Problem 33

min $f(x,y)$ s.t. $x,y \in A$

$$\mathcal{L}(x,y) = \frac{1}{3}y^3 - 4y^2 + 15y - x^2 + 8x + \lambda x + \gamma y - \mu(y-10) + \theta(x+y-8)$$

$$\mathcal{L}'_x = -2x + 8 + \lambda + \theta = 0$$

$$\mathcal{L}'_y = y^2 - 8y + 15 + \gamma - \mu + \theta = 0$$

$\lambda \geq 0, \quad \lambda x = 0$
$\gamma \geq 0, \quad \gamma y = 0$
$\mu \geq 0, \quad \mu(y-10) = 0$
$\theta \geq 0, \quad \theta(x+y-8) = 0$

λ: * $\mathcal{L}'_x \Rightarrow 2x = 8 + \lambda + \theta \geq 8 \Rightarrow \boxed{x \geq 4 \Rightarrow x > 0 \Rightarrow \lambda = 0}$

γ: * If $y = 0$, $y < 10 \Rightarrow \mu = 0$

$\mathcal{L}'_y = 15 + \gamma + \theta > 0$ so $\mathcal{L}'_y = 0$ won't hold

$\boxed{\Rightarrow y > 0 \Rightarrow \gamma = 0}$

μ: * If $y = 10$

$y = 10, x \geq 4 \Rightarrow x + y > 8 \Rightarrow \theta = 0$

$\mathcal{L}_x \Rightarrow x = 4$ $\quad$ $\boxed{\text{so } (4,10) \text{ a candidate}}$

* If $y < 10$, $\mu = 0$

- If $x + y > 8$, $\theta = 0 \Rightarrow x = 4 \Rightarrow y > 4$

$\mathcal{L}'_y = y^2 - 8y + 15 = 0 \Rightarrow y = 3 \text{ or } 5 \Rightarrow y = 5$ as $y > 4$

$\boxed{(4,5) \text{ is a candidate}}$

$-$ If $x+y=8$, $\theta \geq 0$

$\mathcal{L}'_x \Rightarrow \quad \theta = 2x-8$

$\mathcal{L}'_y = y^2 - 8y + 15 + 2x - 8 \qquad , x = 8-y$

$= y^2 - 8y + 7 + 16 - 2y$

$= y^2 - 10y + 23$

$y = \dfrac{10 \pm \sqrt{100-92}}{2} = 5 \pm \dfrac{\sqrt{8}}{2} = 5 \pm \sqrt{2}$

By $\mathcal{L}$: $x \geq 4 \Rightarrow y \leq 4 \Rightarrow y = 5 - \sqrt{2}$

$x = 8-(5-\sqrt{2}) = 3+\sqrt{2}$

$\boxed{(3+\sqrt{2}, 5-\sqrt{2}) \text{ is a candidate}}$

Evaluate $(4,10)$, $(4,5)$, $(3+\sqrt{2}, 5-\sqrt{2})$

$f(4,10) = -\dfrac{1000}{3} + \dfrac{1200}{3} - \dfrac{450}{3} + 16 - 32 = -\dfrac{298}{3}$

$f(4,5) = -\dfrac{125}{3} + \dfrac{300}{3} - 75 + 16 - 32 = -\dfrac{98}{3}$

$f(3+\sqrt{2}, 5-\sqrt{2}) \approx -33.5$

So $(4,10)$ is min, $f = -\dfrac{298}{3}$