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Answers to the examination problems in ECON 3120/4120, 6 June 2006

Problem 1

The first and second order derivatives of f are

$$\begin{aligned}
f_1'(x,y) &= (2x - ay)e^y, \\
f_2'(x,y) &= (x^2 - axy - ax)e^y = x(x - ay - a)e^y, \\
f_{11}''(x,y) &= 2e^y, \\
f_{12}''(x,y) &= (2x - ay - a)e^y, \\
f_{22}''(x,y) &= (x^2 - axy - 2ax)e^y = x(x - ay - 2a)e^y.
\end{aligned}$$

The stationary points are the solutions of the following system:

$$(1) \qquad 2x - ay = 0$$

$$(2) \qquad x(x - ay - a) = 0$$

If $x = 0$, then (1) gives $y = 0$ (because $a \neq 0$). If $x \neq 0$, then (2) gives $x = ay + a$, and then (1) gives $ay + 2a = 0$, i.e. $y = -2$, and so $x = ay + a = -a$.

Conclusion: There are two stationary points, $(0,0)$ and $(-a,-2)$.

(a) To determine the nature of a stationary point (x_0, y_0) we use the second-derivative test, with $A = f_{11}''(x_0,y_0)$, $B = f_{12}''(x_0,y_0)$, and $C = f_{22}''(x_0,y_0)$. The test gives

Point	A	B	C	$AC - B^2$	Result
$(0,0)$	2	$-a$	0	$-a^2$	Saddle point
$(-a,-2)$	$2e^{-2}$	$-ae^{-2}$	a^2e^{-2}	a^2e^{-4}	Local min. pt.

(b) $(x^*, y^*) = (-a,-2)$, and therefore

$$f^*(a) = f(-a,-2) = -a^2e^{-2} \quad \text{and} \quad df^*(a)/da = -2ae^{-2}.$$

On the other hand, $\hat{f}(x,y,a) = (x^2 - axy)e^y$, and

$$\hat{f}_3'(x,y,a) = -xye^y \quad \text{and} \quad \hat{f}_3'(x^*,y^*,a) = -x^*y^*e^{y^*} = -2ae^{-2}.$$

Thus the equation $\hat{f}_3'(x,y,a) = df^*(a)/da$ is true (as the envelope theorem also tells us).

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Problem 2

(a)

Gaussian elimination:

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 2 \\ 1 & t & -1 & 4 \end{pmatrix} \begin{array}{l} -2 \quad -1 \\ \\ \\ \end{array} \sim \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -3 & -6 \\ 0 & t-2 & -4 & 0 \end{pmatrix} \begin{array}{l} \\ \times(-\frac{1}{3}) \\ \\ \end{array}$$

$$\sim \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 1 & 2 \\ 0 & t-2 & -4 & 0 \end{pmatrix} \begin{array}{l} \\ -2 \quad 2-t \\ \\ \end{array} \sim \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & -t-2 & 4-2t \end{pmatrix}$$

The final matrix corresponds to the equation system

$$\begin{aligned} x + z &= 0 \\ y + z &= 2 \\ (-t-2)z &= 4 - 2t \end{aligned}$$

With $t = -2$, the last equation becomes $0 = 8$, which is impossible. Thus, there is *no solution if* $t = -2$.

If $t \neq -2$, then the system has a unique solution: The last equation gives

$$z = \frac{4-2t}{-t-2} = \frac{2t-4}{t+2},$$

and then

$$y = 2 - z = \frac{2(t+2) - (2t-4)}{t+2} = \frac{8}{t+2} \qquad \text{and} \qquad x = -z = \frac{4-2t}{t+2}.$$

(b) $\quad 2x_t \geq y_t \iff 2x_t - y_t \geq 0 \iff \dfrac{2(4-2t)-8}{t+2} \geq 0 \iff \dfrac{4t}{t+2} \leq 0.$

A simple argument with a sign diagram shows that this inequality holds if and only if $-2 < t \leq 0$. ($t = -2$ is excluded because the fractions are not defined there.)

Problem 3

(a) The equation is a linear first-order equation which can be written in standard form as $\dot{x} + a(t)x = b(t)$ with

$$a(t) = \frac{1}{t(t-1)} = \frac{1}{t-1} - \frac{1}{t} \quad \text{and} \quad b(t) = \frac{te^t}{t-1}\,.$$

The general solution can be found by means of formula (5.4.6) in FMEA or (1.4.6) in MA II. We shall need one indefinite integral of $a(t)$ (no arbitrary constant necessary):

$$\begin{aligned} A(t) = \int a(t)\,dt &= \int \left(\frac{1}{t-1} - \frac{1}{t}\right) dt \\ &= \ln|t-1| - \ln|t| = \ln(1-t) - \ln t = \ln\frac{1-t}{t}\,. \end{aligned}$$

(Remember that t lies between 0 and 1.) Then

$$e^{\int a(t)\,dt} = e^{A(t)} = \frac{1-t}{t} \qquad \text{and} \qquad e^{-\int a(t)\,dt} = \frac{t}{1-t}\,.$$

The solution formula in the book now yields the general solution:

$$\underline{\underline{x(t)}} = \frac{t}{1-t}\left(C + \int \frac{1-t}{t}\,\frac{te^t}{t-1}\,dt\right) = \frac{t}{1-t}\left(C - \int e^t\,dt\right) = \underline{\underline{\frac{t(C-e^t)}{1-t}}}\,.$$

(b) It is clear that $\lim_{t\to 0^+} x(t) = 0$ for all values of C. But what about $\lim_{t\to 1^-} x(t)$? The expression for $x(t)$ is a fraction whose denominator, $1-t$, tends to 0 as a limit as $t \to 1^-$. Thus for $x(t)$ to tend to a limit, the numerator, $t(C-e^t)$, must also tend to 0. That is, we must have $C = e$. With this value of C, we get

$$x(t) = t\,\frac{e-e^t}{1-t}\,,$$

and by l'Hôpital's rule,

$$\lim_{t\to 1^-} x(t) = 1 \cdot \lim_{t\to 1^-} \frac{e-e^t}{1-t} = \text{“}\frac{0}{0}\text{”} = \lim_{t\to 1^-} \frac{-e^t}{-1} = e\,.$$

Problem 4

(a) Integration gives

$$S = \int_0^T e^{-rx}(e^{gT-gx} - 1)\, dx = \int_0^T e^{gT-(r+g)x}\, dx - \int_0^T e^{-rx}\, dx$$
$$= -\Big|_0^T \frac{e^{gT-(r+g)x}}{r+g} + \Big|_0^T \frac{e^{-rx}}{r} = \frac{e^{gT} - e^{-rT}}{r+g} + \frac{e^{-rT} - 1}{r}$$

and therefore

$$\begin{aligned} r(r+g)S &= r(e^{gT} - e^{-rT}) + (r+g)(e^{-rT} - 1) \\ &= r(e^{gT} - e^{-rT}) - (r+g)(1 - e^{-rT})\,. \end{aligned}$$

(b) The given equation can be written as $F(r, g, S, T) = 0$, where

$$\begin{aligned} F(r, g, S, T) &= r(e^{gT} - e^{-rT}) - (r+g)(1 - e^{-rT}) - r(r+g)S \\ &= re^{gT} - (r+g) + ge^{-rT} - r(r+g)S\,. \end{aligned}$$

It follows that

$$\begin{aligned} \frac{\partial T}{\partial g} = -\frac{\partial F/\partial g}{\partial F/\partial T} = -\frac{F_2'(r, g, S, T)}{F_4'(r, g, S, T)} &= -\frac{rTe^{gT} - 1 + e^{-rT} - rS}{rge^{gT} - rge^{-rT}} \\ &= \frac{rS + 1 - rTe^{gT} - e^{-rT}}{rg(e^{gT} - e^{-rT})}\,. \end{aligned}$$