positive constant. (You can assume that the maximum value exists.)

Problem 45

Determine the values of a and b such that $\mathbf{A}$ is the inverse of $\mathbf{B}$ when

$$\mathbf{A} = \begin{pmatrix} 2 & -1 & -1 \\ a & 1/4 & b \\ 1/8 & 1/8 & -1/8 \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} 1 & 2 & 4 \\ 0 & 1 & 6 \\ 1 & 3 & 2 \end{pmatrix}$$

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Problem 46

Let t be a real number and let $\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ t & t & t \end{pmatrix}$

(a) Find $|\mathbf{A} - \mathbf{I}|$. ($\mathbf{I}$ is the identity matrix of order 3.)

(b) Put $t = 1$ and find a 3-vector $\mathbf{x}_0 = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ such that $\mathbf{A}\mathbf{x}_0 = \mathbf{x}_0$ and $\mathbf{x}_0$ has length 1.

What is $\mathbf{A}^n\mathbf{x}_0$ for $n = 1, 2, \ldots$?

Problem 47

Let $f(x) = -x^2 + x + e^{-x}$, defined in $[-3, 3]$.

(a) Compute $f'(x)$ and $f''(x)$.

(b) Where is f' (not f) increasing?

(c) Does $f'(x) = 0$ have solution(s) in $[-3, 3]$? How many?

(d) Find the maximum of f over $[-3, 3]$.

Problem 48

(a) Let $\mathbf{B}$ be an $n \times n$ matrix such that $(\mathbf{B} - \mathbf{I})^3 = \mathbf{0}$, where $\mathbf{I}$ is the identity matrix of order n. Show that the matrix $3\mathbf{I} - 3\mathbf{B} + \mathbf{B}^2$ is the inverse matrix of $\mathbf{B}$. (*Hint:* First expand $(\mathbf{B} - \mathbf{I})^3$.)

(b) Find the inverse of the matrix $\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 4 & 0 & 1 \end{pmatrix}$.

Problem 49

(a) Let a and b be positive constants. Show that

$$\int x(x^2 + a^2)^b \, dx = \frac{1}{2(b+1)}(x^2 + a^2)^{b+1} + C$$

(b) Compute the definite integral $\int_0^4 7x\sqrt{x^2 + 9}\, dx$.

Problem 50

Consider the following system of equations:

$$\begin{aligned} x + y - z &= 2 \\ kx + 3y - 2z &= 1 \\ 6x + 2ky - 3kz &= 0 \end{aligned}$$

(a) For what values of k does this system have a unique solution?

(b) Are there solutions for $k = 3$?

Problem 51

Consider the problem

$$\max f(x,y,z) = 4z - x^2 - y^2 - z^2 \quad \text{s.t.} \quad g(x,y,z) = z - xy = 0 \qquad (*)$$

(a) Use Lagrange's method to find necessary conditions for the solution of the problem.

(b) Find all triples (x,y,z) that satisfy the conditions in (a).

(c) The point $(1,1,1)$ is a maximum point in $(*)$. Find an approximate expression for the change in the maximum value of f if the constraint $z - xy = 0$ is changed to $z - xy = 0.1$.

Problem 52

(a) Let $\mathbf{A} = \begin{pmatrix} 2 & 1 & 4 \\ 0 & -1 & 3 \end{pmatrix}$. Compute $\mathbf{AA}'$, $|\mathbf{AA}'|$ and $(\mathbf{AA}')^{-1}$.

(b) The matrix $(\mathbf{AA}')^{-1}$ in (a) is symmetric. Is this a coincidence?

(c) Let $(x_{11}, x_{12}, \dots, x_{1n})$, $(x_{21}, x_{22}, \dots, x_{2n})$, $\dots$, $(x_{m1}, x_{m2}, \dots, x_{mn})$ represent m observations of n quantities, and define the matrix $\mathbf{X}$ by

$$\mathbf{X} = \begin{pmatrix} x_{11} & x_{12} & \dots & x_{1n} \\ x_{21} & x_{22} & \dots & x_{2n} \\ \vdots & \vdots & & \vdots \\ x_{m1} & x_{m2} & \dots & x_{mn} \end{pmatrix}$$

Let $\mathbf{1} = (1, 1, \dots, 1)$ be the $1 \times m$ matrix consisting of only ones. Compute the product $\frac{1}{m}\mathbf{1} \cdot \mathbf{X}$, and give an interpretation of the result.

Problem 53

(a) Compute $\int_0^2 2x^2(2-x)^2\,dx$. Give a rough check of the answer by sketching the graph of $f(x) = 2x^2(2-x)^2$ over $[0,2]$.

(b) The function $x = x(t)$ is differentiable, with $x(0) = 0$ and $\dot{x} = (1 + x^2)t$ for all t. Prove that $t = 0$ is a (global) minimum point for $x(t)$, and show that the function $x(t)$ is convex.

(c) Find the elasticity of y w.r.t. x when $x^a y^b = Ae^{x/y^2}$, where a, b and A are constants.

Problem 54

Let f be a function of two variables, given by

$$f(x,y) = x^2 - y^2 - xy - x^3 \quad \text{for all } x \text{ and } y$$

Find the stationary points of f and classify them.

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Problem 55

Let $\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 3 & 2 & 1 \end{pmatrix}$ and $\mathbf{T} = \dfrac{1}{12}\begin{pmatrix} s & t & 3 \\ 7 & -8 & 3 \\ 1 & t & -3 \end{pmatrix}$, where s and t are real numbers.

(a) Show that for suitable values of s and t, $\mathbf{T} = \mathbf{A}^{-1}$.

(b) The matrix $\mathbf{X}$ satisfies the equation $\mathbf{BX} = 2\mathbf{X} + \mathbf{C}$, where

$$\mathbf{B} = \begin{pmatrix} 3 & 2 & 3 \\ 2 & 3 & 3 \\ 3 & 2 & 3 \end{pmatrix} \quad \text{and} \quad \mathbf{C} = \begin{pmatrix} 2 & 3 & 0 & 1 \\ 1 & 0 & 3 & 1 \\ 0 & 5 & -4 & 1 \end{pmatrix}$$

Use the result from (a) to find $\mathbf{X}$.

(c) $\mathbf{D}$ is an $n \times n$ matrix such that $\mathbf{D}^2 = 2\mathbf{D} + 3\mathbf{I}_n$. Show that $\mathbf{D}^3 = a\mathbf{D} + b\mathbf{I}_n$ for suitable values of a and b. Find corresponding expressions for $\mathbf{D}^6$ and $\mathbf{D}^{-1}$ (that is, expressions of the form $\alpha\mathbf{D} + \beta\mathbf{I}_n$).

Problem 56

The equation

$$y^2 + x^2 e^{ay} = A \qquad (a \text{ and } A \text{ are positive constants}) \qquad (*)$$

represents a curve in the xy-plane.

(a) Find the curve's points of intersection with the coordinate axes.

(b) Find the slope of the tangent to the curve at an arbitrary point (x, y) on the curve.

(c) Let p and q be constants not both 0, and consider the problem

$$\text{maximize } px + qy \qquad \text{subject to} \qquad y^2 + x^2 e^{ay} = A \qquad (**)$$

Write down the necessary conditions for the solution of $(**)$, and show that if (x, y) solves the problem, then $2qxe^{ay} = 2py + pa(A - y^2)$.

(d) At what point (x, y) on the curve given by $(*)$ does x attain its largest value?

Problem 57

Consider the function $f(x, y) = y^3 + 3x^2 y$.

(a) Determine the degree of homogeneity of f and determine a constant k such that

$$xf_1'(x, y) + yf_2'(x, y) = kf(x, y) \quad \text{for all} \quad (x, y)$$

(b) Find the slope of the tangent line to the level curve $y^3 + 3x^2 y = -13$ at an arbitrary point on the curve, and find in particular the equation of the tangent at the point $(2, -1)$.

(c) Examine whether the level curve in (b) is convex or concave around the point $(2, -1)$ by computing y'' at this point.

(d) Show that no point on the level curve in (b) lies above the x-axis. Find the smallest y-coordinate of a point on the curve.

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Problem 58

(a) Compute the determinant $\begin{vmatrix} -2 & 4 & -t \\ -3 & 1 & t \\ t-2 & -7 & 4 \end{vmatrix}$.

(b) For which values of t does the equation system

$$\begin{aligned} -2x + 4y - tz &= t - 4 \\ -3x + y + tz &= 3 - 4t \\ (t-2)x - 7y + 4z &= 23 \end{aligned}$$

have a unique solution?

(c) Show that for $t = 8$ the system of equations in (b) will have a solution with $y = 3$.

(d) Let $\mathbf{B}$ be an $n \times n$ matrix such that $\mathbf{B}^2 = 3\mathbf{B}$. Show that there exists a number s such that the matrix $\mathbf{I}_n + s\mathbf{B}$ is the inverse of the matrix $\mathbf{I}_n + \mathbf{B}$. ($\mathbf{I}_n$ is the identity matrix of order n.)

Problem 59

(a) Let $\mathbf{A} = \begin{pmatrix} 0 & 0 & 0 \\ 4 & 0 & 0 \\ 10 & 5 & 0 \end{pmatrix}$. Compute $\mathbf{A}^2$, $\mathbf{I}_3 + \mathbf{A} + \mathbf{A}^2$ and $(\mathbf{I}_3 - \mathbf{A})(\mathbf{I}_3 + \mathbf{A} + \mathbf{A}^2)$, where $\mathbf{I}_3$ is the identity matrix of order 3.

(b) Compute $(\mathbf{I}_3 - \mathbf{A})^{-1}$ by using the results in (a).

(c) Let $\mathbf{U}$ be the $n \times n$ matrix all of whose elements are 1. Show that

$$(\mathbf{I}_n + a\mathbf{U})(\mathbf{I}_n + b\mathbf{U}) = \mathbf{I}_n + (a + b + nab)\mathbf{U}$$

for all real numbers a and b, where $\mathbf{I}_n$ is the identity matrix of order n.

(d) Use the result in (c) to find the inverse of $\begin{pmatrix} 4 & 3 & 3 \\ 3 & 4 & 3 \\ 3 & 3 & 4 \end{pmatrix}$.

Problem 60

Let $U(x, y)$ be defined for all $x > 0$, $y > 0$ by

$$U(x, y) = A\big[\ln(x^\alpha + y^\alpha) - \ln y^\alpha\big] \qquad (A \text{ and } \alpha \text{ are positive constants})$$

(a) Compute $U_1'(x, y)$, $U_2'(x, y)$ and $U_{12}''(x, y)$.

(b) Is $U(x, y)$ a homogeneous function?

(c) We assume that $U(x, y)$ is a utility function for a society, with x denoting the economic activity level and y the level of pollution. We also assume that the level of pollution y depends on the activity level by the equation

$$y^3 - ax^4 - b = 0 \qquad (a \text{ and } b \text{ positive constants}) \qquad (*)$$

Use Lagrange's method to find the activity level that maximizes the utility function $U(x, y)$ subject to the constraint $(*)$. (You can take it for granted that the maximum exists.)

Problem 61

The equation system

$$\ln(x+u) + uv - y^2 e^v + y = 0$$
$$u^2 - x^v = v$$

defines u and v as C^1 functions of x and y around the point $P : (x, y, u, v) = (2, 1, -1, 0)$.

(a) Differentiate the system.

(b) Find the values of the partial derivatives u'_x, u'_y, v'_x and v'_y at P.

(c) Find an approximate value of $u(1.99,\ 1.02)$.

Problem 62

Let $\mathbf{A}_a = \begin{pmatrix} 1 & -a & -a \\ -a & 1 & -a \\ -a & -a & 1 \end{pmatrix}$ for all reral numbers a.

(a) Compute the determinant $|\mathbf{A}_a|$, and show that $\mathbf{A}_a$ has an inverse if $a \neq -1$ and $a \neq 1/2$.

(b) Show that the inverse of $\mathbf{A}_a$ (when it exists) is

$$\mathbf{A}_a^{-1} = k \begin{pmatrix} 1-a & a & a \\ a & 1-a & a \\ a & a & 1-a \end{pmatrix}$$

where k is a number that depends on a.

(c) Show that if $0 < a < 1/2$ and $\mathbf{x}$ is a 3-vector with only positive components, then $\mathbf{A}_a^{-1}\mathbf{x}$ will also be a vector with only positive components.

Problem 63

(a) The equation

$$3xe^{xy^2} - 2y = 3x^2 + y^2$$

defines y as a differentiable function of x around the point $(x^*, y^*) = (1, 0)$. Find the slope of the graph at this point by implicit differentiation. What is the linear approximation to y around $x^* = 1$?

(b) In an equilibrium model the following system of equations is studied:

$$\begin{aligned} pF'(L) - r &= 0 \\ pF(L) - rL - B &= 0 \end{aligned} \qquad (*)$$

where F is a twice differentiable function with $F'(L) > 0$ and $F''(L) < 0$. All the variables are positive. Consider r and B as exogenous and p and L as endogenous variables, so that p and L are functions of r and B. Find expressions for $\partial p/\partial r$, $\partial p/\partial B$, $\partial L/\partial r$, and $\partial L/\partial B$ by implicit differentiation.

(c) Determine, if possible, the signs of these partial derivatives. Show, in particular, that $\partial L/\partial r < 0$.

Problem 64

Let $f(x, y)$ be defined by $f(x, y) = \ln(x + y) + y$ for all (x, y) with $x + y > 0$. Find the maximum of $f(x, y)$ subject to the constraint $x^2 + 2xy + 2y^2 = 2$. (You can take it for granted that the maximum exists.)

Problem 65

(a) Compute the integral $\int_1^4 e^{-\sqrt{t}}\, dt$.

(b) Let $\mathbf{A}$, $\mathbf{C}$, $\mathbf{D}$, $\mathbf{X}$, and $\mathbf{Y}$ be $n \times n$ matrices that satisfy the equations

$$\mathbf{AX} + \mathbf{Y} = \mathbf{C}$$
$$\mathbf{X} + 2\mathbf{A}^{-1}\mathbf{Y} = \mathbf{D}$$

(We assume that $\mathbf{A}$ has an inverse.) Find $\mathbf{X}$ and $\mathbf{Y}$ expressed in terms of $\mathbf{A}$, $\mathbf{C}$, and $\mathbf{D}$.

Problem 66

For what values of a will the equation system

$$\begin{aligned} x + \ \ y - 2z &= 7 + a \\ 3x - \ \ y + az &= -3 \\ -x + ay - 4z &= \ \ 8 \end{aligned}$$

have (i) exactly one solution, (ii) more than one solution, (iii) no solution?

Problem 67

(a) Find the elasticity of y w.r.t. x when $y^2 e^{x+1/y} = 3$.

(b) The following system of equations defines $u = u(x, y)$ and $v = v(x, y)$ as C^1 functions of x and y around the point $P = (x, y, u, v) = (1, 1, 1, 2)$:

$$u^\alpha + v^\beta = 2^\beta x + y^3$$
$$u^\alpha v^\beta - v^\beta = x - y$$

Here α and β are positive constants. Differentiate the system. Then find the values of $\partial u/\partial x$, $\partial u/\partial y$, $\partial v/\partial x$ and $\partial v/\partial y$ at the point P.

(c) Show that for the function $u(x, y)$ in (b) we have $u(0.99, 1.01) \approx 1 - \dfrac{2^{1-\beta}}{100\alpha}$.

Problem 68

Consider the function h given by $h(x) = \dfrac{e^x}{2 + e^{2x}}$ for all x.

(a) Determine where h is increasing and where it is decreasing. Find the maximum and minimum points for h, if any.

(b) Why must h restricted to $(-\infty, 0)$ have an inverse function? Find a formula for the inverse.

(c) Let $f(x) = \dfrac{g(x)}{2 + (g(x))^2}$, where g is a differentiable function with $g'(x) > 0$ for all x. Does f always have a maximum point?

Problem 69

(a) Compute the determinant of $\mathbf{A}_t = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ 3 & 1 & t \end{pmatrix}$.

(b) Solve the following equation system by Gaussian elimination

$$\begin{aligned} x - y + z &= 2 \\ x + y - z &= 1 \\ 3x + y - z &= 4 \end{aligned}$$

Problem 70

Let the function f be given by $f(x, y) = (x^2 + y^2)(xy + 1)$ for all x and y.

(a) Compute the partial derivatives of f of the first and the second order.

(b) Show that $(0, 0)$, $(\frac{1}{2}\sqrt{2}, -\frac{1}{2}\sqrt{2})$, and $(-\frac{1}{2}\sqrt{2}, \frac{1}{2}\sqrt{2})$ are stationary points of f and classify them. Prove that f has no other stationary points.

(c) Find the maximum of $f(x, y)$ over the set $S = \{(x, y) : x^2 + y^2 \leq a^2\}$, where a is a positive constant.

Problem 71

(a) Sketch the curve $y = \dfrac{4\sqrt{x}}{2 + \sqrt{x}}$, and find the area of the domain bounded by this curve, the x-axis, and the line $x = 4$.

(b) Let a be a positive constant. Find $\displaystyle\lim_{x \to a} \frac{a^x - x^a}{x - a}$.

Problem 72

(a) Let $\mathbf{A} = \begin{pmatrix} a & 1 & 4 \\ 2 & 1 & a^2 \\ 1 & 0 & -3 \end{pmatrix}$. Compute $|\mathbf{A}|$.

(b) For what values of a does the equation system

$$\begin{aligned} ax + y + \ \ 4z &= 2 \\ 2x + y + a^2 z &= 2 \\ x \quad \ \ - \ \ 3z &= a \end{aligned} \qquad (*)$$

have one, none, or infinitely many solutions, respectively? (You are not required to find the solutions.)

(c) Replace the right-hand sides 2, 2, and a in $(*)$ by b_1, b_2, and b_3. Find a necessary and sufficient condition for the new system of equations to have infinitely many solutions.

(d) A 3×3 matrix $\mathbf{B}$ satisfies the equation $\mathbf{B}^3 = -\mathbf{B}$. Show that $\mathbf{B}$ cannot have an inverse.

Problem 73

Find the maximum of $\quad x^2 + y^2 + z^2 \quad$ subject to $\begin{cases} x^2 + y^2 + 4z^2 = 1 \\ x + 3y + 2z = 0 \end{cases}$

Problem 74

A firm produces and sells a good. The cost of producing and selling x units and using y dollars on advertising is $C = cx + y + d$. The demand is given by

$$x = -ap + b + R(y)$$

where p is the price obtained per unit. We assume that $R(0) = 0$, $R'(y) > 0$ and $R''(y) < 0$. The constants a, b, c, and d are all positive.

(a) Show that the profit $\pi(x, y)$ from selling x units and using y dollars on advertisement, is given by

$$\pi(x, y) = -\frac{1}{a}x^2 + \frac{b}{a}x + \frac{1}{a}R(y)x - cx - y - d$$

(b) Show that if $x^* > 0$ and $y^* > 0$ maximize profits, then y^* satisfies the equation

$$(b - ac)R'(y^*) + R(y^*)R'(y^*) = 2a \qquad (*)$$

(c) Equation $(*)$ defines y^* implicitly as function of a, b, and c. Compute $\partial y^*/\partial b$ by implicit differentiation.

(d) Put $R(y) = \alpha y^{1/2}$, where $\alpha > 0$. Find explicit expressions for y^* and x^* in this case.

Problem 75

A firm has a monopoly on the sale of a certain type of vacuum cleaners and has the demand function $p = a - bx$, where p is the price per unit and x is the number of items sold per year. The firm has fixed expenses of r per unit to cover raw material costs, and annual running expenses d for administration, maintenance of buildings, and necessary mechanical equipment.

The firm wishes to automate the production to the extent that it is profitable. The investment in special machines for this purpose is y. Assume that ky is the annual running expenses of depreciation and maintenance of the special machines, and $f(y)$ is the workers' salary per produced vacuum cleaner. Here f is a given C^2 function with $f'(y) < 0$ and $f''(y) > 0$. The constants a, b, r, d, and k are all positive.

(a) Comment on the signs of $f'(y)$ and $f''(y)$. Find the firm's annual net profit $\pi(x, y)$ and compute the partial derivatives of $\pi(x, y)$ of the first and second order.

(b) Show that if $x > 0$, $y > 0$ maximize net profits, then y satisfies the equation

$$2bk + f'(y)(a - r) = f(y)f'(y) \qquad (*)$$

(c) Suppose that $f(y) = \alpha/(y + \beta)$, with $\alpha > 0$ and $\beta > 0$. Show that $(*)$ reduces to a cubic equation in $y + \beta$.

(d) Equation $(*)$ in part (b) defines y implicitly as a function of k. Find an expression for dy/dk.

(e) Suppose that the sufficient second-order conditions for a local maximum of $\pi(x, y)$ are satisfied. Show that then $dy/dk < 0$.

Problem 76

Let $\mathbf{A}_t = \begin{pmatrix} 1 & t & 0 \\ -2 & -2 & -1 \\ 0 & 1 & t \end{pmatrix}$.

(a) Compute $|\mathbf{A}_t|$ and show that $\mathbf{A}_t^{-1}$ exists for every t.

(b) Show that for a certain value of t, $\mathbf{A}_t^3 = \mathbf{I}_3$, where $\mathbf{I}_3$ is identity matrix of order 3, and then find the inverse of $\mathbf{A}_1$.

(c) Suppose that $\mathbf{A}$ and $\mathbf{B}$ are invertible $n \times n$ matrices. Show that if $\mathbf{A}'\mathbf{A} = \mathbf{I}_n$, then $(\mathbf{A}'\mathbf{B}\mathbf{A})^{-1} = \mathbf{A}'\mathbf{B}^{-1}\mathbf{A}$.

Problem 77

Compute the integrals: (i) $\displaystyle\int_{-1}^{6} x(2+x)^{1/3}\,dx$ (ii) $\displaystyle\int e^{\sqrt[3]{x}}\,dx$

Problem 78

Solve the problem

$$\text{minimize } Ax + e^{ax} + e^{by} \quad \text{subject to} \quad e^{ax} + e^{ax+by} = c$$

where A, a, b, and c are positive constants.

Problem 79

The equation system

$$\begin{aligned} e^{x-y}\ln(x+z-1) &= \sqrt{xy} \\ x^2y^3z &= e \end{aligned}$$

defines y and z as differentiable functions of x in a neighborhood of the point $(x, y, z) = (1, 1, e)$.

(a) Find the elasticities of y and z w.r.t. x at the given point.

(b) What are the approximate percentage changes of y and z if x increases from 1 to 1.1?

Problem 80

Let the function g be defined by

$$g(x) = (a-1)x + c^a x^{1-a} - a \quad \text{for } x > 0$$

Here a and c are constants with $a > 1$ and $0 < c < 1$.

(a) Compute $g'(x)$ and $g''(x)$, and examine $g(x)$ as $x \to 0^+$ and as $x \to \infty$.

(b) Show that g has a (global) minimum point, and find the minimum value.

(c) Show that the function g has exactly 2 zeros, and that one of them lies in the interval $(0, c)$ and the other in the interval $(1, \dfrac{a}{a-1})$.

Problem 81

Compute the integrals $\displaystyle\int \frac{dv}{1-v^2}$ and $\displaystyle\int \frac{dx}{\sqrt{1-e^{-x}}}$.

(*Hint:* $\dfrac{2}{1-v^2} = \dfrac{1}{1+v} + \dfrac{1}{1-v}$.)

Problem 82

Let $\mathbf{A}_a = \begin{pmatrix} 1 & a & a \\ a & 2 & a \\ a & a & 3 \end{pmatrix}$, where a is a constant.

(a) Compute the determinant $|\mathbf{A}_a|$ and the matrix $\mathbf{A}_a^2$.

(b) Find a necessary and sufficient condition for the equation system

$$\begin{aligned} x_1 + ax_2 + ax_3 &= 1 \\ ax_1 + 2x_2 + ax_3 &= 2 \\ ax_1 + ax_2 + 3x_3 &= 3 \end{aligned}$$

to have a unique solution. Find the solution when $a = 3$.

(c) Solve the equation system

$$\begin{aligned} x_1 + nx_2 + \cdots + \qquad nx_{n-1} + nx_n &= 1 \\ nx_1 + 2x_2 + \cdots + \qquad nx_{n-1} + nx_n &= 2 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ nx_1 + nx_2 + \cdots + (n-1)x_{n-1} + nx_n &= n-1 \\ nx_1 + nx_2 + \cdots + \qquad nx_{n-1} + nx_n &= n \end{aligned}$$

where n is an integer > 1.

(d) Show that if $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ are $n \times n$ matrices such that $\mathbf{BAC} = \mathbf{I}_n$, then $\mathbf{A}$ has an inverse. Find an expression for $\mathbf{A}^{-1}$.

Problem 83

Consider the problem

$$\text{maximize} \quad U(x_1, x_2, x_3) \quad \text{subject to} \quad p_1x_1 + p_2x_2 + p_3x_3 = m$$

where $U(x_1, x_2, x_3) = \ln(x_1 - 6) + 2\ln(x_2 - 5) + \ln(x_3 - 4)$, and p_1, p_2, p_3 and m are positive constants.

(a) Solve the problem by using Lagrange's method. (You may assume that the problem has a solution.)

(b) Let U^* be the optimal value function for the problem, which means that $U^* = U^*(p_1, p_2, p_3, m)$ is the maximum value of $U(x_1, x_2, x_3)$ as a function of p_1, p_2, p_3, and m. Show by direct computation that $\partial U^*/\partial m = \lambda$, where λ is the Lagrange multiplier from (a).

(c) Put $p_1 = 5$, $p_2 = 2$, $p_3 = 5$ and $m = 100$. Estimate the change in the value of U^* if $m = 100$ is changed to $m = 101$.

Problem 84

In a growth model production Q is a function of capital K and labour L. Suppose that

(i) $\dot{K} = \gamma Q$ (investment is proportional to production)

(ii) $Q = K^\alpha L$

(iii) $\dot{L} = \beta$ (the rate of change of L with respect to t is constant)

Here γ, α, and β are positive constants, $\alpha < 1$.

(a) Derive a differential equation to determine K.

(b) Solve this equation when $K(0) = K_0$ and $L(0) = L_0$.

Problem 85

Find the local and (global) extreme points of $f(x) = e^{x^2} + e^{2-x^2}$, if any.

Problem 86

Let $f(x, y, z) = x^2 + x + y^2 + z^2$.

(a) Find the maximum and minimum of $f(x, y, z)$ subject to $x^2 + 2y^2 + 2z^2 = 16$.

(b) Find the maximum and minimum of $f(x, y, z)$ over the set

$$S = \{ (x, y, z) : x^2 + 2y^2 + 2z^2 \leq 16 \}.$$

Problem 87

Let $\mathbf{A}_a = \begin{pmatrix} a+1 & a+1 & 0 \\ 4 & a+4 & a-1 \\ 3 & 5 & a-1 \end{pmatrix}$ for all real numbers a.

(a) Compute $|\mathbf{A}_a|$.

(b) When does the equation system

$$\begin{aligned} (a+1)x + (a+1)y \quad\quad\quad &= b \\ 4x + (a+4)y + (a-1)z &= 1 \\ 3x + \quad 5y + (a-1)z &= -3 \end{aligned}$$

have a unique solution? (You need not find the solution.) Examine what conditions b must satisfy for the system not to have any solution when $a = 1$ and when $a = 2$.

(c) Compute $|3\mathbf{A}_3|$ and $|\mathbf{A}_5\mathbf{A}_4^{-1}\mathbf{A}_3^2|$.

Problem 88

(a) Find the integrals: (i) $\int 3xe^{-x/2}\,dx$ (ii) $\int_0^{25} \frac{1}{9+\sqrt{x}}\,dx$ (iii) $\int_2^7 t\sqrt{t+2}\,dt$

(b) In auction theory one encounters the differential equation

$$\dot{x} = \frac{4(a-t)}{(2t-a)^2}x, \qquad t > a/2$$

where a is a constant. Find the general solution of this equation.

Problem 89

(a) The equation

$$x^3 \ln x + y^3 \ln y = 2z^3 \ln z$$

defines z as a differentiable function of x and y in a neighborhood of the point $(x, y, z) = (e, e, e)$. Compute $z_1'(e, e)$ and $z_{11}''(e, e)$.

(b) If F is a differentiable function of one variable with $F(0) = 0$ and $F'(0) \neq -1$, find an expression for y' at the point $(x, y) = (1, 0)$, when y is defined implicitly as a differentiable function of x by the equation

$$x^3F(xy) + e^{xy} = x.$$

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Problem 90

Let $a \geq 0$ be a constant and let $f(x) = (2x^2 + a)e^{-x^2-a}$.

(a) Compute $f'(x)$ and find all stationary points of f. (You must distinguish between the cases $0 \leq a < 2$ and $a \geq 2$.)

(b) Show that the graph of f is symmetric about the y-axis. Determine the limits $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$.

(c) For each $a \geq 0$, $f(x)$ has a maximum value, $M(a)$. Show that

$$M(a) = \begin{cases} 2e^{-1-\frac{1}{2}a} & \text{if } 0 \leq a \leq 2 \\ ae^{-a} & \text{if } a > 2 \end{cases}$$

For what value of a does $M(a)$ have its largest value?

(d) Consider the function g defined by $g(x,y) = (2x^2 + y)e^{-x^2-y}$ for all (x,y). Find the stationary points of g, if any. Find the largest value attained by the function over the set $\{(x,y) : y \geq 0\}$. (You may assume that there is a maximum value.)

Problem 91

(a) Find the integral $\displaystyle\int_2^\infty \frac{12x+6}{(x^2+x+2)^{4/3}}\,dx$.

(b) In auction theory one encounters the differential equation

$$(r_2 - r_1)\dot{x} = \left(\frac{r_1}{t-r_2} - \frac{r_2}{t-r_1}\right)x, \quad t > r_2$$

where r_1 and r_2 are constants with $r_2 > r_1$. Find the general solution of this equation. Show that it can be written in the form

$$x = C(t-r_2)^{r_1/(r_2-r_1)}(t-r_1)^{-r_2/(r_2-r_1)} \quad (C \text{ is a constant})$$

Problem 92

(a) Find the slope of the curve $xe^{x^2y} + 3x^2 = 2y + 4$ at the point $(x,y) = (1,0)$.

(b) Consider the equation system

$$\begin{aligned} xe^y + yf(z) &= a \\ xg(x,y) + z^2 &= b \end{aligned}$$

where $f(z)$ and $g(x,y)$ are differentiable functions and a and b are constants. Suppose that the system defines x and y as differentiable functions of z. Find expression for dx/dz and dy/dz.

Problem 93

(a) Consider the equation system

$$\begin{aligned} x_1 + x_2 + x_3 &= b \\ ax_1 + x_2 - x_3 &= 5 \\ x_1 \qquad\; - x_3 &= a \end{aligned}$$

where a and b are given constants. For what values of a is there a unique solution?

(b) Suppose that $\mathbf{A}$ is an $n \times n$ matrix, $\mathbf{B}$ an $m \times m$ matrix, and $\mathbf{C}$ an $n \times m$ matrix. Suppose that $\mathbf{A} - \mathbf{I}$ and $\mathbf{B}$ have inverses. Find a formula for the matrix $\mathbf{X}$ that satisfies the matrix equation $\mathbf{AXB} = \mathbf{XB} + \mathbf{C}$. In particular, let

$$\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/4 \end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix} 1 & 1 & 2 \\ -1 & 3 & 1 \end{pmatrix}$$

and find the matrices $(\mathbf{A} - \mathbf{I})^{-1}$, $\mathbf{B}^{-1}$, and $\mathbf{X}$ in this case.

Problem 94

Find the following integrals: (a) $\displaystyle\int \frac{1}{(u-1)\sqrt{u}}\,du$ (b) $\displaystyle\int \frac{1}{\sqrt{e^y+1}}\,dy$

(*Hint:* You may need the formula $\displaystyle\frac{1}{z^2-1} = \frac{1}{2}\left[\frac{1}{z-1} - \frac{1}{z+1}\right]$.)

Problem 95

Consider the matrices

$$\mathbf{A}_3(t) = \begin{pmatrix} 3-t & -4 & 2 \\ 1 & -t & 0 \\ 0 & 1 & -t \end{pmatrix} \quad \text{and} \quad \mathbf{A}_4(t,a) = \begin{pmatrix} 3-t & -4 & 2 & a \\ 1 & -t & 0 & 0 \\ 0 & 1 & -t & 0 \\ 0 & 0 & 1 & -t \end{pmatrix}$$

(a) Compute the determinants $|\mathbf{A}_3(t)|$ and $|\mathbf{A}_4(t,a)|$.

(b) Find a necessary and sufficient condition on b_1, b_2, and b_3 for the equation system

$$\begin{aligned} 2x_1 - 4x_2 + 2x_3 &= b_1 \\ x_1 - x_2 \phantom{{}+2x_3} &= b_2 \\ x_2 - x_3 &= b_3 \end{aligned}$$

to have solutions, and determine the number of degrees of freedom in that case.

(c) Suppose that the matrix $\mathbf{P}$ has an inverse. Which of the following matrices will then also have inverses? (Give the argument.)

(i) $\mathbf{P}^2$ (ii) $\mathbf{P} + \mathbf{P}$ (iii) $\mathbf{P}'$ (iv) $\mathbf{P} + \mathbf{P}'$

Problem 96

(a) For $x \geq 0$, $y \geq 0$, the equation

$$xy + y^2 + 2x + 2y = C \qquad (C \text{ is a constant})$$

defines y as a C^2 function of x. Compute y' and y''.

(b) A consumer uses an amount m to buy x units of one good at the price 6 kr. per unit and y units of a different good at the price 10 kr. per unit. Here m is positive. The consumer's utility function is $U(x,y) = xy + y^2 + 2x + 2y$, so that her problem is:

$$\text{maximize } (xy + y^2 + 2x + 2y) \text{ subject to } 6x + 10y = m$$

Suppose that $8 < m < 40$. Find the optimal quantities x^* and y^* and the Lagrange multiplier as functions of m.

(c) The maximum value of the utility function is a function of m. Find its derivative for $m = 20$.

(d) What are the solutions for x^* and y^* if $m \leq 8$? What are the solutions if $m \geq 40$?

Problem 97

Let the function φ be defined by $\varphi(x) = \ln(x+1) - \ln(x+2) = \ln\left[\dfrac{x+1}{x+2}\right]$ for all $x \geq 0$.

(a) Find the range of φ.

(b) Find the inverse of φ. Where is it defined?

(c) Find the inverse of φ'. Where is it defined?

Problem 98

Consider the matrices $\mathbf{A}_t = \begin{pmatrix} 1 & 0 & t \\ 2 & 1 & t \\ 0 & 1 & 1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$.

(a) For what values of t does $\mathbf{A}_t$ have an inverse?

(b) Compute $\mathbf{I}_3 - \mathbf{B}\mathbf{A}_t$. For which values of t does this matrix have an inverse? Find a matrix $\mathbf{X}$ such that $\mathbf{B} + \mathbf{X}\mathbf{A}_t^{-1} = \mathbf{A}_t^{-1}$ for $t = 1$.

(c) Find the matrix $\mathbf{Y}$ that satisfies

$$\mathbf{Y}\begin{pmatrix} 1 & 2 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 2 & -1 \\ -1 & 0 & 4 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Problem 99

Suppose that c is a constant and that the equation

$$1 + xy - \ln(e^{xy} + e^{-xy}) = c$$

defines y as a differentiable function of x. Compute dy/dx and d^2y/dx^2.

Problem 100

(a) Find the integral $\displaystyle\int \frac{(x^n - x^m)^2}{\sqrt{x}}\,dx$, where m and n are natural numbers.

(b) Compute $\displaystyle\int_0^{1/3} \frac{dx}{e^x + 1}$.

(c) Solve the differential equation $\dot{x} = \dfrac{\sqrt[3]{ax+b}}{x}\,t^2$, where the constant a is $\neq 0$.

Problem 101

A statistical problem involves the function f defined for all x and y by

$$f(x,y) = 2(1-\rho^2)x^2y^2 - 3x^2 - 3y^2 + 2\rho xy + 4$$

where ρ is a constant in $[-1, 1]$.

(a) Compute the Hessian of f. Show that for $\rho = \pm 1$, f has only one stationary point. Is this point a global or local extreme point or a saddle point for f?

(b) Show that for all ρ in $[-1, 1]$, if (x_0, y_0) is a stationary point, then $x_0^2 = y_0^2$. (*Hint:* Consider $xf_1'(x,y) - yf_2'(x,y)$.)

(c) Find all stationary points of f when $\rho \in (-1, 1)$.

Problem 102

(a) Compute the determinant of $\mathbf{A} = \begin{pmatrix} a & a-1 & a \\ a-1 & 1 & 0 \\ a & 0 & a \end{pmatrix}$.

(b) For what values of a and b will the equation system $\mathbf{A}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} b \\ 0 \\ 1 \end{pmatrix}$ have infinitely many solutions?

(c) For what values of a does there exist a matrix $\mathbf{B}$ such that $\mathbf{AB} = \mathbf{A} + \mathbf{B}$?

Problem 103

The number of liters of petrol in the tank of a car after it has driven x miles is $V(x)$. Suppose that $V(x)$ satisfies the differential equation $V'(x) = -aV(x) - b$, where a and b are positive constants.

(a) Find the general solution of the equation.

(b) Assume that $a = 0.1$ and $b = 0.7$. How many miles can the car travel if it sets out with 20 litres in the tank? What is the minimum number of litres needed at the outset if the car is to run for 15 miles?

Problem 104

Calculate (i) $\displaystyle\lim_{x\to 3}\left(\frac{1}{x-3} - \frac{5}{x^2-x-6}\right)$ (ii) $\displaystyle\int_{8.5}^{41} \frac{dx}{\sqrt{2x-1} - \sqrt[4]{2x-1}}$.

(*Hint:* substitute $z^4 = 2x - 1$ in (ii).)

Problem 105

The equation

$$x^2y^3 + (y+1)e^{-x} = x + 2 \qquad (*)$$

defines y as a differentiable function of x around $(x, y) = (0, 1)$.

(a) Compute y' at this point.

(b) Show that the curve given by $(*)$ intersects the x-axis in exactly one place.

Problem 106

Consider the Lagrange problem

$$\text{maximize} \quad xyz \quad \text{subject to} \quad \begin{cases} x + y + z = 5 \\ xy + xz + yz = 8 \end{cases}$$

(a) Write down the necessary first-order conditions for a solution of the problem.

(b) Show that if the point (x, y, z) satisfies the first-order conditions, then

$$\begin{aligned} z(y-x) &= \mu(y-x) \\ y(x-z) &= \mu(x-z) \\ x(z-y) &= \mu(z-y) \end{aligned}$$

where μ is the Lagrange multiplier associated with the constraint $xy + xz + yz = 8$.

(c) Solve the problem. You may assume that the problem has a solution.

(d) What will be the approximate change in the maximum value of xyz if 5 is changed to 5.01 and 8 is changed to 7.99?

Problem 107

Let f be defined by $f(x) = \ln(2 + e^{x-3})$ for all x.

(a) Show that f is strictly increasing and find the range of f.

(b) Find an expression for the inverse function g of f. Where is g defined?

(c) Verify that $f'(3) = 1/g'(f(3))$.

Problem 108

Consider the 3×3 matrices $\mathbf{A} = \begin{pmatrix} q & -1 & q-2 \\ 1 & -p & 2-p \\ 2 & -1 & 0 \end{pmatrix}$ and $\mathbf{E} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$.

(a) Compute $|\mathbf{A}|$, $\mathbf{AE}$ and $|\mathbf{A} + \mathbf{E}|$.

(b) For what values of p and q does $\mathbf{A} + \mathbf{E}$ have an inverse? Explain why $\mathbf{BE}$ does not have an inverse for any 3×3 matrix $\mathbf{B}$.

(c) Consider the equation system

$$\begin{aligned} qx - \quad y &= q - 2 \\ x - py &= 2 - p \\ 2x - \quad y &= 0 \end{aligned}$$

where x and y are the unknowns and p and q are parameters. For what values of p and q does the system have a unique solution, no solutions, or an infinite number of solutions?

Problem 109

(a) Find the integral $\displaystyle\int_0^1 57x^2 \sqrt[3]{19x^3 + 8}\, dx$.

(b) Solve the differential equation $e^{3t}\dot{x} = \dfrac{x^3 + 1}{x^2}$ $(x > 0)$, with $x(0) = 1$.

Problem 110

(a) Find the maximum and the minimum of $x^2 + y^2 + z$ subject to $x^2 + 2y^2 + 4z^2 = 1$.

(b) Suppose that the constraint is changed to $x^2 + 2y^2 + 4z^2 = 1.02$. What is approximately the change in the maximum value of $f(x, y, z)$?

Problem 111

Find the solution of the differential equation $3x^2\dot{x} = (x^3+9)^{3/2}\ln t$ whose solution curve passes through the point $(t, x) = (1, 3)$.

Problem 112

The revenue from an oil field today ($t = 0$) is 1 billion per year and is expected to increase linearly to 5 billion per year in 10 years. If we measure time in years and let $f(t)$ denote the revenue (measured in billions) per unit time at time t, then $f(t) = 1 + 0.4t$. If $F(t)$ denotes total revenue accumulated during the time interval $[0, t]$, then $F'(t) = f(t)$.

(a) Find the total revenue over the 10 year period (i.e. $F(10)$).

(b) Find the present value of the revenue over the time interval $[0, 10]$ with continuously compounded interest at interest rate $r = 0.05$ per year. (The present value is $\int_0^T f(t)e^{-rt}\,dt$, where T denotes the terminal time.)

Problem 113

For $x > 0$ and $y > 0$, the equation $\dfrac{y^3}{x^3} = (x+a)^p(y+b)^q$ defines y as a differentiable function of x. Find the elasticity of y w.r.t. x. Here $a > 0$, $b > 0$, p, and q are constants.

Problem 114

(a) Let $\mathbf{A} = \begin{pmatrix} 4 & 3 & -2 \\ 13 & 7 & -3 \\ 1 & -2 & 3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$. What conditions must b_1, b_2, and b_3 satisfy for the equation system $\mathbf{Ax} = \mathbf{b}$ to have solutions? How many degrees of freedom are there if the system has solutions?

(b) Is there an invertible matrix $\mathbf{B}$ such that $\mathbf{BA} = \mathbf{0}$?

(c) Find a 3×3 matrix $\mathbf{C}$ such that $\mathbf{C} \neq \mathbf{0}$ and $\mathbf{AC} = \mathbf{0}$.

Problem 115

Consider the problem

$$\text{maximize} \quad f(x,y,z) = xy + e^z \quad \text{s.t.} \quad g(x,y,z) = e^{2z} + x^2 + 4y^2 = 6 \qquad (*)$$

(a) Find all solutions of the first-order conditions, and determine the maximum point in problem $(*)$. You may assume that a maximum point exists.

(b) Estimate the change in the optimal value of f if we change the constraint to $e^{2z} + x^2 + 4y^2 = 6.1$.

(c) If we change the constraint in problem $(*)$ to $e^{2z}+x^2+4y^2 \leq 6$, will the new problem have a solution different from the one you found in (a)?

Problem 116

Define $f(x) = \left(e^{2x} + ae^{-x}\right)^2$, where a is a constant, $a \neq 0$.

(a) Find $f'(x)$ and $f''(x)$.

(b) Determine where f is increasing, and show that f is convex everywhere.

(c) Find the extreme points for f, if any.

(d) Let $a = 4$, and find a minimum point for f over the interval $[1, \infty)$, if there is one.

Problem 117

Find the limit $\lim_{x\to 1} \dfrac{\ln x - x + 1}{(x-1)^2}$.

Problem 118

Find the integrals: (a) $\displaystyle\int \frac{e^{\sqrt{x}}}{\sqrt{x}\,(1+e^{\sqrt{x}})}\,dx$ (b) $\displaystyle\int_1^{e^2} \sqrt{x}\ln x\,dx$

Problem 119

Find $F'(x)$ when $F(x) = \displaystyle\int_4^x \Big(\sqrt{u} + \frac{x}{\sqrt{u}}\Big)\,du.$

Problem 120

(a) Find the general solution of the differential equation $\dot{x} + 2x = 2$.

(b) Find a function $w = w(t)$ such that

$$\ddot{w} + 2\dot{w} = 2, \quad w(0) = 0 \quad \text{and} \quad w\big(-\tfrac{1}{2}\big) = \frac{1}{2} - e.$$

Problem 121

(a) Find the maximum of $e^x y$ subject to $(x-1)^2 + y^2 = 12$.

(b) Suppose that we change the constraint in (a) to $(x-1)^2 + y^2 = 12.03$. What is the approximate corresponding percentage change in the maximum value of $e^x y$?

Problem 122

The function f is defined by $f(x) = x^3 e^{-x^2}$ for all x.

(a) Compute $f'(x)$ and $f''(x)$.

(b) Show that the graph of f is symmetric about the origin. Find $\lim_{x\to\infty} f(x)$.

(c) Find any maximum and minimum points, and sketch the graph.

(d) Compute $\int f(x)\,dx$ and $\int_0^\infty f(x)\,dx$.

(e) Let $0 < a < b$. Explain geometrically or otherwise why the following equality holds:
$\displaystyle\int_{-b}^{-a} f(x)\,dx = -\int_a^b f(x)\,dx.$

Problem 123

(a) Calculate the integral $\displaystyle\int_0^1 \frac{4x^3}{\sqrt{4-x^2}}\,dx.$

(b) Determine the limit $\lim_{x\to 1} \dfrac{a^{2x} - a^2}{2x - 2}$.

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Problem 124

(a) For what values of p and q does the following equation system

$$\begin{aligned} x_1 + x_2 + x_3 &= q \\ px_1 + x_2 - x_3 &= 5 \\ x_1 \qquad - x_3 &= p \end{aligned}$$

have one solution, several solutions, or no solutions?

(b) Find all solutions of the system in the case where the system has several solutions.

(c) Compute the determinant $\begin{vmatrix} 31 & 32 & 33 \\ 32 & 33 & 35 \\ 33 & 34 & 36 \end{vmatrix}$. (*Hint:* Use elementary operations on the rows and/or the columns.)

(d) Show that if $\mathbf{A}$ is an $n \times n$ matrix such that $\mathbf{A}^4 = \mathbf{0}$, then $(\mathbf{I} - \mathbf{A})^{-1} = \mathbf{I} + \mathbf{A} + \mathbf{A}^2 + \mathbf{A}^3$.

Problem 125

Let the function f be defined by $f(x, y) = xye^{4x^2 - 5xy + y^2}$ for all (x, y).

(a) Compute the first-order partial derivatives of f.

(b) Find the three stationary points of f, and show that f has no (global) extreme points.

(c) The level curve $f(x, y) = 1$ passes through the point $(x, y) = (1, 1)$. Find the slope of the tangent line to the level curve at this point.

Problem 126

Let the function f be defined by $f(x) = \dfrac{2}{\sqrt{x} + 1} + \dfrac{1}{2}\sqrt{x}$ for all $x \geq 0$.

(a) Find the maximum and minimum points of f, if any, and sketch the graph of f.

(b) Calculate $\displaystyle\int_0^4 f(x)\, dx$.

Problem 127

(a) Calculate the integral $\displaystyle\int_1^e \frac{\ln x^3}{x^2}\, dx$

(b) Determine the solution of the differential equation

$$x\frac{dx}{dt} = -\frac{1}{2}(x^2 - 25), \qquad x > 5 \tag{$*$}$$

that passes through the point $P = (0, 10)$. What is the slope of the solution curve at P? Show that every solution of $(*)$ is decreasing.

Problem 128

(a) Compute the determinant $\begin{vmatrix} 1 & 1 & a^2 \\ 2 & 3 & -1 \\ 0 & 2 & 1 \end{vmatrix}$.

(b) Consider the equation system

$$\begin{aligned} x + \ \ y + \ \ az + \quad au &= a \\ 2x + \ \ y - a^2 z + \ \ 2au &= 1 \\ 4x + 3y + a^2 z + 4a^2 u &= 1 \end{aligned}$$

where x, y, z, and u are unknowns and a is a constant. Find the number of degrees of freedom for all values of a. (Use Gaussian elimination.)

(c) Suppose that $\mathbf{A}$ and $\mathbf{B}$ are $n \times n$ matrices and that $\mathbf{A}^2\mathbf{B} = \mathbf{AB}$. Prove that $\mathbf{A}^4\mathbf{B} = \mathbf{AB}$.

Problem 129

Consider the problem of maximizing $x + 2z$ subject to $\begin{cases} x + \ \ y + z = 1 \\ x^2 + y^2 + z = 7/4 \end{cases}$

(a) Show that by eliminating the Lagrange multipliers from the first-order conditions, one can derive the equation $4x - 2y = 1$. Then find the only possible solution of the problem.

(b) Can you prove that you have found the solution in (a)?

Problem 130

The equation system

$$\begin{aligned} \ln(e^{x^2 y} + z) &= 1 \\ e^{\ln(x^2+z)-2z} + 2y &= 3 \end{aligned}$$

defines y and z as differentiable functions of x around the point $P = (x, y, z) = (1, 1, 0)$. Find dy/dx and dz/dx at the point P. Also find $\mathrm{El}_x\, y$ at P.

Problem 131

(a) Let $\mathbf{A} = \begin{pmatrix} 1 & a & a \\ a & a & a \\ a & a & 1 \end{pmatrix}$. Compute $|\mathbf{A}|$ and $\mathbf{A}^2$.

(b) Determine the number degrees of freedom for the equation system

$$\begin{aligned} x + ay + az &= 0 \\ ax + ay + az &= 0 \\ ax + ay + \ \ z &= 0 \end{aligned}$$

for all values of a.

(c) Solve the following equation system for all values of a:

$$\begin{aligned} x + ay + az &= 1 \\ ax + ay + az &= a^2 \\ ax + ay + \ \ z &= 1 \end{aligned}$$

Problem 132

Consider the function f defined by

$$f(x, y) = \tfrac{1}{2}x^2 - x + ay(x-1) - \tfrac{1}{3}y^3 + a^2y^2 \quad \text{for all } (x, y)$$

where a is a constant. Find all the stationary points of f and classify them.

Problem 133

A country wants to empty an oil field. It will start the production today at $t = 0$, and has a choice between two extraction profiles f and g giving the production of oil per unit of time. For both extraction profiles the time span is 10 years, and $f(t) = 10t^2 - t^3$, $t \in [0, 10]$, while $g(t) = t^3 - 20t^2 + 100t$, $t \in [0, 10]$.

(a) Sketch the two profiles in the same coordinate system.

(b) Show that $\int_0^t g(\tau)\,d\tau \geq \int_0^t f(\tau)\,d\tau$ for all $t \in [0, 10]$.

(c) The country obtains a price per unit of oil given by $p(t) = 1 + 1/(t+1)$, where t is the number of years. Total revenue is then given by $\int_0^{10} p(t)f(t)\,dt$ and $\int_0^{10} p(t)g(t)\,dt$ respectively. Compute these integrals. Which of the two extraction profiles should be chosen?

Problem 134

Solve the differential equations

(a) $\dot{x} + 4x = 3e^t, \quad x(0) = 1$ (b) $\dot{x} = \dfrac{e^{-3x}}{3 + \sqrt{t+8}}, \quad x(1) = 0$

Problem 135

Let f be defined by $f(x) = \dfrac{x^2 + 4x - 2}{x^2 + 1}$ for all x.

(a) Find $f'(x)$ and determine the local extreme points of f.

(b) Determine $\lim_{x \to \pm\infty} f(x)$ and sketch the graph of f.

(c) Define the function F by the formula $F(x) = \ln f(x)$. Where is F defined? What is the range of F? Sketch the graph of F.

Problem 136

(a) For a ball with radius r, the formulas $V = \frac{4}{3}\pi r^3$ and $O = 4\pi r^2$ give the volume and the surface area respectively. Show that $O = kV^{2/3}$ for a constant k.

(b) A spherical mothball evaporates at a rate proportional to the surface area. If $M(t)$ is the mass at time t, then $dM(t)/dt = -s(M(t))^{2/3}$, where s is a positive constant. Find the solution of this differential equation with $M(0) = 1$.

(c) $M(t)$ is measured in grams and t is measured in days. At $t = 0$ the weight of the mothball is 1 gram, and 75 days later it is 0.5 grams. Determine the value of s. How long does it take for the whole mothball to evaporate?

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Problem 137

Calculate the integrals (a) $\displaystyle\int_4^9 \frac{(\sqrt{x}-1)^2}{x}\,dx$ (b) $\displaystyle\int_0^1 \ln(1+\sqrt{x}\,)\,dx$

Problem 138

Consider the problem of maximizing $e^x + y + z$ subject to $\begin{cases} x + \ y + \ z = 1 \\ x^2 + y^2 + z^2 = 1 \end{cases}$

(a) Find the solution of the problem by using Lagrange's method.

(b) Replace the constraints by $x + y + z = 1.02$ and $x^2 + y^2 + z^2 = 0.98$. What is the approximate change in the maximum value of the objective function?

Problem 139

(a) For what values of a does $\mathbf{A}_a = \begin{pmatrix} 1 & 2 & 3 \\ 0 & a-1 & 1 \\ 1 & 2 & a+1 \end{pmatrix}$ have an inverse?

(b) Find the inverse when $a = 0$.

(c) Let $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{D}$ be $n \times n$ matrices where $|\mathbf{A}| \neq 0$ and $|\mathbf{B}| \neq 0$. Show that there exist matrices $\mathbf{X}$ and $\mathbf{Y}$ such that

$$\begin{aligned} \mathbf{AX} + 2\mathbf{AY} &= \mathbf{C} \\ \mathbf{A}^2\mathbf{XB} + \mathbf{A}^2\mathbf{YB} &= \mathbf{D} \end{aligned}$$

Problem 140

Let f be the function given by $f(x) = 2x^2 - \ln x - 2$, $x > 0$.

(a) Find the extreme points and extreme values for f, if any.

(b) Clearly $x = 1$ is a solution of the equation $f(x) = 0$. Show that this equation has exactly one additional solution $x = x_1$, where x_1 is a number in the interval $(0, 1)$. (You are not supposed to find x_1.) Sketch the graph of f.

(c) Find the local and (global) extreme points for $g(x) = \dfrac{1}{2x^2 - \ln x - 2}$.

(d) Sketch the graph of g.

Problem 141

Let the function f be defined by $f(x, y) = \frac{1}{2}x^2 e^y - \frac{1}{3}x^3 - ye^{3y}$ for all (x, y).

(a) Compute the partial derivatives of f of the first and second order.

(b) Find the stationary points of f, if any, and classify them. Does f have (global) extreme points?

(c) The level curve $f(x, y) = -\frac{2}{3}$ passes through the point $(x, y) = (2, 0)$. Find the slope of the tangent line to the level curve at this point.

Problem 142

(a) Let $\mathbf{A} = \begin{pmatrix} 1 & 3 & 4 \\ 2 & 2 & 1 \\ 3 & -3 & -9 \end{pmatrix}$. Compute $|\mathbf{A}|$.

(b) What conditions must b_1, b_2, and b_3 satisfy for the equation system

$$\begin{aligned} x + 3y + 4z &= b_1 \\ 2x + 2y + z &= b_2 \\ 3x - 3y - 9z &= b_3 \end{aligned}$$

to have solutions?

Problem 143

(a) Solve the differential equation $\dot{x} + 4x = 4e^{-2t}$, $x(0) = 1$.

(b) Suppose that $y = (a+\alpha k)\sqrt{t+1}$ denotes production as a function of capital k, where the factor $\sqrt{t+1}$ is due to technical progress. Suppose that a constant fraction $s \in (0, 1)$ is saved, and that capital accumulation is equal to savings, so that we have the separable differential equation

$$\dot{k} = s(a + \alpha k)\sqrt{t+1}, \quad k(0) = k_0$$

The constants a, α and k_0 are positive. Find the solution.

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Answers

1. (a) $f(x) > 0$ when $0 < x < 1$ and when $x > 2$.
(b) $f'(x) = 3x^2 - 6x + 2$. Let $x_0 = 1 - \frac{1}{3}\sqrt{3}$ and $x_1 = 1 + \frac{1}{3}\sqrt{3}$. Then f increases in $(-\infty, x_0]$ and in $[x_1, \infty)$, and f decreases in $[x_0, x_1]$. x_0 is a local maximum point with $f(x_0) = \frac{2}{9}\sqrt{3}$, while x_1 is a local minimum point with $f(x_1) = -\frac{2}{9}\sqrt{3}$. $f(x)$ is strictly convex in $[1, \infty)$. (c) See Fig. A.1. $\int_0^1 f(x)\,dx = 1/4$.

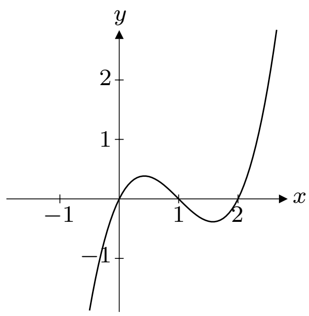

Figure A.1

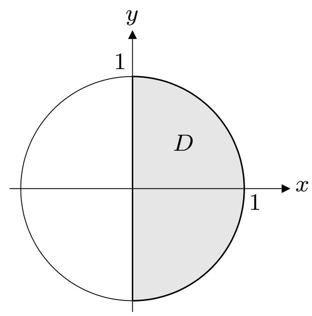

Figure A.2

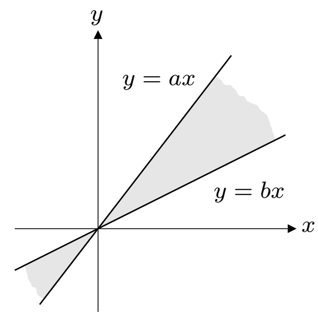

Figure A.3

2. (a) See Fig. A.2. (b) Local maximum at $(0, 0)$, saddle point at $(2/3, 0)$.
(c) Maximum at $(0, 0)$ and at $(1, 0)$. Maximum value 3. Minimum at $(0, -1)$ and at $(0, 1)$. Minimum value 2.
3. $D = -y^2 + (a+b)xy - abx^2 = -(y - ax)(y - bx)$ is equal to 0 along the lines $y = ax$ and $y = bx$. $D > 0$ when (i) $x > 0$ and $bx < y < ax$ or (ii) $x < 0$ and $ax < y < bx$. See Fig. A.3.
4. (a) Direct verification. (b) $L = 10$, $K = 6.25$ million.
5. (a) $|\mathbf{A}| = a(a^2 + 2b^2)$, $\mathbf{A}^2 = \begin{pmatrix} a^2 - b^2 & 2ab & b^2 \\ -2ab & a^2 - 2b^2 & 2ab \\ b^2 & -2ab & a^2 - b^2 \end{pmatrix}$ (c) $a = 0$
6. (a) f is defined in $[-6, 0) \cup (0, \infty)$. (b) $f(x) = 0$ iff $x = -2$ or $x = -6$, $f(x) > 0$ in $(-6, -2)$ and in $(0, \infty)$. (c) Local maximum at $(-4, \frac{1}{2}\sqrt{2}\,)$, local minimum at $(6, \frac{8}{3}\sqrt{3}\,)$ and at $(-6, 0)$. (d) The limits are $-\infty$, ∞, ∞, and 0, respectively. See Fig. A.6
7. (a) Maximum net profits at $x = \sqrt{(a-c)/6}$, $y = b^2/4d^2$.
(b) $\mathrm{El}_y N = \dfrac{1}{N}\Big(\dfrac{1}{2}b\sqrt{y} - dy\Big)$, $\mathrm{El}_y N = 0$ when $y = b^2/4d^2$.
8. (a) $f_1' = 5y - ax^{a-1}y^a$, $f_2' = 5x - ax^a y^{a-1}$,
$f_{11}'' = -a(a-1)x^{a-2}y^a$, $f_{12}'' = 5 - a^2x^{a-1}y^{a-1}$, $f_{22}'' = -a(a-1)x^a y^{a-2}$.
(b) All (x, y) for which $xy = (5/a)^{1/(a-1)}$ are stationary points. The second-derivative test does not apply. (c) f attains its maximum value along $xy = (5/a)^{1/(a-1)}$.
(d) $h(z) = 0$ has no, one, or two roots according as $5(5/a)^{1/(a-1)}\Big(1 - \dfrac{1}{a}\Big) < c$, $= c$, or $> c$. (e) $x = \sqrt{q/p}\,$, $y = \sqrt{p/q}$ solve the minimization problem.

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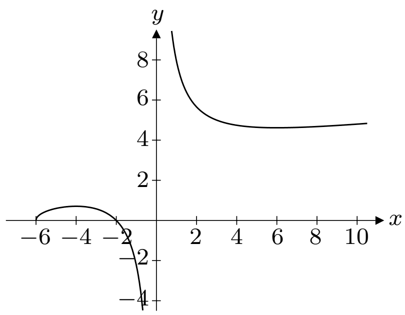

Figure A.6

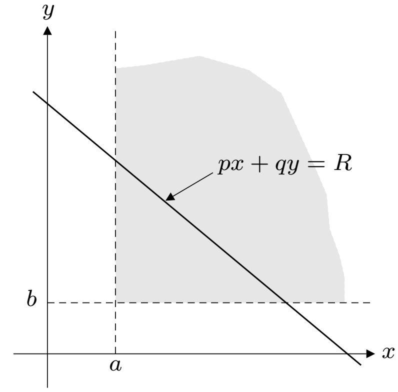

Figure A.10

9. (a) $\dfrac{\partial x}{\partial K} = \dfrac{x}{2K(1+4\ln x)}$, $\dfrac{\partial x}{\partial L} = \dfrac{x}{3L(1+4\ln x)}$, $\dfrac{\partial^2 x}{\partial K \partial L} = \dfrac{x(4\ln x - 3)}{6KL(1+4\ln x)^3}$

10. (a) U is defined for $x > a$ and $y > b$. (c) The condition is $pa + qb < R$. See Fig. A.10.

11. (a) $\lim_{x\to 0} f(x) = \frac{1}{2}$, $\lim_{x\to\infty} f(x) = 0$, $f'(x) = \dfrac{x^2e^x - (e^x-1)^2}{x^2(e^x-1)^2}$

12. (a) $|\mathbf{T}| = \frac{1}{2}pq(1-p-q)$. $\mathbf{T}^{-1}$ exists $\iff p+q \neq 1$. $|\mathbf{S}| = 0$, so $\mathbf{S}$ has no inverse.
(c) $\mathbf{T}^n = \dfrac{1}{2^{n-1}}\mathbf{T} + \dfrac{2^{n-1}-1}{2^{n-1}}\mathbf{S} \to \mathbf{S}$ as $n \to \infty$.

13. (a) With $\mathcal{L}(x,y) = U(x,y) - \lambda(py - w(24-x))$, the first-order conditions $\mathcal{L}'_1 = \mathcal{L}'_2 = 0$ yield (2) (b) Differentiate (1) and (2) w.r.t. w keeping p constant.
(c) $\partial x/\partial w = 4(1-\ln 16)/(1+\ln 16)$.

14. $x = A\dfrac{e^t}{t^2} + \dfrac{e^{2t}}{t} - \dfrac{e^{2t}}{t^2}$; $A = 0$

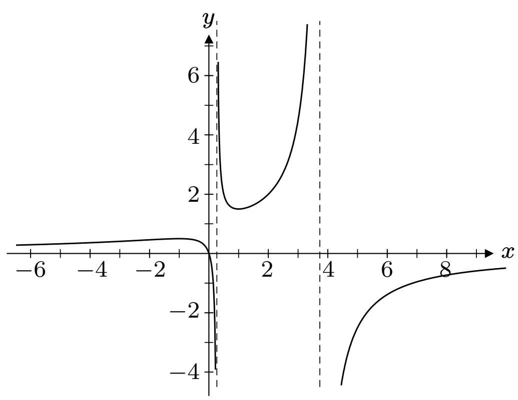

Figure A.15

15. (a) $f(q)$ is defined $\iff 2q - (p-q)^2 \neq 0 \iff q \neq p + 1 \pm \sqrt{2p+1}$
$f(q) \to 0$ as $q \to \infty$ and as $q \to -\infty$. (b) $f'(q) = \dfrac{2\hat{z}(q+p)(q-p)}{(2q-(p-q)^2)^2}$. $q = -p$ is a stationary point and a local maximum point. $q = p$ is a stationary point and a local minimum point. (c) Figure A.15 shows the graph when $p = 1$ and $\hat{z} = 1.5$.

16. (a) $f'_1 = 2(x+y-2) + 2(x^2+y-2)2x$, $f'_2 = 2(x+y-2) + 2(x^2+y-2)$, $f''_{11} = 12x^2 + 4y - 6$, $f''_{12} = f''_{21} = 4x + 2$, $f''_{22} = 4$. (b) $(0,2)$ and $(1,1)$ are local (and actually global) minimum points, and $(1/2, 13/8)$ is a saddle point.
(c) $g'(t) = 4p^4t^3 + 6p^2qt^2 + (-6p^2 + 4q^2 + 4pq)t - 4p - 8q$

17. (a) $|\mathbf{D}| = a + 2b + c$, $\quad \mathbf{C} \cdot \mathbf{D} = \begin{pmatrix} a-32 & b+35 & c-38 \\ 2a-66 & 2b+71 & 2c-76 \\ a-33 & b+35 & c-37 \end{pmatrix}$.
We see that $\mathbf{D} = \mathbf{C}^{-1}$ if $a = 33$, $b = -35$ and $c = 38$.
(b) $\mathbf{Y} = \mathbf{A}^{-1} \cdot \mathbf{C} \cdot \mathbf{H}$. ($\mathbf{A}$ has an inverse since $|\mathbf{A}| = -2 \neq 0$.)

18. 1. (Use l'Hôpital's rule.)

19. (a) $f_1' = \dfrac{2}{2x+y+2} - 2$, $f_2' = \dfrac{1}{2x+y+2} - 1$,
$f_{11}'' = \dfrac{-4}{(2x+y+2)^2}$, $f_{12}'' = f_{21}'' = \dfrac{-2}{(2x+y+2)^2}$, $f_{22}'' = \dfrac{-1}{(2x+y+2)^2}$.
(b) The stationary points are all the points on the line $2x + y = -1$.
(c) See Fig. A.19. The function has the maximum value $\ln\left(2 - \frac{1}{2}\sqrt{2}\right) + \frac{1}{2}\sqrt{2}$, attained at $\left(-\frac{1}{2}\sqrt{2}, \frac{1}{2}\sqrt{2}\right)$.

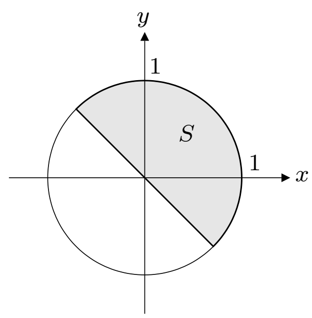

Figure A.19

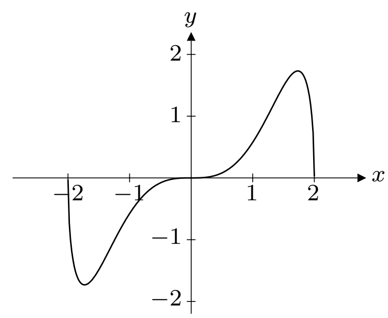

Figure A.26

20. $\begin{pmatrix} f_{11}'' & f_{12}'' \\ f_{21}'' & f_{22}'' \end{pmatrix} = \begin{pmatrix} \frac{1}{2}e^{-x-y} - e^{-x} & \frac{1}{2}e^{-x-y} \\ \frac{1}{2}e^{-x-y} & \frac{1}{2}e^{-x-y} - e^{-y} \end{pmatrix}$

21. (a) $\frac{1}{6}(2x-1)^3 + \frac{1}{2}e^{2x-2} + C$ (b) $\frac{1}{2}x^2 - x - \ln|x-1| + C$ (c) $\ln 4/3 \approx 0.29$

22. (a) $U'(x) = aAe^{-ax} - bBe^{bx}$ (b) Concave for all x. (c) $C = aAe^{-ax^*} = bBe^{bx^*}$
(d) *Hint:* $U(x) \approx U(x^*) + U'(x^*)(x - x^*) + \frac{1}{2}U''(x^*)(x - x^*)^2$

23. $\mathrm{El}_x\, y = a + 2b \ln x + c/\ln x$. y is defined as a function of x for $x > 1$.

24. (a) $(-1, 0)$ is local maximum point. (c) Maximum at $x = \sqrt[4]{2} - 1$, $y = 1/\sqrt[4]{2}$.
(d) The minimization problem has no solution.

25. (a) $\mathbf{A}_t$ has an inverse $\iff t \neq -1$. $\mathbf{I} - \mathbf{B}\mathbf{A}_t$ has no inverse for any t.
(b) $\mathbf{X} = \mathbf{I} - \mathbf{B}\mathbf{A}_1 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & -1 \\ -2 & -1 & 0 \end{pmatrix}$

26. (a) f is defined in $[-2, 2]$. $f(x) + f(-x) = 0$, so the graph of f is symmetric about the origin. (b) $f'(x) = \dfrac{4x^2(3 - x^2)}{3\sqrt{4 - x^2}}$. f increases in $[-\sqrt{3}, \sqrt{3}]$ and decreases in $[-2, -\sqrt{3}]$ and in $[\sqrt{3}, 2]$. (c) See Fig. A.26. (d) $g'\left(\frac{1}{3}\sqrt{3}\right) = 3\sqrt{3}/8$

27. $-1/6$

28. (a) $g''(x) = -ae^{-x}(x-1)(x-3)$. g is convex in $[1, 3]$. (b) $g(x) \to \infty$ as $x \to \infty$.
(e) $\dfrac{dx_0}{da} = \dfrac{1 + x_0^2}{2e^{x_0} + a(x_0 - 1)^2}$ (f) 1/2

29. (a) $k = b - a$ (c) $\dfrac{dE}{dt} = \left(\dfrac{-a}{p}\dfrac{dp}{dt} + \dfrac{b}{r}\dfrac{dr}{dt}\right)E$ (d) $b \geq a\,\dfrac{\ln 1.06}{\ln 1.08}$

30. (a) $(x, y) = (-2\sqrt{b}, 0)$ and $(x, y) = \left(4/3, \pm\sqrt{b - 4/9}\right)$ solve the maximization and the minimization problem, respectively.

31. (a) (i) Homogeneous of degree 4. (ii) Not homogeneous. (iii) Homogeneous of degree 0.

32. (a) $f'(x) = \dfrac{e^{2x}(2x^2+2x+1)}{(x+1)^2}$. No local extreme points.
(b) $f(x)$ tends to $-\infty$, ∞, 0, and ∞, respectively. (d) f is concave in $(-1, x_0)$.

33. (b) Minimum $= -298/3$ at $(4, 10)$.

34. $\mathbf{X} = (\mathbf{B}-\mathbf{C})^{-1}(\mathbf{E}-\mathbf{A})\mathbf{D}^{-1}$

35. (a) The system has one degree of freedom.
(b) $\dfrac{dy}{dg} = \dfrac{L'(r)S'_g}{D}$, $\dfrac{dr}{dg} = \dfrac{-lPS'_g}{D}$, where $D = lP(S'_r - I'_r) - L'(r)(S'_y - I'_y)$.

36. (a) $f'(x) = 1 + (\alpha+\beta)e^{-x} - 2\alpha e^{-2x}$, $\quad f''(x) = -(\alpha+\beta)e^{-x} + 4\alpha e^{-2x}$.
(b) $\bar{x} = \ln \dfrac{4\alpha}{\alpha+\beta} > 0$ because $4\alpha > \alpha+\beta$ when $\alpha > \beta$. (e) $\alpha > \beta + 1$

37. (a) $|\mathbf{A}| = 1$, $\quad \mathbf{A}^2 = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix}$, $\quad \mathbf{A}^3 = \begin{pmatrix} 1 & 1 & 2 \\ 2 & 2 & 3 \\ 1 & 2 & 2 \end{pmatrix}$.

38. (a) $du = -(8/3)\,dx + (10/3)\,dy$, $dv = 2\,dx - dy$. $\partial u/\partial y = 10/3$, $\partial v/\partial x = 2$.
(b) $\Delta u \approx du = -2.8/3 \approx -0.93$, $\quad \Delta v \approx dv = 0.4$.

39. (b) $(x_0, y_0) = \left(\frac{1}{8}(3+\sqrt{17}),\, \frac{1}{8}(-1+\sqrt{17})\right)$
(c) (x_0, y_0) is a maximum point. Minimum does not exist.

40. (a) $x - \dfrac{2}{3}x^3 + \dfrac{1}{5}x^5 + C$ (b) $a(P_L - P_N) - \dfrac{b}{2-\alpha}\left(P_L^{2-\alpha} - P_N^{2-\alpha}\right)$

41. (a) $f'_1 = e^{x+y} + e^{x-y} - 3/2$, $\quad f'_2 = e^{x+y} - e^{x-y} - 1/2$,
$f''_{11} = e^{x+y} + e^{x-y}$, $\quad f''_{12} = e^{x+y} - e^{x-y}$, $\quad f''_{22} = e^{x+y} + e^{x-y}$.
(b) $(-\frac{1}{2}\ln 2, \frac{1}{2}\ln 2)$ is the minimum point.

42. (a) $|\mathbf{A}_t| = 2t - 4$. $\mathbf{A}_t$ has an inverse $\iff t \neq 2$. (b) Show that $\mathbf{A}_1 \cdot \mathbf{A}_1^{-1} = \mathbf{I}_3$.
(c) $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 4 & 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$. Solution: $x = -5/2$, $y = -1$, $z = 11/2$.

43. $\dfrac{dX}{dN} = g(u) + g'(u)\big(\varphi'(N) - u\big)$, $\dfrac{d^2X}{dN^2} = \dfrac{g''(u)}{N}\big(\varphi'(N) - u\big)^2 + g'(u)\varphi''(N)$,
where $u = \dfrac{\varphi(N)}{N}$.

44. (a) $f'_1(x,y) = e^{-x/y}(y - x)$, $f'_2(x,y) = e^{-x/y}x(1 + x/y)$
(b) $\mathrm{El}_x\, f(x,y) = 1 - x/y$, $\mathrm{El}_y\, f(x,y) = 1 + x/y$.
(c) $f(x,x) = x^2e^{-1} \to \infty$ as $x \to \infty$. (d) $x = c(1 - \frac{1}{2}\sqrt{2})$, $y = \frac{1}{2}c\sqrt{2}$

45. $a = -3/4$ and $b = 3/4$. (Then $\mathbf{AB} = \mathbf{I}_3$.)

46. (a) $|\mathbf{A} - \mathbf{I}| = t - 1$. (b) $\mathbf{x}_0 = \pm\dfrac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ (c) $\mathbf{A}^n\mathbf{x}_0 = \mathbf{x}_0$ for all n.

47. (a) $f'(x) = -2x + 1 - e^{-x}$, $f''(x) = -2 + e^{-x}$. (b) f' is strictly increasing in $[-3, -\ln 2]$. (c) $f'(x) = 0$ has exactly two solutions in $[-3, 3]$. One lies in $(-3, -\ln 2)$ and the other in $(-\ln 2, 3)$. (d) The maximum point for $f(x)$ over $[-3, 3]$ is $x = -3$, and the maximum value is $f(-3) = e^3 - 12$.

48. (a) $(\mathbf{B} - \mathbf{I})^3 = \mathbf{B}^3 - 3\mathbf{B}^2 + 3\mathbf{B} - \mathbf{I}$, etc. (b) $\mathbf{A}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -4 & 0 & 1 \end{pmatrix}$

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49. (a) Differentiate the right-hand side. (b) 686/3. (*Hint:* Substitute $u = x^2 + 9$.)

50. (a) Unique solution when $k \neq 2$ and $k \neq 3$. (b) No solution when $k = 3$.

51. (a) The necessary conditions are $-2x + \lambda y = 0$, $-2y + \lambda x = 0$, $4 - 2z - \lambda = 0$ and $z = xy$. (b) $(0,0,0)$, $(1,1,1)$ and $(-1,-1,1)$. (c) $\Delta f^* \approx \lambda \cdot 0.1 = 0.2$.

52. $\mathbf{AA}' = \begin{pmatrix} 21 & 11 \\ 11 & 10 \end{pmatrix}$, $|\mathbf{AA}'| = 89$, $(\mathbf{AA}')^{-1} = \dfrac{1}{89}\begin{pmatrix} 10 & -11 \\ -11 & 21 \end{pmatrix}$
(b) No, $\mathbf{AA}'$ is symmetric for every matrix $\mathbf{A}$, and the inverse of a symmetric matrix is again symmetric.
(c) The matrix $(1/m)\mathbf{1} \cdot \mathbf{X}$ is the $1 \times n$ matrix $\left(\frac{1}{m}(x_{11} + x_{21} + \cdots + x_{m1}), \ldots, \frac{1}{m}(x_{1n} + x_{2n} + \cdots + x_{mn})\right)$, whose i-th component, $\frac{1}{m}(x_{1i} + x_{2i} + \cdots + x_{mi})$, is the arithmetical mean of the m observations of quantity i.

53. (a) $32/15 \approx 2.133$. See Fig. A.53. The area of the triangle OAB is 2.
(b) Look at the sign of $\dot{x}$. $\ddot{x} = (1 + x^2)(2xt^2 + 1) > 0$ for all x.
(c) $\mathrm{El}_x\, y = \dfrac{x - ay^2}{2x + by^2}$

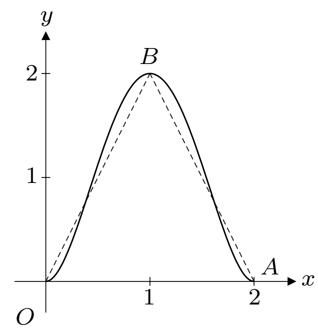

Figure A.53

54. $(0,0)$ is a saddle point. $(5/6, -5/12)$ is a local maximum point.

55. (a) $t = 4, s = -5$ (b) $\mathbf{X} = \begin{pmatrix} -1/2 & 0 & 0 & 1/6 \\ 1/2 & 3 & -3 & 1/6 \\ 1/2 & -1 & 2 & 1/6 \end{pmatrix}$
(c) $\mathbf{D}^6 = 182\mathbf{D} + 183\mathbf{I}_n$, $\mathbf{D}^{-1} = \frac{1}{3}\mathbf{D} - \frac{2}{3}\mathbf{I}_n$

56. (a) $(\sqrt{A}, 0)$, $(-\sqrt{A}, 0)$, $(0, \sqrt{A}\,)$ and $(0, -\sqrt{A}\,)$ (b) $y' = \dfrac{-2xe^{ay}}{2y + ax^2e^{ay}}$
(d) $x = \dfrac{\sqrt{2}}{a} e^{-\frac{1}{2}(1-\sqrt{1+a^2A})}\sqrt{\sqrt{1 + a^2A} - 1}$, $y = \left(1 - \sqrt{1 + a^2A}\,\right)/a$

57. (a) f is homogeneous of degree 3. $k = 3$. (b) $y' = -\dfrac{2xy}{x^2 + y^2}$. The tangent line is given by $y = \dfrac{4}{5}x - \dfrac{13}{5}$ (c) $y'' = -\dfrac{78}{125}$ (d) $y_{\min} = -\sqrt[3]{13}$

58. (a) $5t^2 - 45t + 40$ (b) Unique solution if $t \neq 1$ and $t \neq 8$. (c) $x = 8$, $y = 3$, $z = -1$
(d) $s = -1/4$

59. (a) $\mathbf{A}^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 20 & 0 & 0 \end{pmatrix}$, $\mathbf{I}_3 + \mathbf{A} + \mathbf{A}^2 = \begin{pmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 30 & 5 & 1 \end{pmatrix}$,
$(\mathbf{I}_3 - \mathbf{A})(\mathbf{I}_3 + \mathbf{A} + \mathbf{A}^2) = \mathbf{I}_3$. (b) $(\mathbf{I}_3 - \mathbf{A})^{-1} = \mathbf{I}_3 + \mathbf{A} + \mathbf{A}^2$.
(c) *Hint:* Show that $\mathbf{U}^2 = n\mathbf{U}$. (d) $\dfrac{1}{10}\begin{pmatrix} 7 & -3 & -3 \\ -3 & 7 & -3 \\ -3 & -3 & 7 \end{pmatrix}$

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60. (a) $U_1'(x,y) = A\dfrac{\alpha x^{\alpha-1}}{x^\alpha + y^\alpha}$, $U_2'(x,y) = -A\dfrac{\alpha x^\alpha}{y(x^\alpha + y^\alpha)}$,
$U_{12}''(x,y) = -A\dfrac{\alpha^2 x^{\alpha-1} y^{\alpha-1}}{(x^\alpha + y^\alpha)^2}$.
(b) U is homogeneous of degree 0. (c) $x = \sqrt[4]{3b/a}$ (with $y = \sqrt[3]{4b}$).

61. (a) Differentiation gives the equations

$$\frac{1}{x+u}(dx + du) + u\,dv + v\,du - 2ye^v\,dy - y^2 e^v\,dv + dy = 0$$
$$2u\,du - \frac{x^v v}{x}\,dx - x^v \ln x\,dv = dv$$

(b) $u_x' = -\dfrac{1+\ln 2}{5+\ln 2}$, $u_y' = \dfrac{1+\ln 2}{5+\ln 2}$, $v_x' = \dfrac{2}{5+\ln 2}$, $v_y' = -\dfrac{2}{5+\ln 2}$.
(c) $u(1.99, 1.02) \approx u(2,1) + u_x'(2,1)\cdot(-0.01) + u_y'(2,1)\cdot 0.02) \approx -0.9911$.

62. (a) $|\mathbf{A}_a| = -2a^3 - 3a^2 + 1 = (a+1)^2(1-2a)$ (b) $k = 1/(1-a-2a^2)$

63. (a) Slope: $-3/2$. Linear approximation: $y(x) \approx -\frac{3}{2}x + \frac{3}{2}$.
(b) $\dfrac{\partial p}{\partial r} = \dfrac{L}{F(L)}$, $\dfrac{\partial p}{\partial B} = \dfrac{1}{F(L)}$, $\dfrac{\partial L}{\partial r} = \dfrac{F(L) - LF''(L)}{pF(L)F''(L)}$, $\dfrac{\partial L}{\partial B} = -\dfrac{F'(L)}{pF(L)F''(L)}$.
(c) Since $F(L) = (rL+B)/p > 0$, $F'(L) > 0$, and $F''(L) < 0$, it follows that $\partial p/\partial r > 0$, $\partial p/\partial B > 0$, $\partial L/\partial r < 0$ and $\partial L/\partial B > 0$.

64. $f_{\max} = f(0,1) = 1$.

65. (a) $4e^{-1} - 6e^{-2}$. (*Hint:* Substitute $u = \sqrt{t}$.) (b) $\mathbf{X} = 2\mathbf{A}^{-1}\mathbf{C} - \mathbf{D}$, $\mathbf{Y} = \mathbf{AD} - \mathbf{C}$.

66. Unique solution when $a \neq -9$ and $a \neq 2$. Solutions with 1 degree of freedom when $a = -9$. No solutions when $a = 2$.

67. (a) $\mathrm{El}_x\, y = xy/(1-2y)$. (b) Differentiation yields the equations

$$\alpha u^{\alpha-1}\,du + \beta v^{\beta-1}\,dv = 2^\beta\,dx + 3y^2\,dy$$
$$\alpha u^{\alpha-1} v^\beta\,du + u^\alpha \beta v^{\beta-1}\,dv - \beta v^{\beta-1}\,dv = dx - dy$$

At P, $\dfrac{\partial u}{\partial x} = \dfrac{2^{-\beta}}{\alpha}$, $\dfrac{\partial u}{\partial y} = -\dfrac{2^{-\beta}}{\alpha}$, $\dfrac{\partial v}{\partial x} = \dfrac{2^\beta - 2^{-\beta}}{\beta 2^{\beta-1}}$, $\dfrac{\partial v}{\partial y} = \dfrac{2^{-\beta}+3}{\beta 2^{\beta-1}}$.
(c) $u(0.99, 1.01) \approx u(1,1) + u_x'(1,1)\cdot(-0.01) + u_y'(1,1)\cdot 0.01$, and so on.

68. (a) h is strictly increasing in $(-\infty, \frac{1}{2}\ln 2]$ and strictly decreasing in $[\frac{1}{2}\ln 2, \infty)$. $x = \frac{1}{2}\ln 2$ is a maximum point.
(b) The inverse function is $h^{-1}(x) = \ln(1 - \sqrt{1-8x^2}\,) - \ln(2x)$, $x \in (0, \frac{1}{3})$.
(c) No. If, for example, $g(x) = e^x/(1+e^x)$, then $f'(x) > 0$ for all x.

69. (a) $|\mathbf{A}_t| = 2(t+1)$ (b) $x = 3/2$, $y = s - 1/2$, $z = s$, $s \in \mathbb{R}$

70. (a) $f_1' = 2x(xy+1) + (x^2+y^2)y$, $f_2' = 2y(xy+1) + (x^2+y^2)x$,
$f_{11}'' = 6xy + 2$, $f_{12}'' = f_{21}'' = 3x^2 + 3y^2$, $f_{22}'' = 6xy + 2$
(b) $(0,0)$ is a local minimum point and the others are saddle points.
(c) Maximum $a^2(1 + \frac{1}{2}a^2)$ at $(\frac{1}{2}a\sqrt{2}, \frac{1}{2}a\sqrt{a}\,)$ and at $(-\frac{1}{2}a\sqrt{2}, -\frac{1}{2}a\sqrt{a}\,)$.

71. (a) Area $= 16(2\ln 2 - 1) \approx 6.18$ (b) $a^a(\ln a - 1)$

72. (a) $|\mathbf{A}| = (a-1)(a-2)$ (b) Unique solution iff $a \neq 1$ and $a \neq 2$. For $a = 1$ there is no solution, for $a = 2$ there are infinitely many solutions. (c) $a = 2$ and $b_1 = b_2$, or $a = 1$ and $b_1 - b_2 + b_3 = 0$ (d) Take the determinant of each side.

73. Maximum $= 1$ at $(3/\sqrt{10}, -1/\sqrt{10}, 0)$ and at $(-3/\sqrt{10}, 1/\sqrt{10}, 0)$.

74. (a) $\pi(x,y) = px - (cx + y + d)$. Then use $p = -x/a + b/a + R(y)/a$.
(c) $\dfrac{\partial y^*}{\partial b} = \dfrac{-R'(y^*)}{(b-ac)R''(y^*) + (R'(y^*))^2 + R(y^*)R''(y^*)}$
(d) $x^* = \dfrac{2a(b-ac)}{4a-\alpha^2}$, $y^* = \dfrac{\alpha^2(b-ac)^2}{(4a-\alpha^2)^2}$

75. (a) One would expect $f'(y)$ to be negative, for if investment in special machines is increased, workers' salary per unit produced should decrease. One would also expect $f''(y) > 0$, because the decrease in workers' salary per unit will become gradually less as investment in special equipment is increased. Annual net profits: $\pi(x,y) = -bx^2 + (a-r)x - d - ky - xf(y)$. $\pi_1' = -2bx + (a-r) - f(y)$, $\pi_2' = -k - xf'(y)$, with $\pi_{11}' = -2b$, $\pi_{12}' = \pi_{21}' = -f'(y)$, $\pi_{22}' = -xf''(y)$.
(b) Use the first-order conditions. (c) $2bk(y+\beta)^3 - \alpha(a-r)(y+\beta) + \alpha^2 = 0$
(d) $\dfrac{dy}{dk} = \dfrac{2b}{(f'(y))^2 + f''(y)(f(y)-(a-r))}$

76. (a) $|\mathbf{A}_t| = 2t^2 - 2t + 1 \neq 0$ for all t, so $\mathbf{A}_t^{-1}$ exists for every t.
(b) For $t = 1$, $\mathbf{A}_1^{-1} = \mathbf{A}_1^2 = \begin{pmatrix} -1 & -1 & -1 \\ 2 & 1 & 1 \\ -2 & -1 & 0 \end{pmatrix}$

77. (i) 447/14. (*Hint:* Substitute $u = (2+x)^{1/3}$.)
(ii) $3x^{2/3}e^{x^{1/3}} - 6x^{1/3}e^{x^{1/3}} + 6e^{x^{1/3}} + C$. (*Hint:* Substitute $u = \sqrt[3]{x}$.)

78. $x = \dfrac{1}{a}\ln\Big(\dfrac{\sqrt{A^2+4a^2c} - A}{2a}\Big)$, $y = \dfrac{1}{b}\ln\Big(\dfrac{2ac}{\sqrt{A^2+4a^2c} - A} - 1\Big)$

79. (a) $\mathrm{El}_x\, y = 2/(9e) - 1/3 \approx -0.25$, $\mathrm{El}_x\, z = -1 - 2/(3e) \approx -1.25$.
(b) If x increases from 1 to 1.1, i.e. with 10 %, then y decreases by about 2.5 % and z decreases by about 12.5 %.

80. (a) $g'(x) = (a-1)(1 - c^a x^{-a})$, $g''(x) = a(a-1)c^a x^{-a-1}$.
$\lim_{x\to 0^+} g(x) = \infty$, $\lim_{x\to\infty} g(x) = \infty$.
(b) $g_{\min} = g(c) = a(c-1)$.
(c) Use the intermediate value theorem.

81. $\displaystyle\int \frac{dv}{1-v^2} = \frac{1}{2}\ln\left|\frac{1+v}{1-v}\right| + C$
$\displaystyle\int \frac{dx}{\sqrt{1-e^{-x}}} = \ln\left(\frac{1+\sqrt{1-e^{-x}}}{1-\sqrt{1-e^{-x}}}\right) + C_1$. (*Hint:* Substitute $u = \sqrt{1-e^{-x}}$.)

82. (a) $|\mathbf{A}_a| = 2a^3 - 6a^2 + 6$ and $\mathbf{A}_a^2 = \begin{pmatrix} 2a^2+1 & a^2+3a & a^2+4a \\ a^2+3a & 2a^2+4 & a^2+5a \\ a^2+4a & a^2+5a & 2a^2+9 \end{pmatrix}$
(b) The system has a unique solution iff $|\mathbf{A}_a| \neq 0$, i.e. iff $2a^3 - 6a^2 + 6 \neq 0$. For $a = 3$ the solution is $x_1 = 1$, $x_2 = 1$, $x_3 = -1$.
(c) $x_1 = x_2 = \cdots = x_{n-1} = 1$, $x_n = 2 - n$. (d) $\mathbf{A}^{-1} = \mathbf{CB}$.

83. (a) $x_1 = (m + 18p_1 - 5p_2 - 4p_3)/4p_1$, $x_2 = (m - 6p_1 + 5p_2 - 4p_3)/2p_2$,
$x_3 = (-6p_1 - 5p_2 + 12p_3)/4p_3$.
(b) $U^* = -3\ln 4 - \ln p_1 - 2\ln p_2 - \ln p_3 + 4\ln(m - 6p_1 - 5p_2 - 4p_3)$,
$\partial U^*/\partial m = 4(m - 6p_1 - 5p_2 - 4p_3)^{-1} = \lambda$. (c) $\Delta U^* \approx \lambda \Delta m = 0.1 \cdot 1 = 0.1$.

84. (a) $\dot K = \gamma K^\alpha(\beta t + L_0)$ (b) $K = \Big((1-\alpha)\gamma\big(\tfrac{\beta}{2}t^2 + L_0 t\big) + K_0^{1-\alpha}\Big)^{1/(1-\alpha)}$

85. $x = \pm 1$ are (global) minimum points, $x = 0$ is a local maximum point.

86. (a) $(x,y,z) = (4,0,0)$ gives the maximum, while all points $(x,y,z) = (-1,y,z)$ with $y^2 + z^2 = 15/2$ give the minimum. (b) The maximum point is the same as in (a). Minimum at $(x,y,z) = (-1/2, 0, 0)$.

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87. (a) $|\mathbf{A}_a| = (a+1)(a-1)(a-2)$. (b) The system has a unique solution iff $|\mathbf{A}_a| \neq 0$, i.e. iff $a \neq \pm 1$ and $a \neq 2$. If $a = 1$ and $b \neq 2$, the system has no solution. If $a = 2$ and $b \neq 12$, the system has no solution. (c) $|3\mathbf{A}_3| = 216$, $|\mathbf{A}_5\mathbf{A}_4^{-1}\mathbf{A}_3^2| = 768/5$.

88. (a) (i) $-6e^{-x/2}(x+2) + C$. (*Hint:* Integration by parts.) (ii) $10 - 18\ln(14/9)$. (*Hint:* Substitute $u = 9 + \sqrt{x}$.) (iii) $886/15$. (*Hint:* Substitute $u = \sqrt{t+2}$.)
(b) $x(t) = \dfrac{B}{2t-a}e^{-a/(2t-a)}$, where B is a constant.

89. (a) $z_1'(e,e) = 1/2$, $z_{11}''(e,e) = 11/(16e)$ (b) $y' = 1/(F'(0)+1)$

90. (a) $f'(x) = 4x(1 - \frac{1}{2}a - x^2)e^{-x^2-a}$. For $a < 2$, f has three stationary points: $x = 0$ and $x = \pm\sqrt{1 - a/2}$. For $a \geq 2$, $x = 0$ is the only stationary point.
(b) The graph is symmetric about y-axis, since $f(-x) = f(x)$ for all x. $f(x)$ tends to 0 as $x \to \pm\infty$. (c) $M(a)$ is largest for $a = 0$.
(d) Only stationary point: $(0,1)$. The maximum value is $2e^{-1}$, attained at $(\pm 1, 0)$.

91. (a) 9. (*Hint:* Substitute $u = x^2 + x + 2$.) (b) *Hint:* The equation is separable.

92. (a) $y' = 7$.
(b) $\dfrac{dx}{dz} = \dfrac{1}{D}\Big[2z\big(xe^y + f(z)\big) - xyf'(z)\dfrac{\partial g}{\partial y}\Big]$, $\dfrac{dy}{dz} = \dfrac{1}{D}\Big[yf'(z)\Big(g + x\dfrac{\partial g}{\partial x}\Big) - 2ze^y\Big]$,
where $D = \begin{vmatrix} e^y & xe^y + f(z) \\ g + x\dfrac{\partial g}{\partial x} & x\dfrac{\partial g}{\partial y} \end{vmatrix} = xe^y\dfrac{\partial g}{\partial y} - \big(xe^y + f(z)\big)\Big(g + x\dfrac{\partial g}{\partial x}\Big)$.

93. (a) Unique solution for $a \neq 3$.
(b) $\mathbf{X} = (\mathbf{A} - \mathbf{I})^{-1}\mathbf{C}\mathbf{B}^{-1}$. With the given values for $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ we get
$(\mathbf{A} - \mathbf{I})^{-1} = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix}$, $\mathbf{B}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}$, $\mathbf{X} = \begin{pmatrix} 0 & 4 & 6 \\ 1 & -2 & 2 \end{pmatrix}$.

94. (a) $\ln\left|\dfrac{\sqrt{u}-1}{\sqrt{u}+1}\right| + C$. (b) $\ln\left|\dfrac{\sqrt{e^y+1}-1}{\sqrt{e^y+1}+1}\right| + C$.

95. (a) $|\mathbf{A}_3(t)| = -t^3 + 3t^2 - 4t + 2$, $|\mathbf{A}_4(t,a)| = t^4 - 3t^3 + 4t^2 - 2t - a$.
(b) Solution iff $b_1 - 2b_2 + 2b_3 = 0$. One degree of freedom.
(c) $\mathbf{P}^2$, $\mathbf{P} + \mathbf{P}$ and $\mathbf{P}'$ are all invertible. $\mathbf{P} + \mathbf{P}'$ is not necessarily invertible.

96. (a) $y' = -\dfrac{y+2}{x+2y+2}$, $y'' = \dfrac{2(y+2)(x+y)}{(x+2y+2)^2}$.
(b) $x^* = 5/3 - m/24$, $y^* = m/8 - 1$, $\lambda = m/48 + 1/6$. (c) $dU^*(20)/dm = 7/12$.
(d) For $m \leq 8$, $x^* = m/6$ and $y^* = 0$. For $m \geq 40$, $x^* = 0$ and $y^* = m/10$.

97. (a) $V_\varphi = [-\ln 2, 0)$ (b) $\varphi^{-1}(x) = \dfrac{2e^x - 1}{1 - e^x}$ for x in $[-\ln 2, 0)$.
(c) $(\varphi')^{-1}(x) = -\dfrac{3}{2} + \sqrt{\dfrac{1}{4} + \dfrac{1}{x}}$.

98. (a) $t \neq -1$ (b) $\mathbf{I}_3 - \mathbf{B}\mathbf{A}_t = \begin{pmatrix} 0 & 0 & -t \\ 0 & 0 & -1 \\ -2 & -1 & 1-t \end{pmatrix}$ has no inverse for any value of t.
$\mathbf{X} = \mathbf{I}_3 - \mathbf{B}\mathbf{A}_1 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & -1 \\ -2 & -1 & 0 \end{pmatrix}$ (c) $\mathbf{Y} = \begin{pmatrix} 1 & 0 & 2 \\ -1 & 2 & 1 \end{pmatrix}$

99. $y' = -y/x$, $y'' = 2y/x^2$

100. (a) $\dfrac{2x^{2n+1/2}}{4n+1} - \dfrac{4x^{n+m+1/2}}{2n+2m+1} + \dfrac{2x^{2m+1/2}}{4m+1} + C$ (b) $\frac{1}{3} - \ln(e^{1/3} + 1) + \ln 2$.
(c) The solution is given implicitly by $\dfrac{3}{5a^2}(ax+b)^{5/3} - \dfrac{3b}{2a^2}(ax+b)^{2/3} = \dfrac{1}{3}t^3 + C$,

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where C is a constant. The left-hand side of this equation can also be written as $\frac{3x}{2a}(ax+b)^{2/3} - \frac{9}{10a^2}(ax+b)^{5/3}$ or $\frac{3}{10a^2}(ax+b)^{2/3}(2ax-3b)$. The differential equation also has the constant solution $x \equiv -b/a$ (provided $b \neq 0$).

101. (a) The Hessian is $\mathbf{H} = \begin{pmatrix} 4(1-\rho^2)y^2 - 6 & 8(1-\rho^2)xy + 2\rho \\ 8(1-\rho^2)xy + 2\rho & 4(1-\rho^2)x^2 - 6 \end{pmatrix}$.

If $\rho = \pm 1$, f has only $(0,0)$ as a stationary point, and it is a maximum point.

(b) $x_0 f_1'(x_0, y_0) - y_0 f_2'(x_0, y_0) = -6x_0^2 + 6y_0^2$.

(c) $(0,0)$, (p,p), $(-p,-p)$, $(q,-q)$, $(-q,q)$, with $p = \sqrt{\dfrac{3-\rho}{2(1-\rho^2)}}$, $q = \sqrt{\dfrac{3+\rho}{2(1-\rho^2)}}$.

102. (a) $|\mathbf{A}| = -a(a-1)^2$ (b) $a = b = 1$ (c) $\mathbf{B}$ exists iff $a \neq 1$.

103. (a) $V(x) = Ae^{-ax} - b/a$. (b) With 20 liters, the car can drive $10\ln(27/7) \approx 13.5$ miles. In order to drive 15 miles, the car must start with at least $7e^{1.5} - 7 \approx 24.37$ liters.

104. (i) $1/5$. (ii) $7 + 2\ln 2$.

105. (a) $y' = 3$. (b) *Hint:* The curve cuts the x-axis at points where $y = 0$, and there we must have $e^{-x} = x + 2$.

106. (a) With the Lagrangian $\mathcal{L}(x,y,z) = xyz - \lambda(x+y+z-5) - \mu(xy+xz+yz-8)$, the necessary first-order conditions are

$$\text{(1)} \qquad \mathcal{L}_1'(x,y,z) = yz - \lambda - \mu(y+z) = 0$$
$$\text{(2)} \qquad \mathcal{L}_2(x,y,z) = xz - \lambda - \mu(x+z) = 0$$
$$\text{(3)} \qquad \mathcal{L}_3(x,y,z) = xy - \lambda - \mu(x+y) = 0$$

(c) The maximum value of $f(x,y,z)$ is $112/27$, which is attained at the three points $(7/3, 4/3, 4/3)$, $(4/3, 7/3, 4/3)$, $(4/3, 4/3, 7/3)$, all with $\lambda = -16/9$ and $\mu = 4/3$.
(d) $\Delta f^* \approx \lambda \cdot 0.01 + \mu(-0.01) = -\frac{28}{9} \cdot 0.01 \approx -0.031$.

107. (a) $f'(x) = e^{x-3}/(2 + e^{x-3}) > 0$. $V_f = (\ln 2, \infty)$. (b) $g(x) = 3 + \ln(e^x - 2)$ for all $x > \ln 2$. ($D_g = V_f$.) (c) $f'(3) = 1/3$, $g'(\ln 3) = 3$.

108. (a) $|\mathbf{A}| = (q-2)(p+1)$, $|\mathbf{A} + \mathbf{E}| = 2(1-p)(2-q)$.

$$\mathbf{AE} = \begin{pmatrix} 2q-3 & 2q-3 & 2q-3 \\ 3-2p & 3-2p & 3-2p \\ 1 & 1 & 1 \end{pmatrix}$$

(b) $\mathbf{A} + \mathbf{E}$ has an inverse iff $p \neq 1$ and $q \neq 2$. $|\mathbf{BE}| = 0$, so $\mathbf{BE}$ has no inverse.

$$\text{(c)} \begin{cases} p \neq -1 \text{ and } q \neq 2: & \text{No solution.} \\ p = -1: & \text{Unique solution.} \\ p = 1/2: & \text{No solution.} \\ p \neq 1/2 \text{ and } q = 2: & \text{Unique solution.} \end{cases}$$

109. (a) $195/4$. (*Hint:* Substitute $u = \sqrt[3]{19x^3 + 8}$.) (b) $x = (2e^{1-e^{-3t}} - 1)^{1/3}$

110. (a) f attains its maximum value $f_{\max} = 17/16$ at $(\pm\sqrt{15}/4, 0, 1/8)$ (with $\lambda = 1$) and its minimum value $f_{\min} = -1/2$ at $(0, 0, -1/2)$ (with $\lambda = -1/4$).

(b) $\Delta f_{\max} \approx \lambda \cdot 0.02 = 0.02$.

111. $x = \left(\dfrac{4}{\varphi(t)^2} - 9\right)^{1/3}$, where $\varphi(t) = t\ln t - t + 2/3$.

112. (a) 30. (b) $\int_0^{10} (1 + 0.4t)e^{-0.05t}\, dt = 180 - 260e^{-0.5} \approx 22.302$.

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113. $\text{El}_x\, y = \dfrac{3 + \dfrac{px}{x+a}}{3 - \dfrac{qy}{y+b}}$.

114. (a) Solution(s) iff $-3b_1 + b_2 - b_3 = 0$. 1 degree of freedom.
(b) No, then $\mathbf{A}$ would have to be the zero matrix.
(c) $\mathbf{C} = \begin{pmatrix} -5s & -5t & -5u \\ 14s & 14t & 14u \\ 11s & 11t & 11u \end{pmatrix}$, where s, t and u are not all 0.

115. There are three solutions of the first-order conditions: $(x_1, y_1, z_1) = (0, 0, \ln\sqrt{6})$, $(x_2, y_2, z_2) = (1, \frac{1}{2}, \ln 2)$, $(x_3, y_3, z_3) = (-1, -\frac{1}{2}, \ln 2)$, with $\lambda_1 = \sqrt{6}/12$ and $\lambda_2 = \lambda_3 = 1/4$, respectively. $f_{\max} = 5/2$ attained at (x_2, y_2, z_2) and at (x_3, y_3, z_3). (b) $\Delta f^* \approx \frac{1}{4}\cdot 0.1 = 0.025$. (c) No.

116. (a) $f'(x) = 2(e^{2x} + ae^{-x})(2e^{2x} - ae^{-x})$, $f''(x) = 16e^{4x} + 2ae^{x} + 4a^2e^{-2x}$
(b) If $a > 0$, then f is increasing in $[\frac{1}{3}\ln(\frac{1}{2}a), \infty)$; if $a < 0$, then f is increasing in $[\frac{1}{3}\ln(-a), \infty)$. (c) If $a > 0$, then $x = \frac{1}{2}\ln(\frac{1}{2})a$ is a minimum point, and if $a < 0$, $x = \frac{1}{3}\ln(-a)$ is a minimum point. (d) Minimum at $x = 1$.

117. $-1/2$.

118. (a) $2\ln(1 + e^{\sqrt{x}}) + C$. (*Hint:* Substitute $u = 1 + e^{\sqrt{x}}$.)
(b) $(8e^3 + 4)/9$. (*Hint:* Integration by parts.)

119. $4\sqrt{x} - 4$.

120. (a) $x = Ce^{-2t} + 1$. (b) $w(t) = -e^{-2t} + t + 1$.

121. (a) $f_{\max} = f(4, \sqrt{3}) = e^4\sqrt{3}$. (b) 0.5 %.

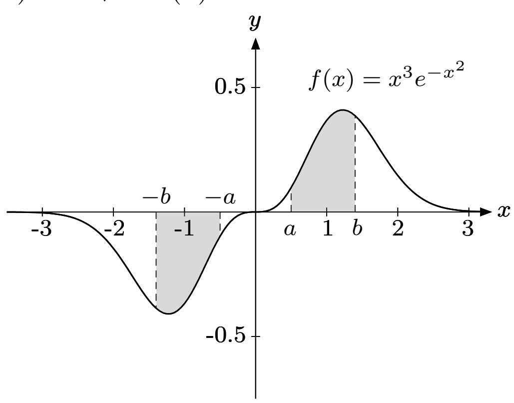

Figure A.122

122. (a) $f'(x) = 2x^2e^{-x^2}(\frac{3}{2} - x^2)$, $f''(x) = 2xe^{-x^2}(3 - 7x^2 + 2x^4)$
(b) $f(-x) = -f(x)$. $\lim_{x\to\infty} f(x) = 0$. (c) (Global) minimum point at $x = -\sqrt{6}/2$, (global) maximum point at $x = \sqrt{6}/2$. ($x = 0$ is an inflection point, not an extreme point.) See Fig. A.122. (d) $\int f(x)\,dx = \frac{1}{2}\big(-x^2e^{-x^2} - e^{-x^2}\big) + C$, $\int_0^\infty f(x)\,dx = \frac{1}{2}$. (e) The two areas are shown in Fig. A.122. Since the graph is symmetric about the origin, these two areas must be equal. Alternatively, we can use the identity $f(-x) = -f(x)$ and the substitution $t = -x$ to show that $\int_{-b}^{-a} f(x)\,dx = -\int_a^b f(t)\,dt$.

123. (a) $12\sqrt{3} - 44/3$. (b) $a^2\ln a$.

124. (a) Unique solution for $p \neq 3$. For $p = 3$ and $q \neq -1$ there is no solution. For $p = 3$ and $q = -1$ there are solutions with 1 degree of freedom.
(b) $x_1 = t + 3$, $x_2 = -2t - 4$, $x_3 = t$, where t is arbitrary.
(c) 1. (d) *Hint:* $(\mathbf{I} - \mathbf{A})(\mathbf{I} + \mathbf{A} + \mathbf{A}^2 + \mathbf{A}^3) = \mathbf{I} - \mathbf{A}^4$.

125. (a) $f_1'(x,y) = ye^{4x^2-5xy+y^2}(8x^2-5xy+1)$, $f_2'(x,y) = xe^{4x^2-5xy+y^2}(2y^2-5xy+1)$. (b) Stationary points: $(0,0)$, $(\frac{1}{2}\sqrt{2}, \sqrt{2})$, $(-\frac{1}{2}\sqrt{2}, -\sqrt{2})$. $f(x,1) \to \infty$ as $x \to \infty$, $f(x,1) \to -\infty$ as $x \to -\infty$. (c) $y' = 2$.

126. (a) $x = 1$ is a minimum point. No maximum exists. (b) $32/3 - 4\ln 3$. (*Hint:* Substitute $u = \sqrt{x} + 1$.)

127. (a) $3 - 6/e$. (*Hint:* Integration by parts.) (b) $x = 5\sqrt{3e^{-t} + 1}$, $\dot{x}(0) = -15/4$

128. (a) $4a^2 + 3$ (b) If $a \neq 0$ and $a \neq 1$, one degree of freedom. If $a = 0$, two degrees of freedom. If $a = 1$, no solutions. (c) $\mathbf{A}^4\mathbf{B} = \mathbf{A}^2(\mathbf{A}^2\mathbf{B}) = \mathbf{A}^2(\mathbf{AB})$, etc.

129. (a) Only possible solution: $(x,y,z) = (0, -1/2, 3/2)$. (b) Eliminate z from the constraints and show that x and y must be bounded. Then show that z must also be bounded. Moreover, the set of admissible points is closed. (It is a curve in $\mathbb{R}^3$.)

130. $\dfrac{dy}{dx} = \dfrac{-2(e+1)}{e+2}$, $\dfrac{dz}{dx} = \dfrac{-2e}{e+2}$, $\text{El}_x\, y = \dfrac{-2(e+1)}{e+2}$

131. (a) $|\mathbf{A}| = a^3 - 2a^2 + a = a(a-1)^2$, $\mathbf{A}^2 = \begin{pmatrix} 2a^2+1 & 2a^2+a & a^2+2a \\ 2a^2+a & 3a^2 & 2a^2+a \\ a^2+2a & 2a^2+a & 2a^2+1 \end{pmatrix}$

(b) If $a \neq 0$ and $a \neq 1$, then $|\mathbf{A}| \neq 0$, and the unique solution is $x = y = z = 0$, with 0 degrees of freedom. For $a = 0$, there is one degree of freedom. For $a = 1$, there are two degrees of freedom. (c) For $a = 1$, the solution is $x = 1 - t - s$, $y = t$, $z = s$, where s and t are arbitrary. For $a = 0$, we have $x = 1$ and $z = 1$, with y arbitrary. For $a \neq 0$ and $a \neq 1$, the solution is $x = 1 + a$, $y = -a - 2$ and $z = 1 + a$.

132. For $a \neq 0$, $(1,0)$ is a local minimum point and $(1 - a^3, a^2)$ is a saddle point. For $a = 0$, $(1,0)$ is a saddle point. (The second-derivative test does not apply when $a = 0$.)

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